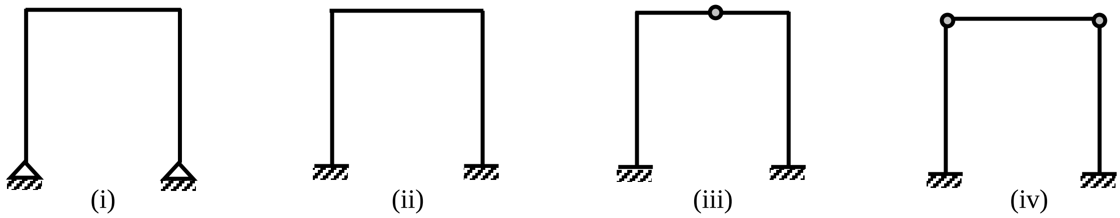
Approximate Lateral Load Analysis by Portal Method

Portal Frame

Portal frames, used in several Civil Engineering structures like buildings, factories, bridges have the primary purpose of transferring horizontal loads applied at their tops to their foundations. Structural requirements usually necessitate the use of statically indeterminate layout for portal frames, and approximate solutions are often used in their analyses.



Portal Frame Structures

Assumptions for the Approximate Solution

In order to analyze a structure using the equations of statics only, the number of independent force components must be equal to the number of independent equations of statics.

If there are n more independent force components in the structure than there are independent equations of statics, the structure is statically indeterminate to the n^{th} degree. Therefore to obtain an approximate solution of the structure based on statics only, it will be necessary to make n additional independent assumptions. A solution based on statics will not be possible by making fewer than n assumptions, while more than n assumptions will not in general be consistent.

Thus, the first step in the approximate analysis of structures is to find its degree of statical indeterminacy (dosi) and then to make appropriate number of assumptions.

For example, the dosi of portal frames shown in (i), (ii), (iii) and (iv) are 1, 3, 2 and 1 respectively. Based on the type of frame, the following assumptions can be made for portal structures with a *vertical axis of symmetry* that are *loaded horizontally at the top*

1. The horizontal support reactions are equal
2. There is a point of inflection at the center of the unsupported height of each fixed based column

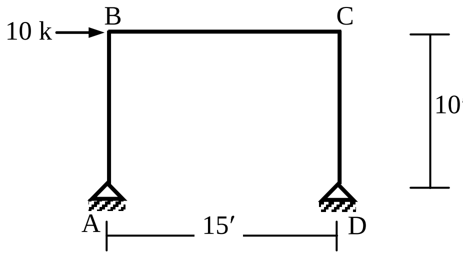
Assumption 1 is used if dosi is an odd number (i.e., = 1 or 3) and Assumption 2 is used if dosi > 1.

Some additional assumptions can be made in order to solve the structure approximately for different loading and support conditions.

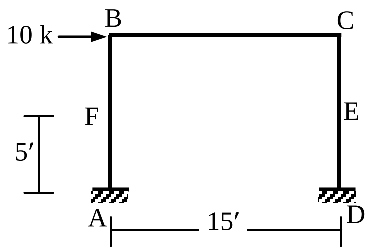
3. Horizontal body forces not applied at the top of a column can be divided into two forces (i.e., applied at the top and bottom of the column) based on simple supports
4. For hinged and fixed supports, the horizontal reactions for fixed supports can be assumed to be four times the horizontal reactions for hinged supports

Example

Draw the axial force, shear force and bending moment diagrams of the frames loaded as shown below.



(i)



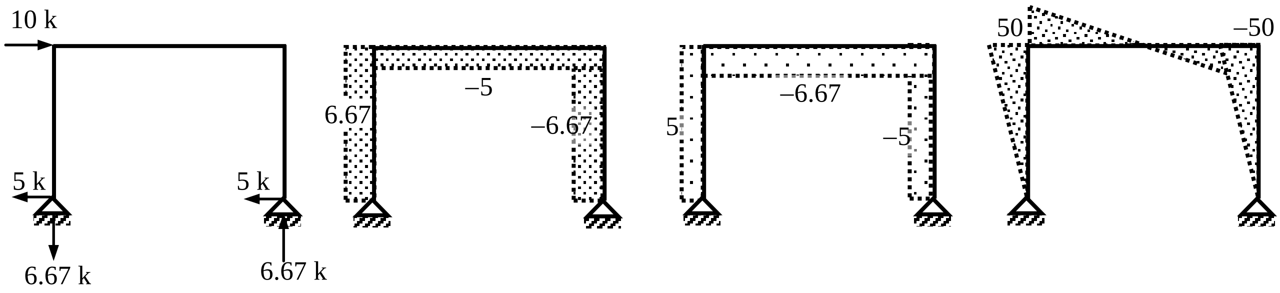
(ii)

Solution

(i) For this frame, dosi = $3 \times 3 + 4 - 3 \times 4 = 1$; i.e., Assumption 1 $\Rightarrow H_A = H_D = 10/2 = 5$ k

$\therefore \sum M_A = 0 \Rightarrow 10 \times 10 - V_D \times 15 = 0 \Rightarrow V_D = 6.67$ k

$\therefore \sum F_y = 0 \Rightarrow V_A + V_D = 0 \Rightarrow V_A = -6.67$ k



Reactions AFD (k) SFD (k) BMD (k-ft)

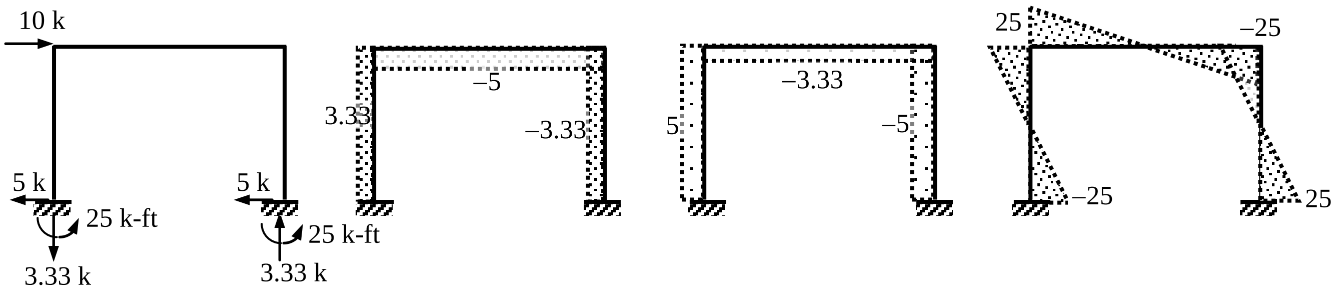
(ii) dosi = $3 \times 3 + 6 - 3 \times 4 = 3$

Assumption 1 $\Rightarrow H_A = H_D = 10/2 = 5$ k, Assumption 2 $\Rightarrow BM_E = BM_F = 0$

$\therefore BM_F = 0 \Rightarrow H_A \times 5 + M_A = 0 \Rightarrow M_A = -25$ k-ft; Similarly $BM_E = 0 \Rightarrow M_D = -25$ k-ft

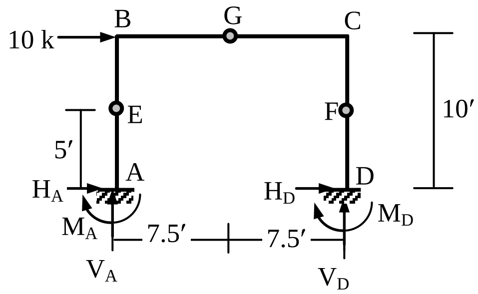
$\therefore \sum M_A = 0 \Rightarrow -25 - 25 + 10 \times 10 - V_D \times 15 = 0 \Rightarrow V_D = 3.33$ k

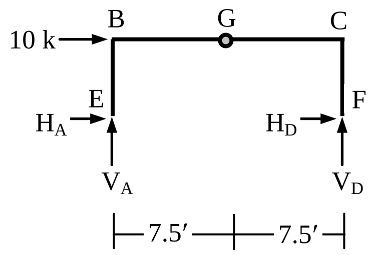
$\therefore \sum F_y = 0 \Rightarrow V_A + V_D = 0 \Rightarrow V_A = -3.33$ k



Reactions AFD (k) SFD (k) BMD (k-ft)

(iii)





dosi = 3 × 4 + 6 – 3 × 5 – 1 = 2; ∴ Assumption 1 and 2 ⇒ $BM_E = BM_F = 0$

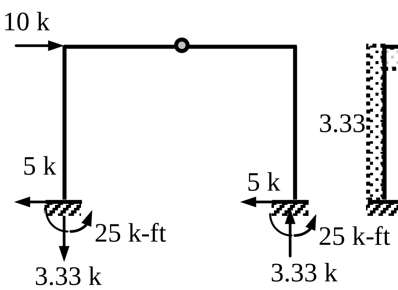
∴ $BM_E = 0$ (bottom) ⇒ $-H_A \times 5 + M_A = 0 \Rightarrow M_A = 5H_A$; Similarly $BM_F = 0 \Rightarrow M_D = 5H_D$

Also $BM_E = 0$ (free body of EBCF) ⇒ $10 \times 5 - V_D \times 15 = 0 \Rightarrow V_D = 3.33$ k

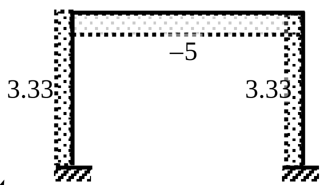
∴ $\sum F_y = 0 \Rightarrow V_A + V_D = 0 \Rightarrow V_A = -V_D = -3.33$ k

$BM_G = 0$ (between E and G) ⇒ $V_A \times 7.5 - H_A \times 5 = 0 \Rightarrow H_A = -5$ k ⇒ $M_A = 5H_A = -25$ k-ft

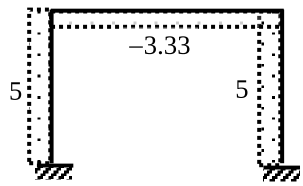
$\sum F_x = 0$ (entire structure) ⇒ $H_A + H_D + 10 = 0 \Rightarrow -5 + H_D + 10 = 0 \Rightarrow H_D = -5$ k ⇒ $M_D = 5H_D = -25$ k-ft



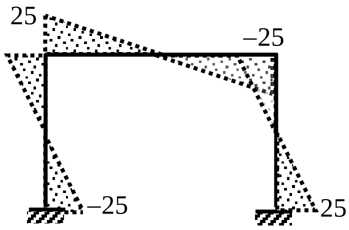
Reactions



AFD (k)



SFD (k)



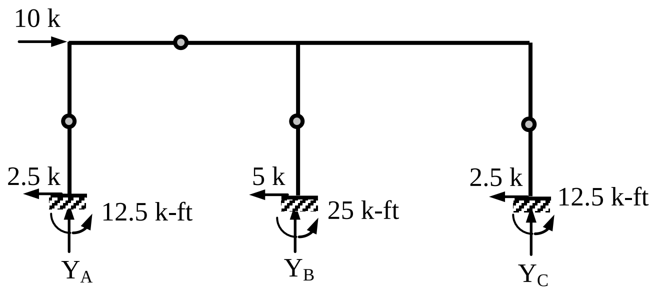
BMD (k-ft)

(iv) dosi = 3 × 5 + 9 – 3 × 6 = 6 ⇒ 6 Assumptions needed to solve the structure

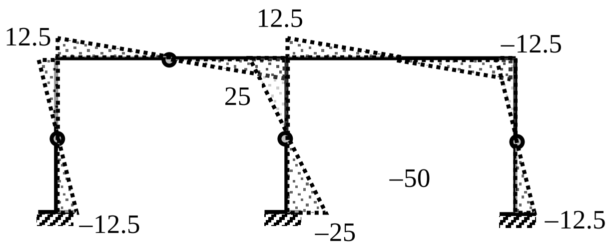
Assumption 1 and 2 ⇒ H_A: H_B: H_C = 1: 2: 1 ⇒ H_A = 10/4 = 2.5 k, H_B = 5 k, H_C = 2.5 k

∴ $M_A = M_C = 2.5 \times 5 = 12.5$ k-ft, $M_B = 5 \times 5 = 25$ k-ft

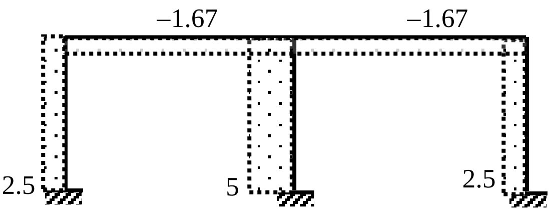
The other 4 assumptions are the assumed internal hinge locations at midpoints of columns and one beam



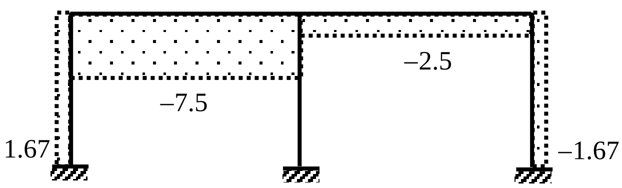
Reactions



BMD (k-ft)



SFD (k)



AFD (k)

Analysis of Multi-storied Structures by Portal Method

Approximate methods of analyzing multi-storied structures are important because such structures are statically highly indeterminate. The number of assumptions that must be made to permit an analysis by statics alone is equal to the degree of statical indeterminacy of the structure.

Assumptions

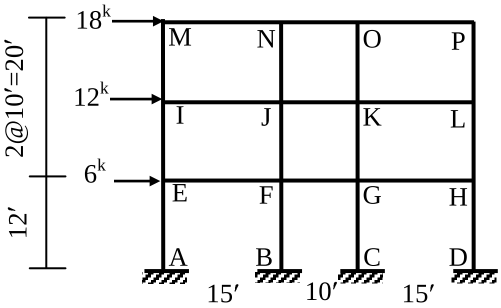
The assumptions used in the approximate analysis of portal frames can be extended for the lateral load analysis of multi-storied structures. The *Portal Method* thus formulated is based on three assumptions

1. The shear force in an interior column is twice the shear force in an exterior column.
2. There is a point of inflection at the center of each column.
3. There is a point of inflection at the center of each beam.

Assumption 1 is based on assuming the interior columns to be formed by columns of two adjacent bays or portals. Assumption 2 and 3 are based on observing the deflected shape of the structure.

Example

Use the Portal Method to draw the axial force, shear force and bending moment diagrams of the three-storied frame structure loaded as shown below.

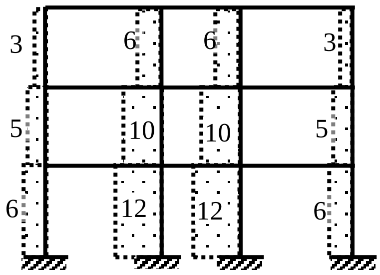


Column shear forces are at the ratio of 1:2:2:1.

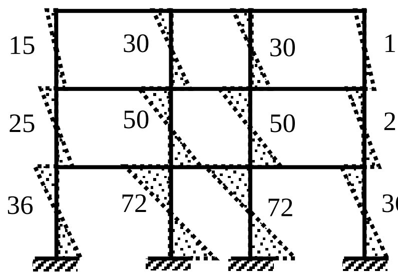
$\therefore$ Shear force in (V) columns IM, JN, KO, LP are
$[18 \times 1/(1 + 2 + 2 + 1) =]\ 3^k$, $[18 \times 2/(1 + 2 + 2 + 1) =]\ 6^k$, 6^k, 3^k respectively. Similarly,
$V_{EI} = 30\times1/(6) = 5^k$, $V_{FJ} = 10^k$, $V_{GK} = 10^k$, $V_{HL} = 5^k$; and
$V_{AE} = 36\times1/(6) = 6^k$, $V_{BF} = 12^k$, $V_{CG} = 12^k$, $V_{DH} = 6^k$

Bending moments are
$M_{IM} = 3\times10/2 = 15^{k'}$, $M_{JN} = 30^{k'}$, $M_{KO} = 30^{k'}$, $M_{LP} = 15^{k'}$
$M_{EI} = 5\times10/2 = 25^{k'}$, $M_{FJ} = 50^{k'}$, $M_{GK} = 50^{k'}$, $M_{HL} = 25^{k'}$
$M_{AE} = 6\times10/2 = 30^{k'}$, $M_{FJ} = 60^{k'}$, $M_{GK} = 60^{k'}$, $M_{HL} = 30^{k'}$

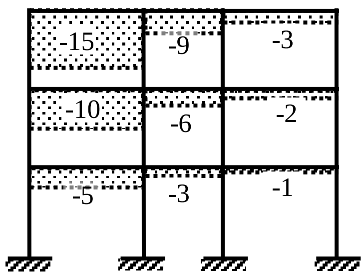
The rest of the calculations follow from the free-body diagrams



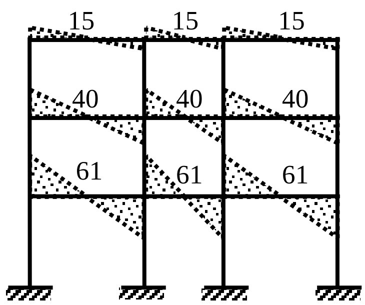
Column SFD (k)



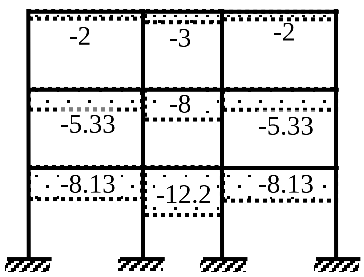
Column BMD (k-ft)



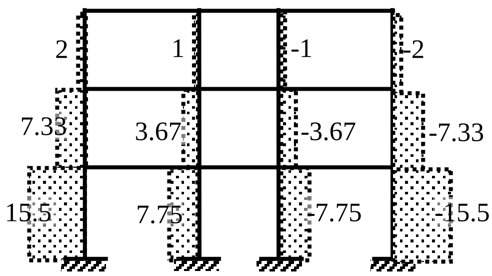
Beam AFD (k)



Beam BMD (k-ft)



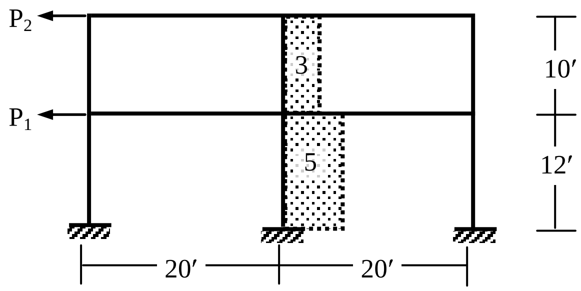
Beam SFD (k)



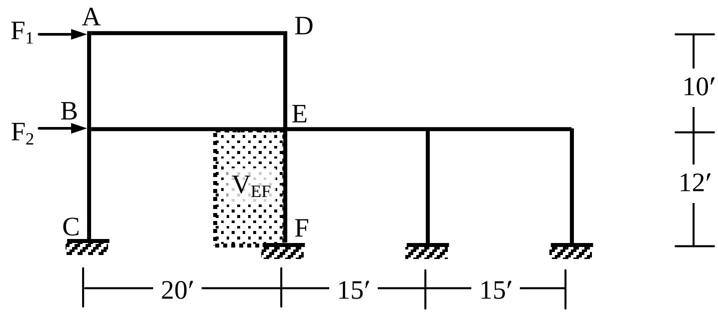
Column AFD (k)

Problems on Lateral Load Analysis by Portal Method

1. The figure below shows the shear forces (kips) in the interior columns of a two-storied frame. Use the Portal Method to calculate the corresponding
   (i) applied loads P_1 and P_2, (ii) column bending moments, (iii) beam axial forces.

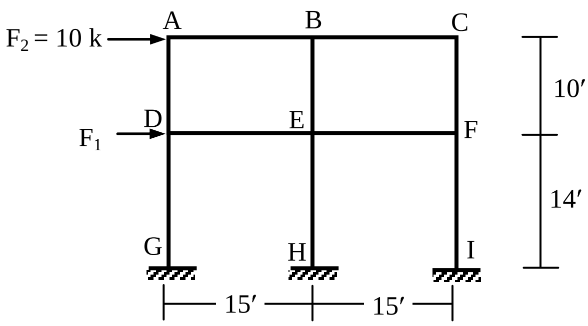


2. The figure below shows the applied loads (F_1, F_2) and shear force (V_{EF}) in column EF of a two-storied frame. If F_2 = 10 k, and V_{EF} = 5 k, use the Portal Method to calculate the
   (i) applied load F_1, (ii) maximum column bending moments.

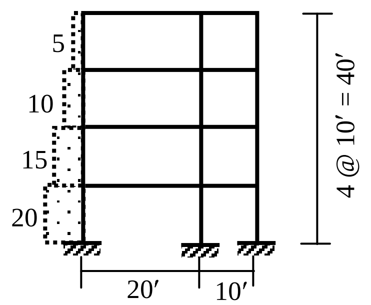


3. For the structure shown in Question 2, use the Portal Method to calculate the lateral loads F_1, F_2 if the axial forces in beams AD and BE are 10 kips and 15 kips respectively.

4. For the structure shown below, use the Portal Method to
   (i) draw the bending moment diagrams of the top floor beams AB and BC
   (i) calculate the applied load F_1 if the maximum bending moment in column EH is 30 k-ft.



5. The figure below shows the exterior column shear forces (kips) in a four-storied fame.
   Calculate (i) the applied loads, (ii) beam shear forces.



Analysis of Multi-storied Structures by Cantilever Method

Although the results using the *Portal Method* are reasonable in most cases, the method suffers due to the lack of consideration given to the variation of structural response due to the difference between sectional properties of various members. The *Cantilever Method* attempts to rectify this limitation by considering the cross-sectional areas of columns in distributing the axial forces in various columns of a story.

Assumptions

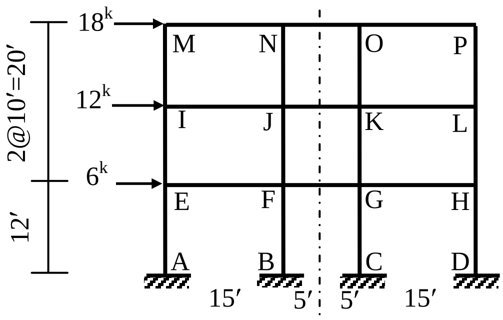
The *Cantilever Method* is based on three assumptions

1. The axial force in each column of a storey is proportional to its horizontal distance from the centroidal axis of all the columns of the storey.
2. There is a point of inflection at the center of each column.
3. There is a point of inflection at the center of each beam.

Assumption 1 is based on assuming that the axial stresses can be obtained by a method analogous to that used for determining the distribution of normal stresses on a transverse section of a cantilever beam. Assumption 2 and 3 are based on observing the deflected shape of the structure.

Example

Use the Cantilever Method to draw the axial force, shear force and bending moment diagrams of the three - storied frame structure loaded as shown below.



The dotted line is the column centerline (at all floors)

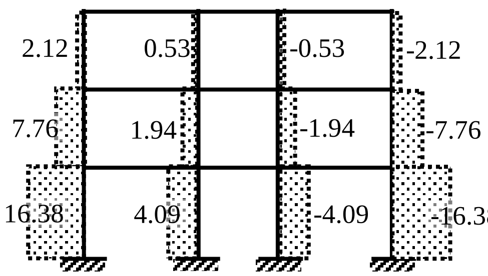
$\therefore$ Column axial forces are at the ratio of 20: 5: –5: –20.

$\therefore$ Axial force in (P) columns IM, JN, KO, LP are $[18 \times 5 \times 20/\{20^2 + 5^2 + (-5)^2 + (-20)^2\} =]\ 2.12^k$, $[18 \times 5 \times 5/(20^2 + 5^2 + (-5)^2 + (-20)^2\} =]\ 0.53^k$, -0.53^k, -2.12^k respectively.

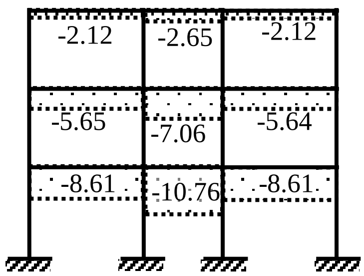
Similarly, $P_{EI} = 330\times20/(850) = 7.76^k$, $P_{FJ} = 1.94^k$, $P_{GK} = -1.94^k$, $P_{HL} = -7.76^k$; and

$P_{AE} = 696\times20/(850) = 16.38^k$, $P_{BF} = 4.09^k$, $P_{CG} = -4.09^k$, $P_{DH} = 16.38^k$

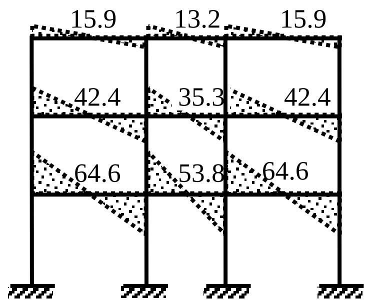
The rest of the calculations follow from the free-body diagrams



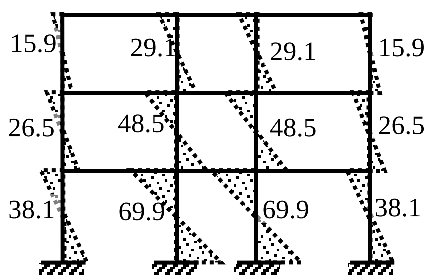
Column AFD (k)



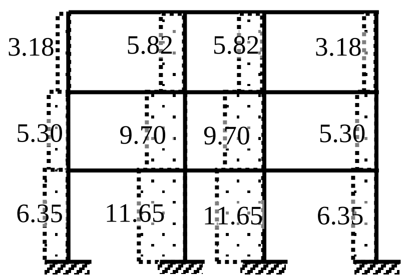
Beam SFD (k)



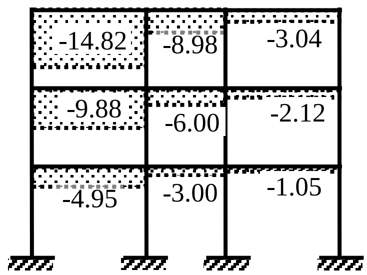
Beam BMD (k-ft)



Column BMD (k-ft)

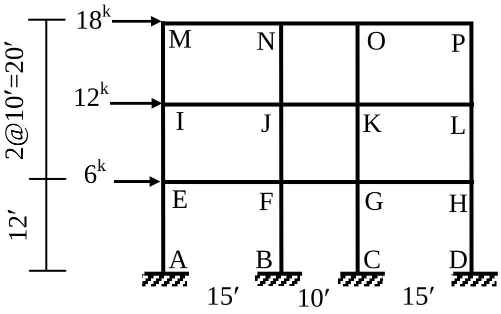


Column SFD (k)

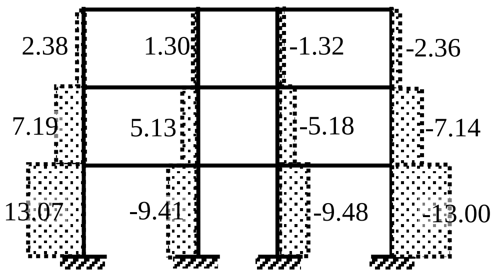


Beam AFD (k)

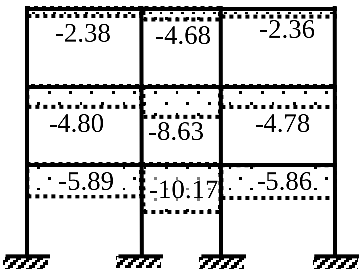
Results from 'Exact' Structural Analysis



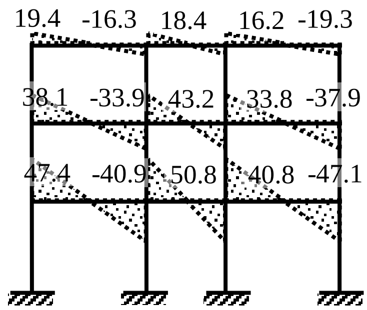
All members have equal cross-sections



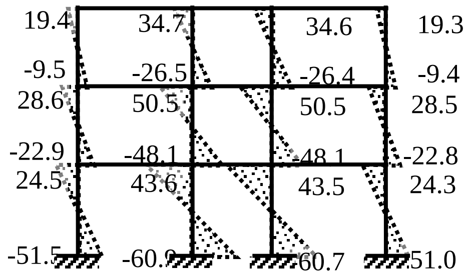
Column AFD (k)



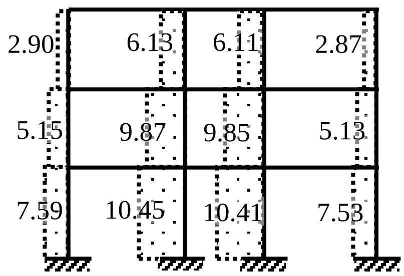
Beam SFD (k)



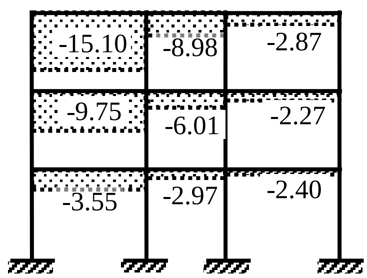
Beam BMD (k-ft)



Column BMD (k-ft)

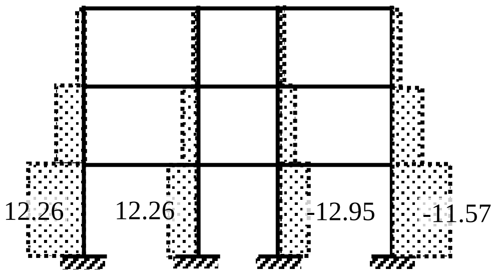


Column SFD (k)

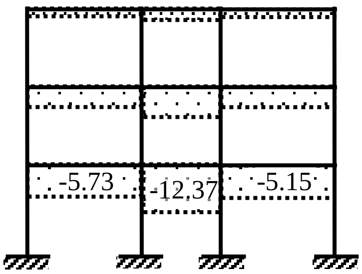


Beam AFD (k)

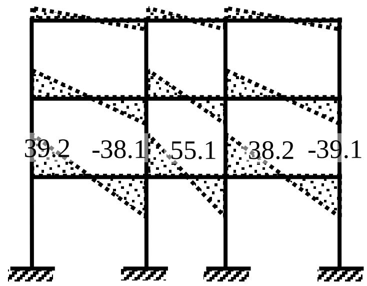
Interior columns have twice the area of exterior columns



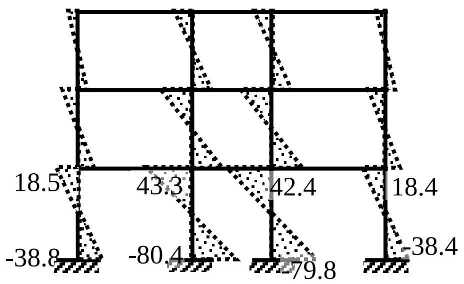
Column AFD (k)



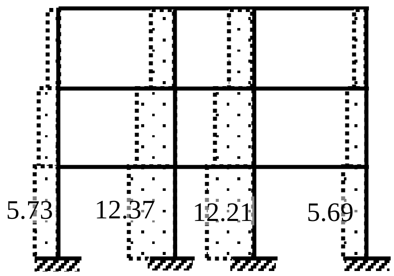
Beam SFD (k)



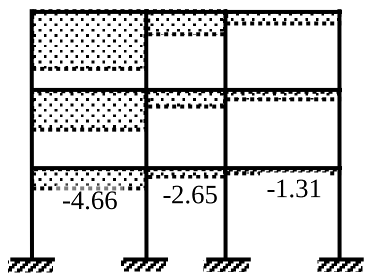
Beam BMD (k-ft)



Column BMD (k-ft)



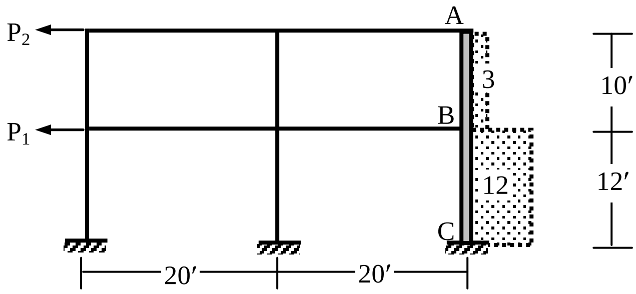
Column SFD (k)



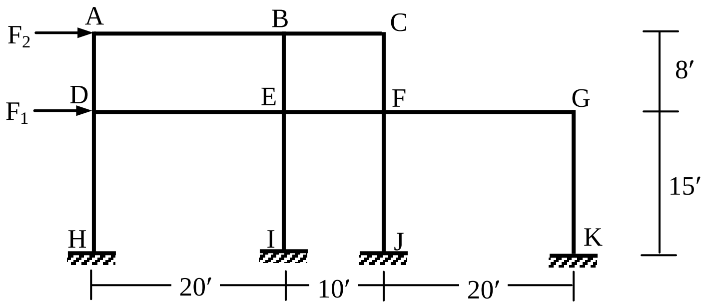
Beam AFD (k)

Problems on Lateral Load Analysis by Cantilever Method

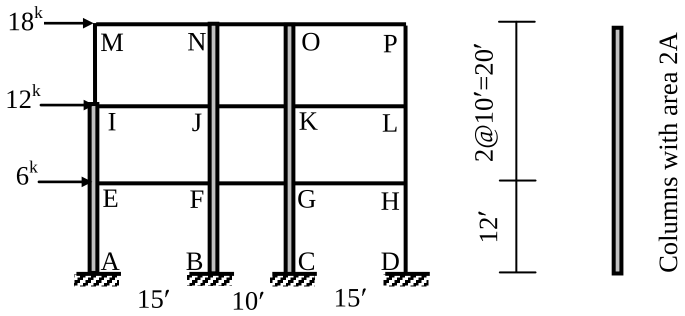
1. The figure below shows the axial forces (kips) in the exterior columns of a two-storied frame. If the cross-sectional area of column ABC is twice the area of the other columns, use the Cantilever Method to calculate the corresponding applied loads P_1 and P_2.



2. For the structure shown below, use the Cantilever Method to calculate the lateral loads F_1, F_2 if the shear forces in beams AB and DE are 10 kips and 15 kips respectively. Assume all the columns have the same cross-sectional area.



3. Use the Cantilever Method to draw the axial force, shear force and bending moment diagrams of the three-storied structure loaded as shown below.



4. Figure (a) below shows the exterior column axial forces (kips) in a three-storied fame. Use the Cantilever Method to calculate (i) the applied loads, (ii) beam bending moments, (iii) column bending moments. Assume all the columns to have equal cross-sectional areas.

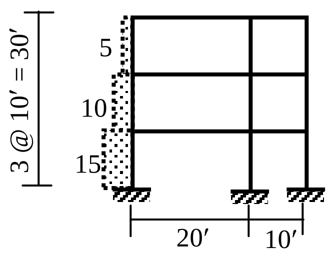


Fig. (a)

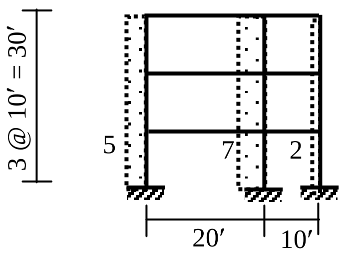


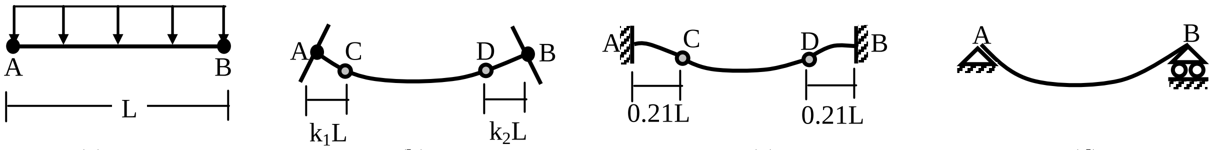
Fig. (b)

5. Figure (b) above shows the column shear forces (kips) in a three-storied fame. Calculate the column BM, beam BM, beam SF and column AF. Also check if they satisfy the conditions for Cantilever Method (for equal column areas).

Approximate Vertical Load Analysis

Approximation based on the Location of Hinges

If a beam AB is subjected to a uniformly distributed vertical load of w per unit length [Fig. (a)], both the joints A and B will rotate as shown in Fig. (b), because although the joints A and B are partly restrained against rotation, the restraint is not complete. Had the joints A and B been completely fixed against rotation [Fig. (c)] the points of inflection would be located at a distance 0.21L from each end. If, on the other hand, the joints A and B are hinged [Fig. (d)], the points of zero moment would be at the end of the beam. For the actual case of partial fixity, the points of inflection can be assumed to be somewhere between 0.21 L and 0 from the end of the beam. *For approximate analysis, they are often assumed to be located at one-tenth (0.1 L) of the span length from each end joint.*



(a) (b) (c) (d)

Depending on the support conditions (i.e., hinge ended, fixed ended or continuous), a beam in general can be statically indeterminate up to a degree of three. Therefore, to make it statically determinate, the following three assumptions are often made in the vertical load analysis of a beam

1. The axial force in the beam is zero
2. Points of inflection occur at the distance 0.1 L measured along the span from the left and right support.

Bending Moment and Shear Force from Approximate Analysis

Based on the approximations mentioned (i.e., points of inflection at a distance 0.1 L from the ends), the maximum positive bending moment in the beam is calculated to be

$M_{(+)} = w(0.8L)^2/8 = 0.08\ wL^2$, at the midspan of the beam

$\therefore$ The maximum negative bending moment is

$M_{(-)} = wL^2/8 - 0.08\ wL^2 = 0.045\ wL^2$, at the joints A and B of the beam

The shear forces are maximum (positive or negative) at the joints A and B and are calculated to be

$V_A = wL/2$, and $V_B = -wL/2$

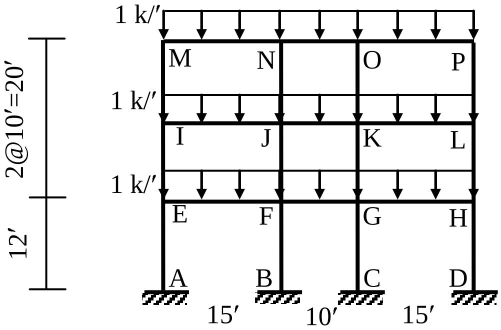
Moment and Shear Values using ACI Coefficients

Maximum allowable LL/DL = 3, maximum allowable adjacent span difference = 20%

1. Positive Moments
   (i) For End Spans
      (a) If discontinuous end is unrestrained, $M_{(+)} = wL^2/11$
      (b) If discontinuous end is restrained, $M_{(+)} = wL^2/14$
   (ii) For Interior Spans, $M_{(+)} = wL^2/16$
2. Negative Moments
   (i) At the exterior face of first interior supports
      (a) Two spans, $M_{(-)} = wL^2/9$
      (b) More than two spans, $M_{(-)} = wL^2/10$
   (ii) At the other faces of interior supports, $M_{(-)} = wL^2/11$
   (iii) For spans not exceeding 10′, of where columns are much stiffer than beams, $M_{(-)} = wL^2/12$
   (iv) At the interior faces of exterior supports
      (a) If the support is a beam, $M_{(-)} = wL^2/24$
      (b) If the support is a column, $M_{(-)} = wL^2/16$
3. Shear Forces
   (i) In end members at first interior support, V = 1.15wL/2
   (ii) At all other supports, V = wL/2
   [where L = clear span for $M_{(+)}$ and V, and average of two adjacent clear spans for $M_{(-)}$]

Example

Analyze the three-storied frame structure loaded as shown below using the approximate location of hinges to draw the axial force, shear force and bending moment diagrams of the beams and columns.



The maximum positive and negative beam moments and shear forces are as follows.

For the 15′ beam, $M_{(+)} = 0.08 \times 1 \times 15^2 = 18$ k-ft

$M_{(-)} = 0.045 \times 1 \times 15^2 = 10.13$ k-ft

$V_{(\pm)} = 1 \times 15/2 = 7.5$ k

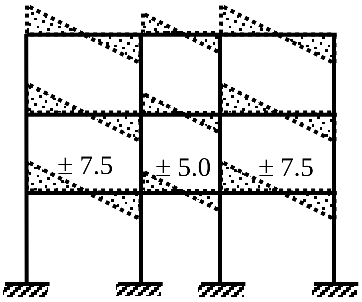
For the 10′ beam, $M_{(+)} = 0.08 \times 1 \times 10^2 = 8$ k-ft

$M_{(-)} = 0.045 \times 1 \times 10^2 = 4.5$ k-ft

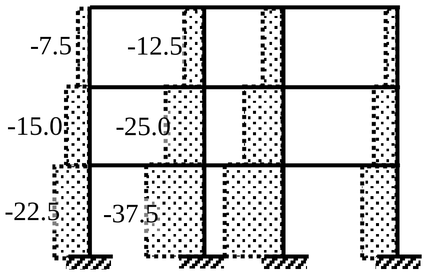
$V_{(\pm)} = 1 \times 10/2 = 5$ k

Axial Force P in all the beams = 0

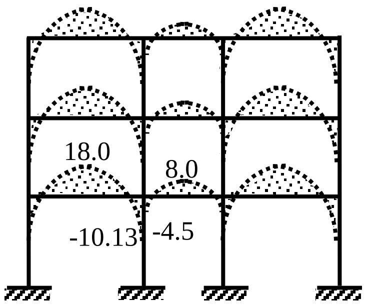
The rest of the calculations follow from the free-body diagrams



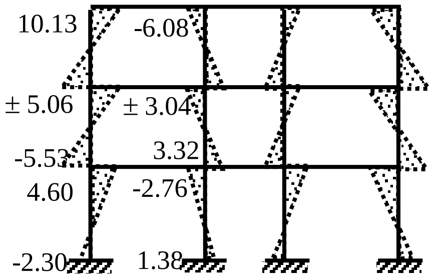
Beam SFD (k)



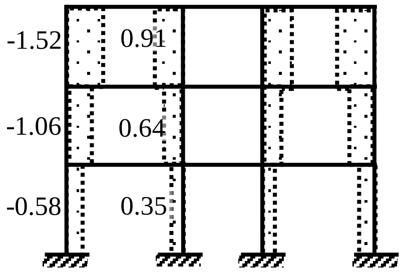
Column AFD (k)



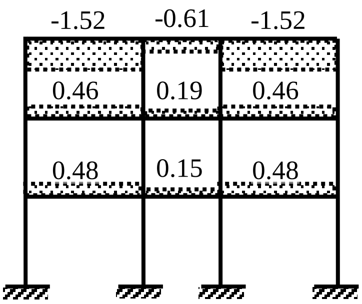
Beam BMD (k-ft)



Column BMD (k-ft)

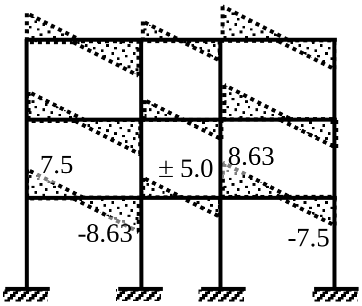


Column SFD (k)

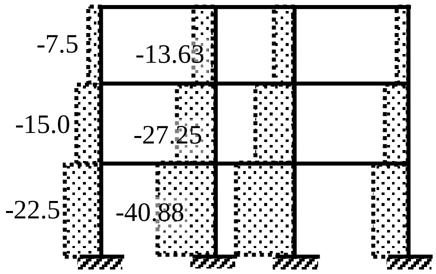


Beam AFD (k)

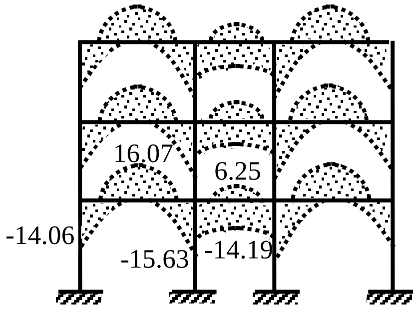
Using the ACI coefficients (for pattern loading)



Beam SFD (k)



Column AFD (k)

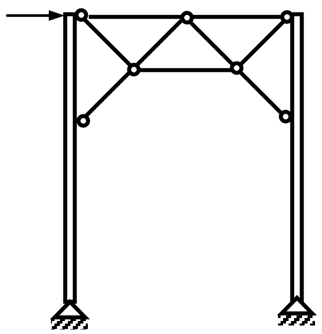


Beam BMD (k-ft)

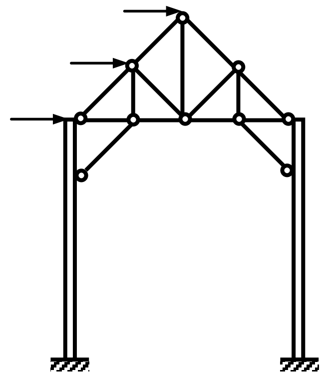
Approximate Analysis of Bridge Portal and Mill Bent

<u>Bridge Portals and Mill Bents</u>

Portals for bridges or bents for mill buildings are often arranged in a manner to include a truss between two flexural members. In such structures, the flexural members are continuous from the foundation to the top and are designed to carry bending moment, shear force as well as axial force. The other members that constitute the truss at the top of the structure are considered pin-connected and to carry axial force only.



Bridge Portal



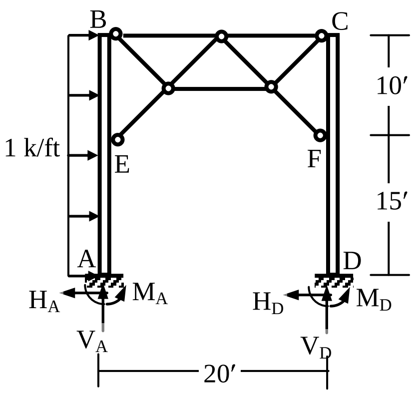
Mill Bent

Such a structure can be statically indeterminate to the first or third degree, depending on whether the supports are assumed hinged of fixed. Therefore, the same three assumptions made earlier for portal frames can be made for the approximate analysis of these structures also; i.e., for a load applied at the top

1. The horizontal support reactions are equal
2. There is a point of inflection at the center of the unsupported height of each fixed based column

<u>Example</u>

In the bridge portal loaded as shown below, draw the bending moment diagrams of columns AB and CD.



Assuming the total load to be applied equally (i.e., 25/2 = 12.5 k and 12.5 k) at A and B, the horizontal reactions are $H_A = 12.5 + 12.5/2 = 18.75$ k, $H_B = 12.5/2 = 6.25$ k

Also, BM = 0 at the midpoint of the free height; i.e., at 15/2 = 7.5′ from the bottom.

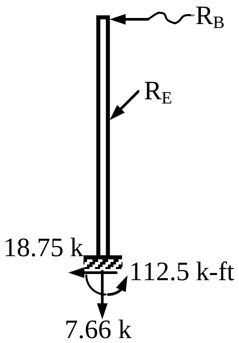
$\therefore -M_A + 18.75 \times 7.5 - 7.5 \times 7.5/2 = 0 \Rightarrow M_A = 112.5$ k-ft

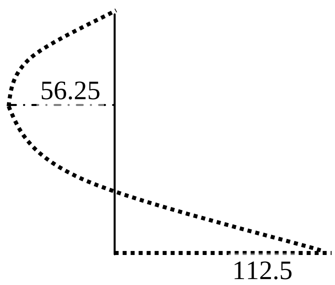
$-M_D + 6.25 \times 7.5 = 0 \Rightarrow M_D = 46.88$ k-ft

$\therefore \sum M_A = 0 \Rightarrow -112.5 - 46.88 + 25 \times 12.5 - V_D \times 20 = 0$

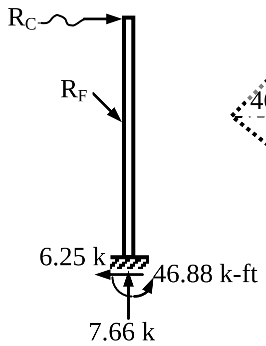
$\Rightarrow V_D = 7.66$ k

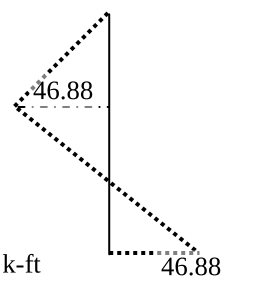
$\therefore \sum F_y = 0 \Rightarrow V_A + V_D = 0 \Rightarrow V_A = -7.66$ k





BMD (k-ft) of AB





BMD (k-ft) of CD

Approximate Analysis of Statically Indeterminate Trusses

Two approximate methods are commonly used for the analysis of statically indeterminate trusses. The methods are based on two basic assumptions

Method 1: Diagonal members take equal share of the sectional shear force

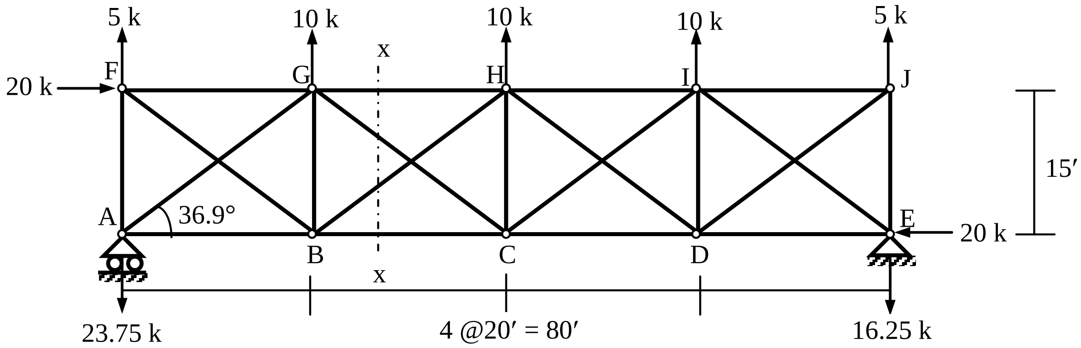
Method 2: Diagonal members can take tension only (i.e., they cannot take any compression)

Example

Calculate member forces GC, BH, GH, BC of the statically indeterminate truss shown below assuming

(i) Diagonal members take equal share of the sectional shear force,

(ii) Diagonal members can take tension only.



$\sum F_x = 0 \Rightarrow E_x + 20 = 0 \Rightarrow E_x = -20$

$\sum M_E = 0 \Rightarrow 20 \times 15 + (5 + 10 + 10 + 10 + 5) \times 40 + A_y \times 80 = 0 \Rightarrow A_y = -23.75$ k

$\sum F_y = 0 \Rightarrow A_y + E_y + 5 + 10 + 10 + 10 + 5 = 0 \Rightarrow E_y = -16.25$ k

(i) At section x-x,

$\sum F_x = 0 \Rightarrow F_{GH} + F_{BC} + F_{BH} \cos 36.9° + F_{GC} \cos 36.9° + 20 = 0$

$\Rightarrow F_{GH} + F_{BC} + 0.8\, F_{BH} + 0.8\, F_{GC} + 20 = 0$

$\sum F_y = 0 \Rightarrow F_{BH} \sin 36.9° - F_{GC} \sin 36.9° + 5 + 10 - 23.75 = 0$

$\Rightarrow 0.6\, F_{BH} - 0.6\, F_{GC} = 8.75$

Assuming diagonal members to take equal share of the sectional shear force

$\Rightarrow 0.6\, F_{BH} = -0.6\, F_{GC} = 8.75/2 = 4.375 \Rightarrow F_{BH} = 7.29$ k, $F_{GC} = -7.29$ k

$\sum M_B = 0 \Rightarrow -23.75 \times 20 + 5 \times 20 + 20 \times 15 - 0.8 \times 7.29 \times 15 + F_{GH} \times 15 = 0 \Rightarrow F_{GH} = 10.83$ k

$\therefore \sum F_x = 0 \Rightarrow F_{GH} + F_{BC} + 0.8\, F_{BH} + 0.8\, F_{GC} + 20 = 0 \Rightarrow F_{BC} = -30.83$ k

(ii) Assuming the diagonal member GC to fail in compression (i.e., to be non-existent)

At section x-x,

$\sum F_x = 0 \Rightarrow F_{GH} + F_{BC} + F_{BH} \cos 36.9° + 20 = 0 \Rightarrow F_{GH} + F_{BC} + 0.8\, F_{BH} + 20 = 0$

$\sum F_y = 0 \Rightarrow F_{BH} \sin 36.9° + 5 + 10 - 23.75 = 0 \Rightarrow F_{BH} = 14.58$ k

$\sum M_B = 0 \Rightarrow -23.75 \times 20 + 5 \times 20 + 20 \times 15 + F_{GH} \times 15 = 0 \Rightarrow F_{GH} = 5$ k

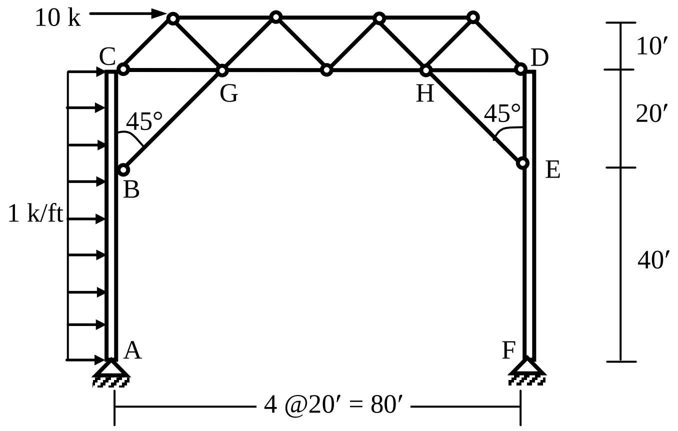
$\therefore \sum F_x = 0 \Rightarrow F_{GH} + F_{BC} + 0.8\, F_{BH} + 20 = 0 \Rightarrow F_{BC} = -36.67$ k

Note: The actual values from GRASP (assuming identical member sections) are

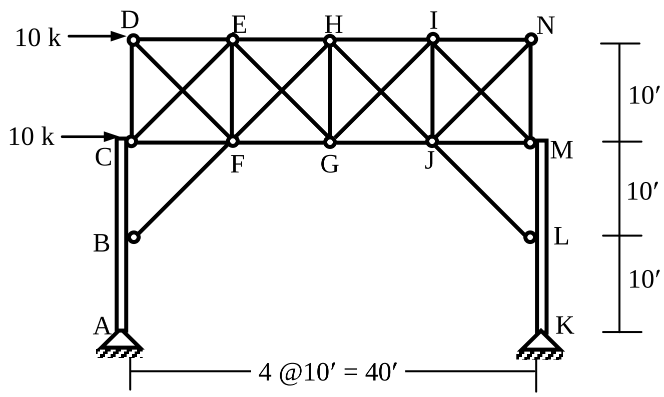
$F_{BH} = 4.88$ k, $F_{GC} = -9.71$ k, $F_{GH} = 12.77$ k, $F_{BC} = -28.90$ k

Problems on Approximate Analysis of Bridge Portal, Mill Bent and Truss

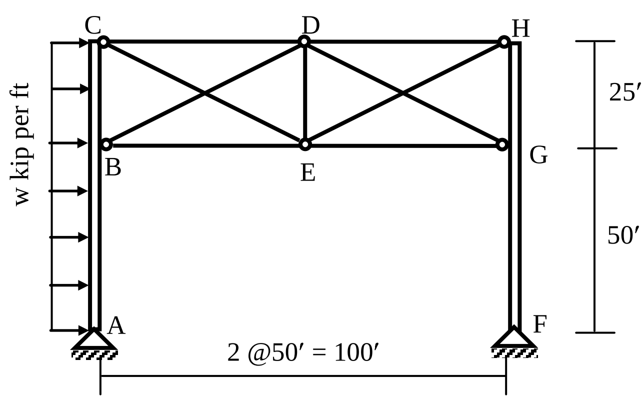
1. In the mill bent shown below, use the portal method to calculate the axial forces in members BG and EH and draw the shear force and bending moment diagrams of ABC and DEF.



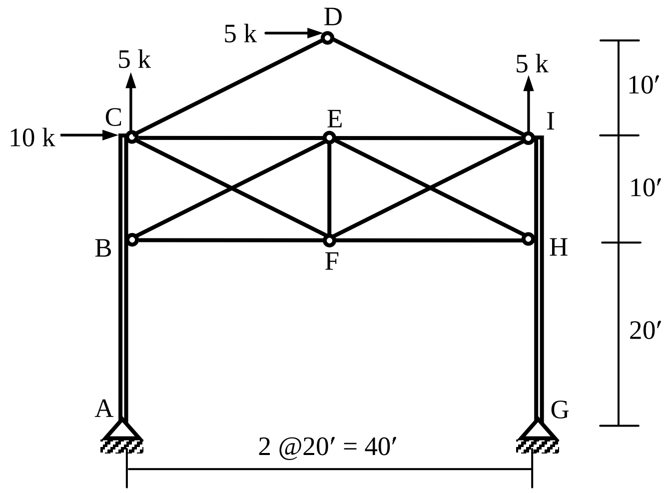
2. In the mill bent shown below,
   (i) Use the Portal Method to draw the bending moment diagram of the member KLM.
   (ii) Calculate the forces in EG and FH, assuming them to take equal share of the sectional shear.



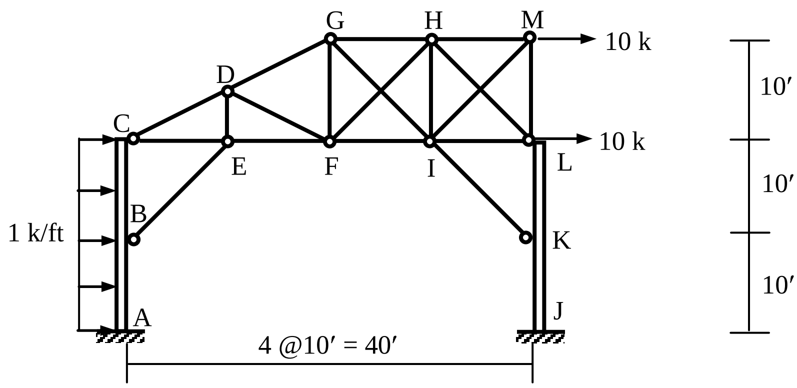
3. In the bridge portal shown below, compression in member DG is 10 kips. Use the Portal Method to
   (i) calculate the load w per unit length, assuming the diagonal members to share the sectional shear force equally.
   (ii) draw the BMD and SFD of the member FGH for the value of w calculated in (i).



4. In the structure shown below,
   (i) Use the Portal Method to calculate the reactions at support A, G and draw the BMD of ABC.
   (ii) Calculate the forces in members CD, BE, CF, assuming diagonal members to take tension only.



5. In the bridge portal shown below,
   (i) Use the Portal Method to calculate the reactions at support A and force in member BE.
   (ii) Calculate the forces in members GI and FH, assuming diagonal members to take tension only.



Deflection Calculation by the Method of Virtual Work

Method of Virtual Work

Another way of representing the equilibrium equations is by energy methods, which is based on the law of conservation of energy. According to the principle of virtual work, if a system in equilibrium is subjected to virtual displacements, the virtual work done by the external forces (δW_E) is equal to the virtual work done by the internal forces (δW_I)

$$\delta W_E = \delta W_I \qquad \text{......................(1)}$$

where the symbol δ is used to indicate 'virtual'. This term is used to indicate hypothetical increments of displacements and works that are assumed to happen in order to formulate the problem.

Consider the body loaded as shown in Fig. 1. Under the given loading conditions, the point A deflects an amount Δ in the direction shown in the Figure. Moreover the same load causes the element B within the body to extend an amount dL in the direction shown.

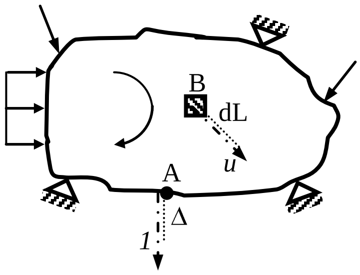


Fig. 1

If a *virtual unit load* (i.e., a load of magnitude *1*), when applied in the direction of Δ, causes a *virtual internal force u* in the element B in the direction of dL, the virtual work done by the external forces

$$\delta W_E = \mathit{1}.\ \Delta \qquad \text{......................(2)}$$

while the virtual work done by the virtual internal force (u) on B is = u. dL ....(3)

$\therefore$ The total internal virtual work done is $\delta W_I = \Sigma\ u.\ dL$ ....(4)

where the symbol Σ indicates the summation over the lengths of all the elements within the body.
In this formulation, the terms in *italic* indicate *virtual loads* or *internal forces*.

$\therefore$ The principle of virtual work [Eq. (1)] $\Rightarrow \mathit{1}.\ \Delta = \Sigma\ u.\ dL \Rightarrow \Delta = \Sigma\ u.\ dL$ ....(5)

It is to be noted here that the term Δ above can indicate the deflection or rotation of the body, depending on which the *virtual load (1)* can be a unit force or a unit moment applied in the direction of Δ.

Deflection of Truss due to External Loads

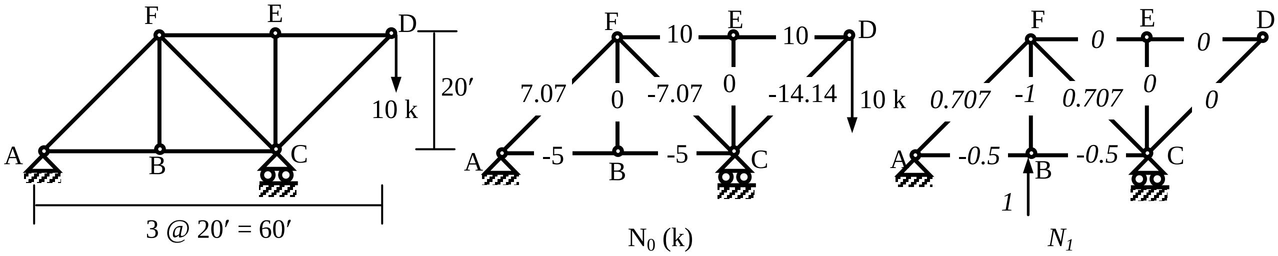
The above principle can be applied to calculate the deflection of a truss due to axial deformation of its members. This axial deformation can be caused be caused by external loads on the truss, temperature change or misfit of member length. The axial deformation due to external loads is caused by the internal forces within the truss members, the resulting extension of a truss member being

$$dL = N_0L/EA \qquad \text{......................(6)}$$

where N_0, L, E and A stand for the axial force (due to external loads), length, modulus of elasticity and cross-sectional area of a truss member. The internal force u due to the *unit virtual load* is often expressed by N_1, from which the equation for truss deflection [Eq. (5)] becomes $\Delta = \Sigma\ N_1.\ N_0L/EA$ ....(7)

Example

Calculate the vertical deflection of the point B of the truss ABCDEF due to the external loads applied [Given: EA/L = 500 kip/ft, for all the truss members].



Using member forces N_0 and N_1 from the above analyses, $\Delta = \Sigma\ N_0\ N_1\ L/EA$ ....(7)

$\therefore$ Ignoring zero force members,

$$\Delta_{B,v} = \{(7.07)\ (0.707) + (-7.07)\ (0.707) + (-5)\ (-0.5) + (-5)\ (-0.5)\}/500 = 0.01\ \text{ft} \uparrow$$

Deflection of Truss due to Temperature Change and Misfit

In addition to external loads, a truss joint may deflect due to change in member lengths (i.e., become longer or shorter than its original length) caused by change in temperature or geometrical misfit of any truss member (being longer or shorter than its specified length).

$\therefore$ In Eq. (5); i.e., $\Delta = \Sigma\, u.\, dL$ …..…………………(5)

the tem dL (elongation of a truss member) can also be due to temperature change or fabrication defect of any truss member.

The change in length due to increase in the temperature ΔT is $= \alpha \Delta T\, L$ …..…………………(8)

where α = Coefficient of thermal expansion; i.e., change of length of a member of unit length due to unit change of temperature, ΔT = Change of temperature of a member of length L.

Adding to it a geometric misfit (due to fabrication defect) of ΔL, the total elongation of a truss member

$$dL = N_0 L/EA + \alpha \Delta T\, L + \Delta L \qquad \text{…..…………………(9)}$$

from which the equation for truss deflection [Eq. (5)] becomes

$$\Delta = \Sigma\, N_1\, dL = \Sigma\, N_1\, (N_0 L/EA + \alpha \Delta T\, L + \Delta L) \qquad \text{…..…………………(10)}$$

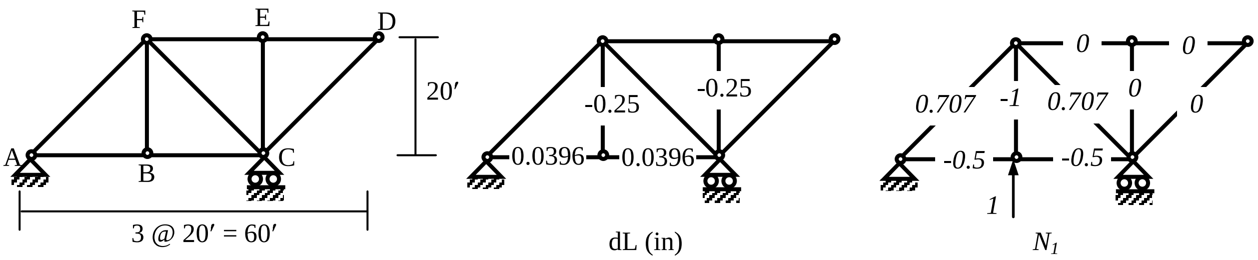
Example

Calculate the vertical deflection of joint B of the truss ABCDEF shown below due to

(i) temperature rise of 30°F in the bottom cord members AB and BC,

(ii) fabrication defects resulting in vertical members BF and CE to be made 0.25″ shorter than specified

[Given: Coefficient of thermal expansion $\alpha = 5.5 \times 10^{-6}$/°F, for all the truss members].



(i) For members AB and BC, $\alpha = 5.5 \times 10^{-6}$/°F, $\Delta T = 30$°F, L = 20 ft = 240 in

$\therefore dL = \alpha \Delta T\, L = (5.5 \times 10^{-6})\,(30)\,(240) = 0.0396$ in

$\therefore$ Ignoring zero force members, $\Delta_{B,v} = (0.0396)\,(-0.5) + (0.0396)\,(-0.5) = -0.0396$ in $\downarrow$

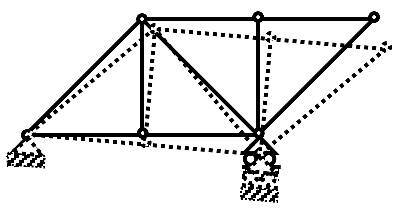
(ii) For members BF and CE, dL = –0.25 in

$\therefore$ Ignoring zero force members, $\Delta_{B,v} = (-0.25)\,(-1) + (-0.25)\,(0) = 0.25$ in $\uparrow$

Support Settlement

Settlement of supports due to consolidation or instability of the subsoil/foundation is a major reason of deflection of structures. There is a fundamental difference between the effect of support settlement on statically determinate and indeterminate structures. While it causes deflection due to geometrical changes only in statically determinate structures, it induces internal stresses in statically indeterminate structures (which may even be more significant than the forces due to external loads).

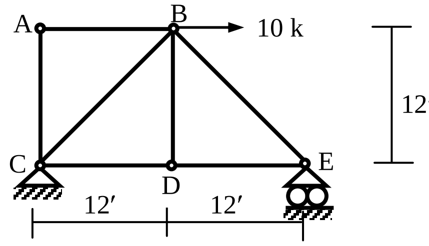
The effect of support settlement on statically indeterminate structures is dealt separately but the following figure shows the deflected shape of the truss ABCDEF shown above due to settlement of support C.



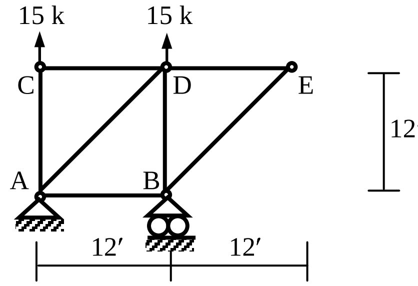
Problems on Truss Deflection by the Method of Virtual Work

Assume EA/L = 500 k/ft, $\alpha = 5.5 \times 10^{-6}$/°F for the following trusses.

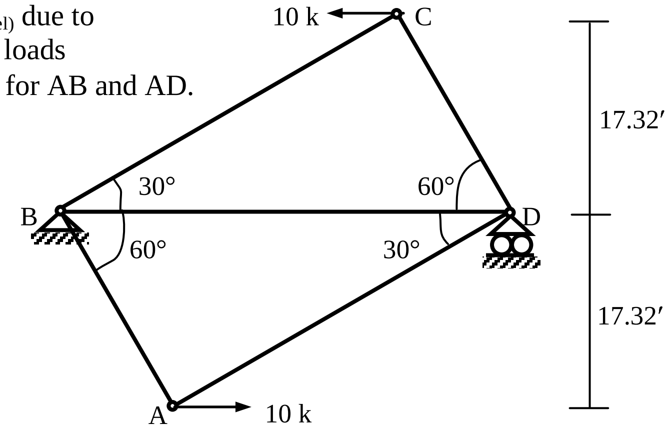
1. Calculate $\Delta_{E,h}$ due to
   (i) The external load
   (ii) ΔT = 50°F for CD and CE.



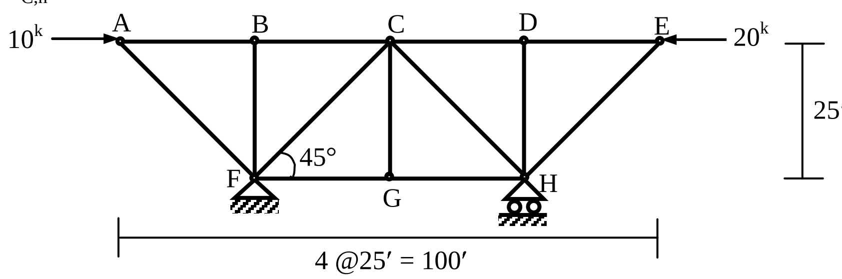
2. Calculate $\Delta_{B\text{-}C(rel)}$ due to
   (i) The external loads
   (ii) ΔL = 0.5″ for CD and CE.



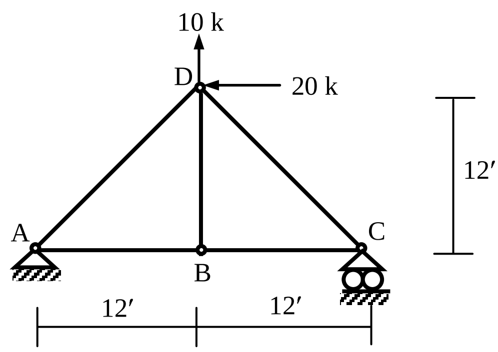
3. Calculate $\Delta_{A\text{-}C(rel)}$ due to
   (i) The external loads
   (ii) ΔT = –50°F for AB and AD.



4. Calculate $\Delta_{C,v}$ and $\Delta_{C,h}$ due to the external loads.



5. Calculate $\Delta_{B,v}$ and $\Delta_{D(along\ AD)}$ due to the external loads.



Deflection due to Flexural Deformations

Flexural deformation is the main source of deflection in many civil engineering structures, like beams, slabs and frames; i.e., those designed primarily against bending moment. It is often much more significant than other causes of deflection like axial, shear and torsional deformation. From Eq. (5) of the previous section, the principle of virtual work $\Rightarrow \Delta = \Sigma\, u.\, dL$ …………………(5)
where the term Δ above can indicate the deflection or rotation of the body, u is the *virtual internal force* in an element within the body, which deforms by an amount dL in the direction of u.

<u>Deflection of Beam/Frame due to External Loads</u>
For flexural deformation, u is be the *virtual internal moment* m_1 in the element while dL is the rotation $d\theta$ caused by external forces; i.e., $dL = d\theta = \text{curvature} \times ds = (m_0/EI)\, ds$.

$$\therefore \Delta = \int m_1\, m_0/EI\, ds \qquad \text{…………………(11)}$$

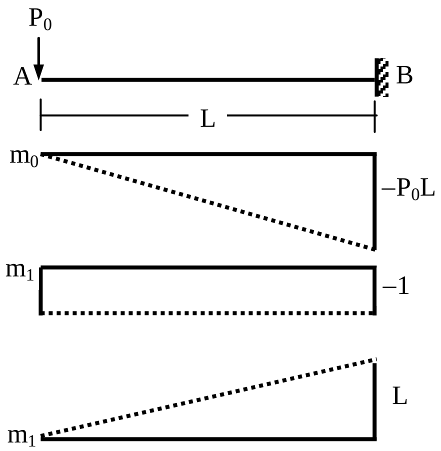
where m_0 is the bending moment caused by external forces and EI is called the flexural rigidity of the member. Here, the integration sign $\int$ is used instead of summation Σ because the bending moments vary within the length of each member (unlike the trusses, where axial forces do not vary within the members).

<u>Integration Table</u>
In order to facilitate the integration shown in Eq. (11), the following table is used between functions f_1 and f_2, both of which can be uniform or vary linearly or parabolically along the length (L) of a member.

Integration of Product of Functions ($I = \int f_1\, f_2\, dS$)

f_2 \ f_1	A, L	B, L	A, L	A, B, L	A, C, B, L
a, L	AaL	BaL/2	AaL/2	(A+B)aL/2	[A+4C+B]aL/6
b, L	AbL/2	BbL/3	AbL/6	[A+2B]bL/6	[2C+B]bL/6
a, L	AaL/2	BaL/6	AaL/3	[2A+B]aL/6	[A+2C]aL/6
a, b, L	A(a+b)L/2	B(a+2b)L/6	A(2a+b)L/6	[A(2a+b)+B(a+2b)]L/6	[Aa+Bb+ 2C(a+b)]L/6

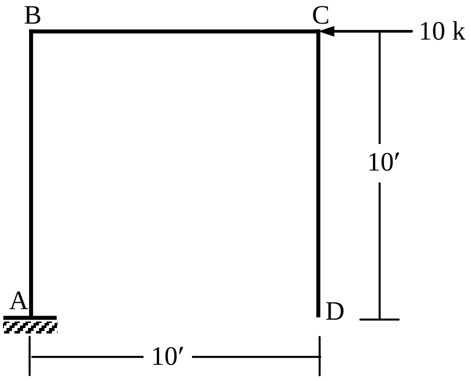
<u>Example</u>: Calculate the tip rotation and deflection of the beam shown below [Given: EI = const].



Using the m_0 diagram along with m_1 for unit anticlockwise moment at A
$\theta_A = (-1)\,(-P_0L/EI)\, L/2 = P_0L^2/2EI$

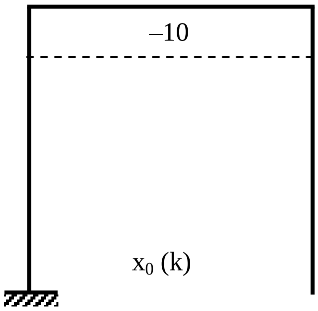
Using the m_0 diagram and m_1 for unit upward load at A
$v_A = (L)\,(-P_0L/EI)\, L/3 = -P_0L^3/3EI$

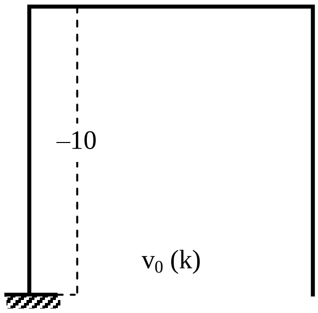
Deflection due to Combined Flexural, Shear and Axial Deformations

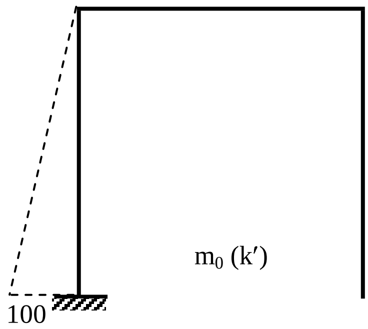


$EA = 400\times10^3$ k, $GA^* = 125\times10^3$ k, $EI = 40\times10^3$ k-ft^2

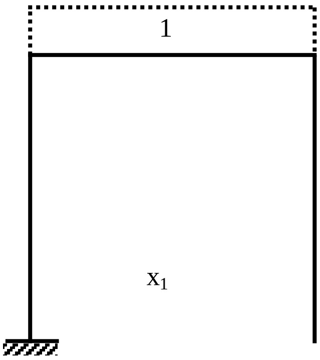
Calculate the
(i) horizontal deflection at C ($\Delta_{C,h}$)
(ii) vertical deflection at D ($\Delta_{D,v}$).

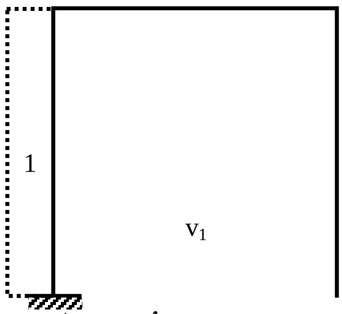


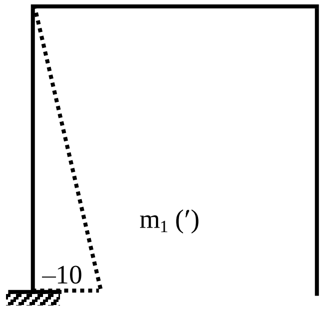




<u>For horizontal deflection at C ($\Delta_{C,h}$)</u>:

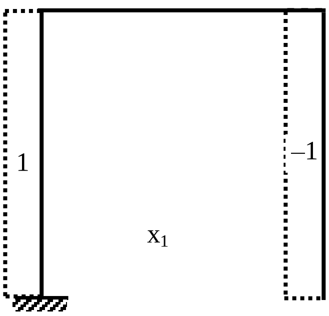


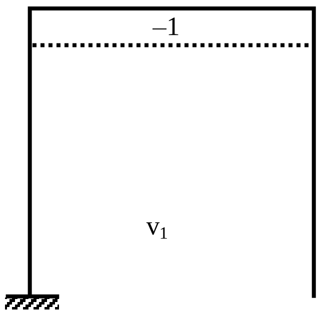


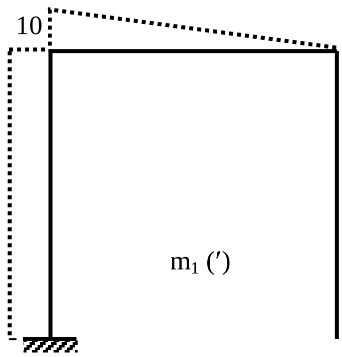


$\Delta_{D,v} = \int (x_1\, x_0/EA)\, dS + \int (v_1\, v_0\,/GA^*)\, dS + \int (m_1\, m_0\,/EI)\, dS$
$= 10(-10)(1)/(400\times10^3) + 10(-10)(1)/(125\times10^3) + 10/3(100)(-10)/(40\times10^3) = (-0.25-0.80-83.33)\times10^{-3}$
$= -0.0844$ ft $\longleftarrow$

<u>For vertical deflection at D ($\Delta_{D,v}$)</u>:







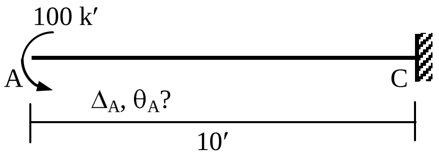
$\Delta_{D,v} = \int (x_1\, x_0/EA)\, dS + \int (v_1\, v_0\,/GA^*)\, dS + \int (m_1\, m_0\,/EI)\, dS$
$= 0 + 0 + 10/2\,(10)(100)/(40\times10^3) = 0 + 0 + 125\times10^{-3} = 0.125$ ft $\uparrow$

Problems on Deflection of Beams/Frames using Method of Virtual Work (Unit Load Method)

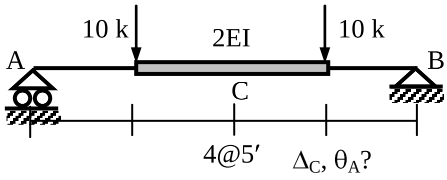
Assume EA = 400×10^3 k, GA* = 125×10^3 k, EI = 40×10^3 k-ft^2

<u>Beams</u>

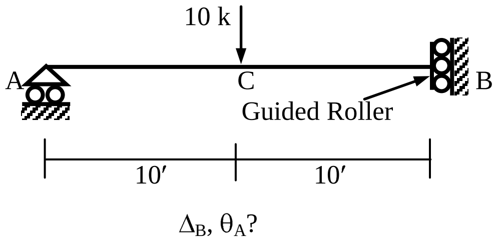
1.



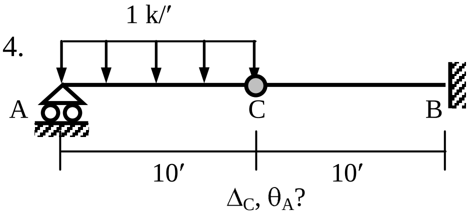
2.



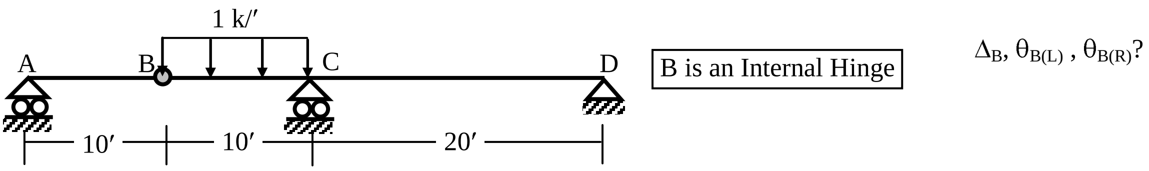
3.



4.

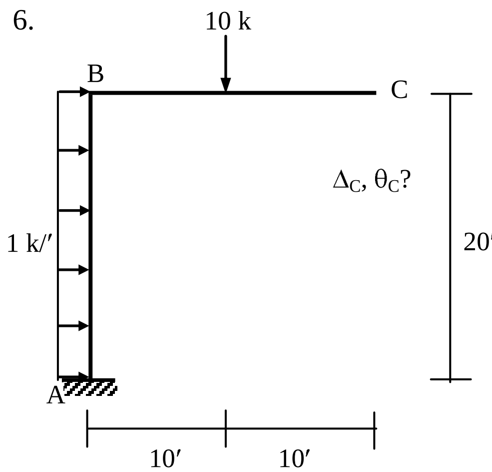


5.

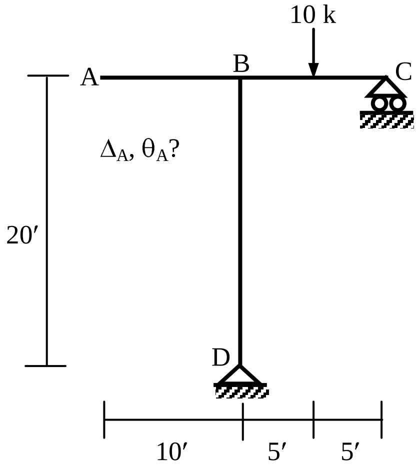


<u>Frames</u>

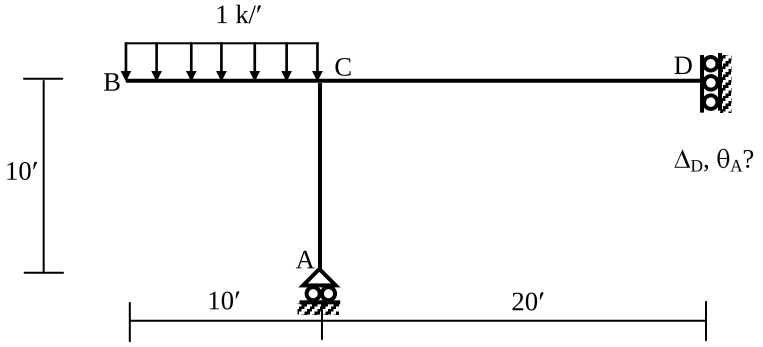
6.



7.

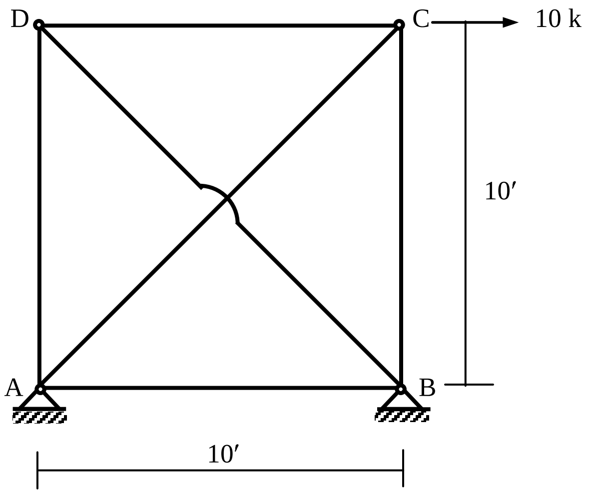


8.



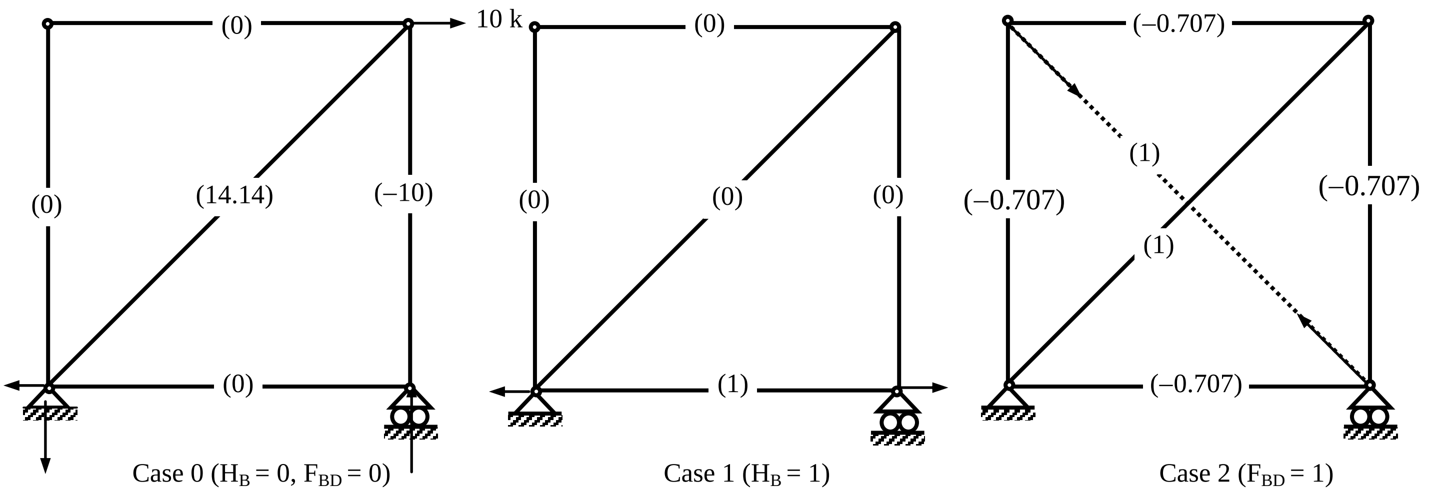
The

Flexibility Method for 2-degree Indeterminate Trusses



EA/L = constant = 1000 k/ft
(Note: $EA \neq$ constant)

dosi = $1\times6 + 4 - 2\times4 = 2$
The horizontal reaction H_B and member force F_{BD} are taken as the two redundants.



Case 0 ($H_B = 0$, $F_{BD} = 0$) [Forces N_0 (k)]

Case 1 ($H_B = 1$) [Forces N_1]

Case 2 ($F_{BD} = 1$) [Forces N_2]

$\Delta_{1,0} = \sum\{N_1\, N_0 /(EA/L)\} = \{0\times0 + 0\times0 + 0\times14.14 + 0\times(-10) + 1\times0\}/1000 = 0$ ft
$\Delta_{2,0} = \sum\{N_2\, N_0 /(EA/L)\} = \{14.14\times1 + (-10)\times(-0.707)\}/1000 = 21.21\times10^{-3}$ ft
$\Delta_{1,1} = \sum\{N_1\, N_1 /(EA/L)\} = \{0^2 + 0^2 + 0^2 + 0^2 + 1^2\}/1000 = 1\times10^{-3}$ ft/k
$\Delta_{1,2} = \Delta_{2,1} = \sum\{N_1\, N_2 /(EA/L)\} = \{1\times(-0.707)\}/1000 = -0.707\times10^{-3}$ ft/k
$\Delta_{2,2} = \sum\{N_2\, N_2 /(EA/L)\} = \{4\times (-0.707)^2 + 2\times1^2\}/1000 = 4\times10^{-3}$ ft/k

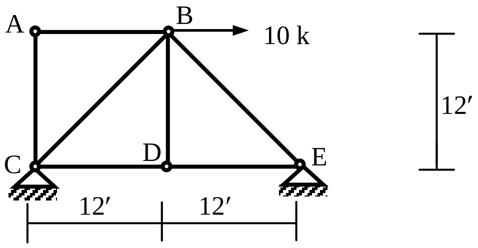
$\therefore (1\times10^{-3})\, H_B + (-0.707\times10^{-3})\, F_{BD} = 0$
$(-0.707\times10^{-3})\, H_B + (4\times10^{-3})\, F_{BD} = -21.21\times10^{-3}$
$\Rightarrow H_B = -4.29$ k, and $F_{BD} = -6.06$ k

$\therefore N = N_0 + N_1\, H_B + N_2\, F_{BD}$
$\Rightarrow F_{AB} = 0 + 1\times(-4.29) + (-0.707)\times(-6.06) = 0$, $F_{BC} = -10 + 0 + (-0.707)\times(-6.06) = -5.71$ k
$F_{CD} = 0 + 0 + (-0.707)\times(-6.06) = 4.29$, $F_{DA} = 0 + 0 + (-0.707)\times(-6.06) = 4.29$ k
$F_{AC} = 14.14 + 0 + (1)\times(-6.06) = 8.08$ k, $F_{BD} = -6.06$ k

Problems on Flexibility Method for Trusses (from past exams)

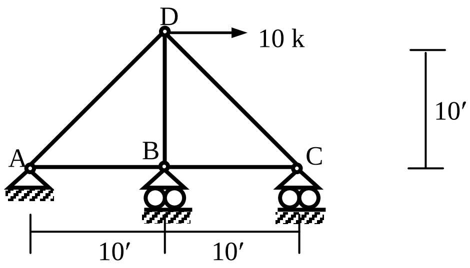
1.

Also solve if C moves 0.10′ to right

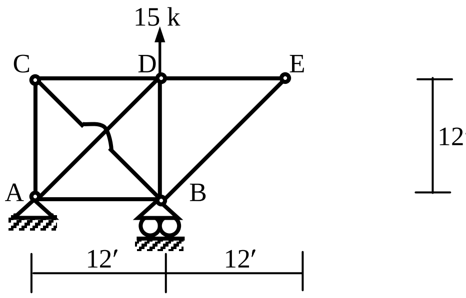


2.

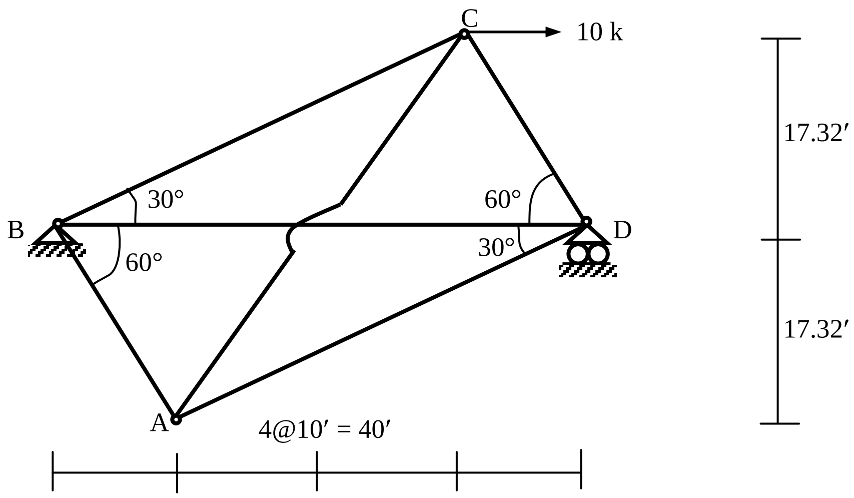
Also solve if B settles 0.10′



3.

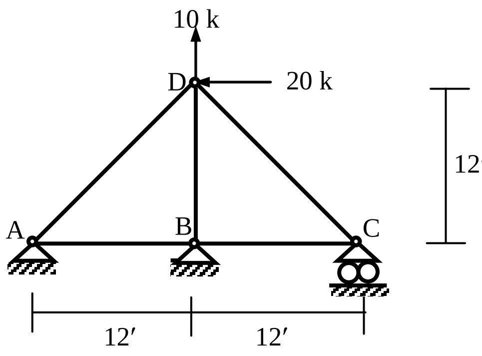


4.

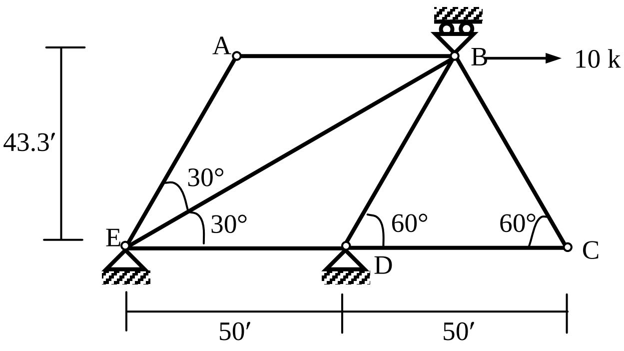


5.

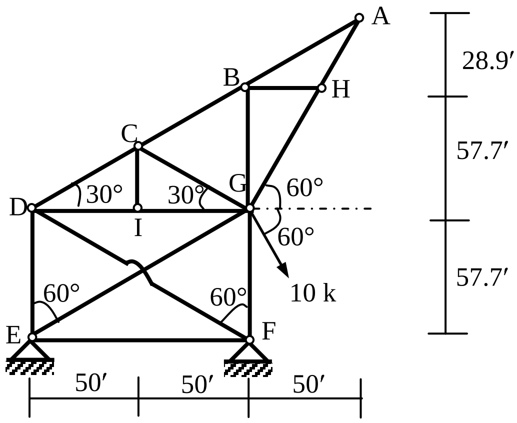
Also solve if C settles 0.10′



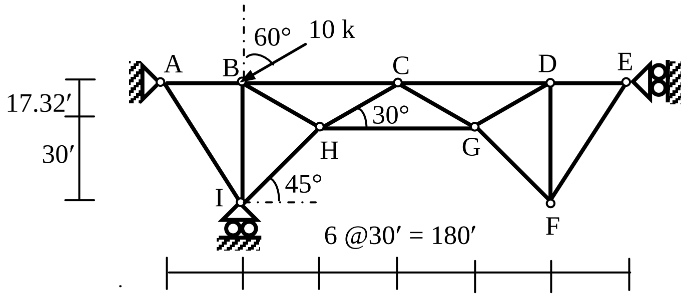
6.



7.

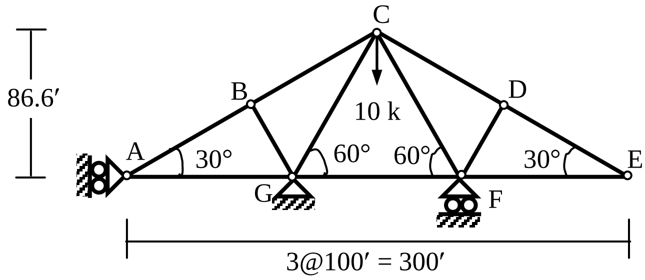


8.



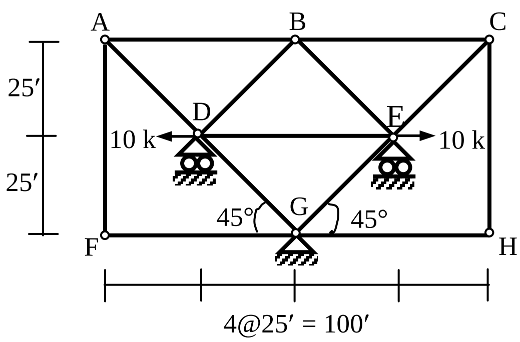
Support E moves 0.10′ rightwards

9.



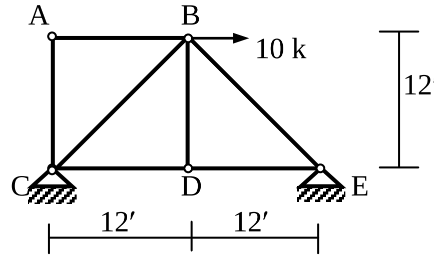
Support A moves 0.10′ rightwards

10.

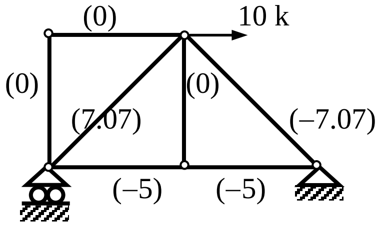


Solutions for Problems on Flexibility Method for Trusses

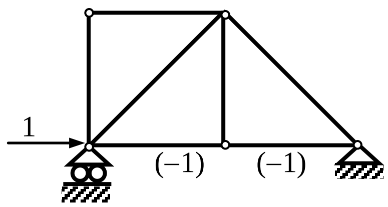
1.



Also support C moves 0.10′ rightwards
Assume EA/L = Constant = 500 k/ft
dosi = 7 + 4 –2×5 = 1



N_0 (k) ($X_C = 0$)



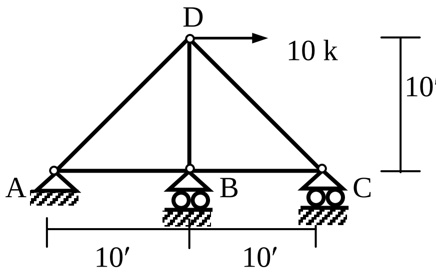
N_1 ($X_C = 1$)

$\Delta_{1,0} = \{(-5)(-1) + (-5)(-1)\}/500 = 0.02'$, $\Delta_{1,1} = \{(-1)^2 + (-1)^2\}/500 = 0.004$

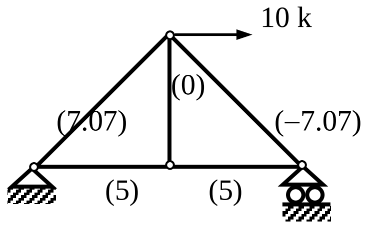
$\therefore 0.004\, X_C + 0.02 = 0.10 \Rightarrow X_C = 20$ k

$\therefore N = N_0 + X_C N_1 \Rightarrow P_{AB} = 0$, $P_{AC} = 0$, $P_{BC} = 7.07$ k, $P_{BD} = 0$, $P_{BE} = -7.07$ k, $P_{CD} = -25$ k, $P_{BE} = -25$ k

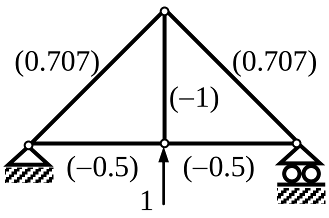
2.



Also support B settles 0.10′
Assume EA/L = Constant = 500 k/ft
dosi = 5+ 4 –2×4 = 1



N_0 (k) ($Y_B = 0$)



N_1 ($Y_B = 1$)

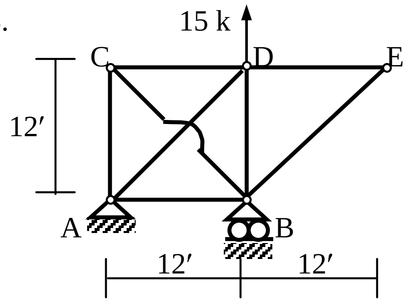
$\Delta_{1,0} = \{(5)(-0.5) + (5)(-0.5) + (7.07)(0.707) + (-7.07)(0.707)\}/500 = -0.01'$

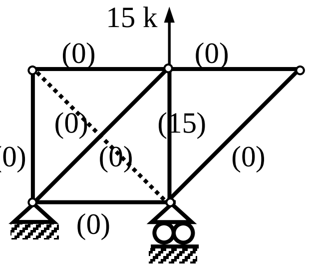
$\Delta_{1,1} = \{(-0.5)^2 + (-0.5)^2 + (0.707)^2 + (-1)^2 + (0.707)^2\}/500 = 0.005$

$\therefore 0.005\, Y_B - 0.01 = -0.10 \Rightarrow Y_B = -18$ k

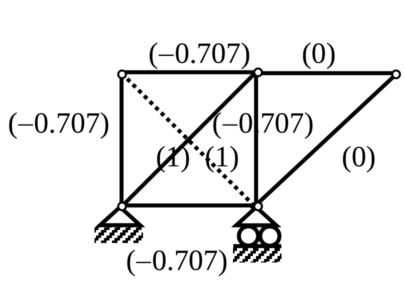
$\therefore N = N_0 + Y_B N_1 \Rightarrow P_{AB} = 14$ k, $P_{BC} = 14$ k, $P_{AD} = -5.66$ k, $P_{BD} = 18$ k, $P_{CD} = -19.80$ k

3.





N_0 (k) ($P_{BC} = 0$)



N_1 ($P_{BC} = 1$)

EA/L = Constant = 500 k/ft, dosi = 8+3 −2×5 = 1

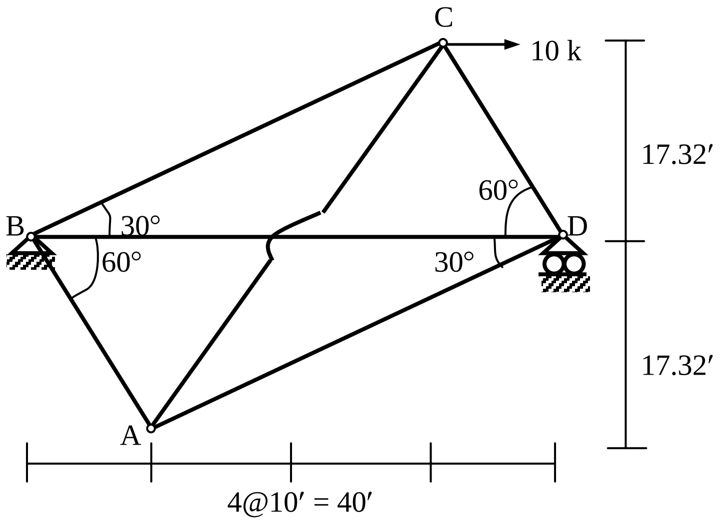
$\Delta_{1,0} = \{(15)\ (-0.707)\}/500 = -0.0212'$, $\Delta_{1,1} = \{4\times(-0.707)^2 + 2\times\ (1)^2\}/500 = 0.008$

$\therefore 0.008\ P_{BC} - 0.0212 = 0 \Rightarrow P_{BC} = 2.65$ k

$\therefore N = N_0 + P_{BC}\ N_1$

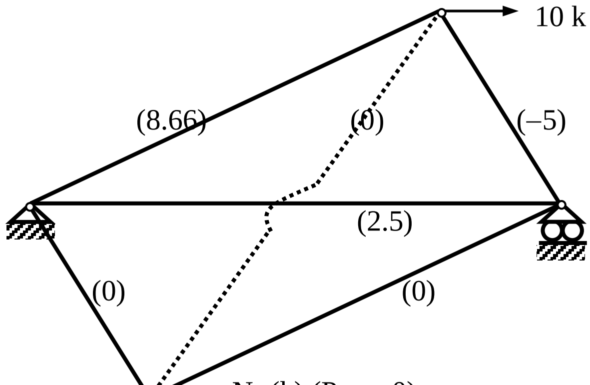
$\Rightarrow P_{AB} = -1.87$ k, $P_{AC} = -1.87$ k, $P_{CD} = -1.87$ k, $P_{BD} = 15.13$ k, $P_{BC} = 2.65$ k, $P_{AD} = 2.65$ k, $P_{DE} = 0$, $P_{BE} = 0$

4.

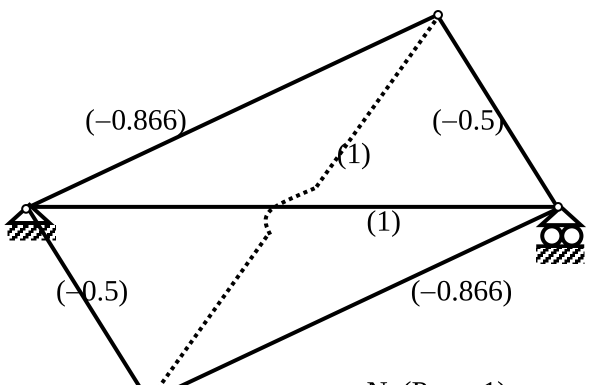


EA/L = Constant = 1000 k/ft

dosi = 6 +3−2×4 = 1



N_0 (k) ($P_{AC} = 0$)



N_1 ($P_{AC} = 1$)

$\Delta_{1,0} = \{(8.66)\ (-0.866)+(-5)\ (-0.5)+(2.5)\ (1)\}/1000 = -0.0025'$

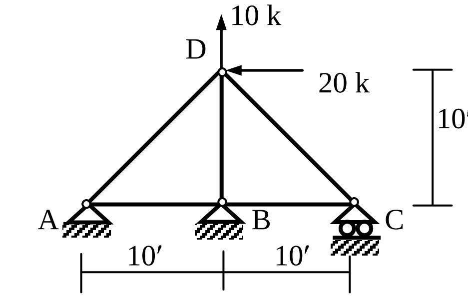
$\Delta_{1,1} = \{2\times(-0.5)^2+2\times(-0.866)^2+2\times\ (1)^2\}/1000 = 0.004$

$\therefore 0.004\ P_{AC} - 0.0025 = 0 \Rightarrow P_{AC} = 0.63$ k

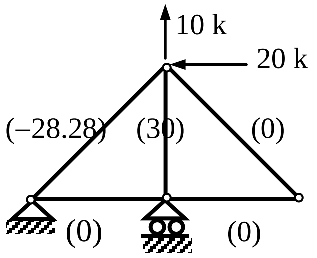
$\therefore N = N_0 + P_{AC}\, N_1$

$\Rightarrow P_{AB} = -0.31$ k, $P_{BC} = 8.12$ k, $P_{CD} = -5.31$ k, $P_{DA} = -0.54$ k, $P_{AC} = 0.63$ k, $P_{BD} = 3.13$ k

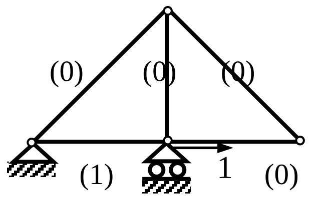
5.



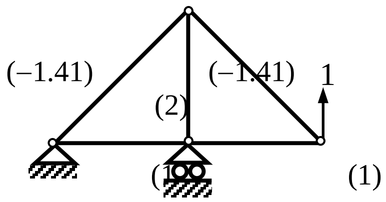
Also support C settles 0.10′
Assume EA/L = Constant = 500 k/ft
dosi = 5+5−2×4 = 2



N_0 (k)



N_1 (X_B = 1)



N_2 (Y_C = 1)

$\Delta_{1,0} = 0$, $\Delta_{2,0} = \{(-28.28)(-1.41) + (30)(2)\}/500 = 0.20'$

$\Delta_{1,1} = (1)^2/500 = 0.002$, $\Delta_{1,2} = \Delta_{2,1} = (1)\times(1)/500 = 0.002$, $\Delta_{2,2} = \{2\times(-1.41)^2 + (2)^2 + 2\times(1)^2\}/500 = 0.02$

$\therefore 0.002\,X_B + 0.002\,Y_C + 0 = 0$

$0.002\,X_B + 0.02\,Y_C + 0.20 = -0.10$

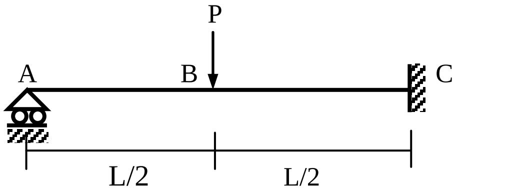
$\therefore X_B = 16.67$ k, $Y_C = -16.67$ k

$\therefore N = N_0 + X_B N_1 + Y_C N_2$

$\Rightarrow P_{AB} = 0$, $P_{BC} = -16.67$ k, $P_{AD} = -4.71$ k, $P_{BD} = -3.33$ k, $P_{CD} = 23.57$ k

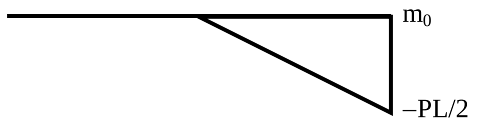
Flexibility Method for 1-degree Indeterminate Beams

Example 1

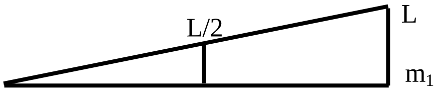


EI = constant
dosi = 3 × 1 + 4 – 3 × 2 = 1
Take R_A as the redundant

$\therefore \Delta_{1,0} + R_A \Delta_{1,1} = \Delta_A = 0$ ………….…(i)

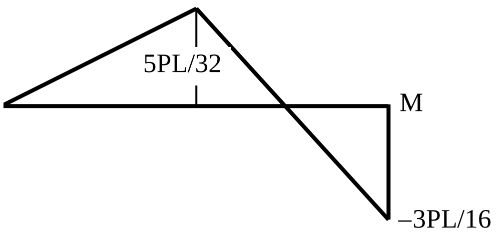


$\Delta_{1,0} = (L/2)/6 \times [2L + L/2]\ (–PL/2)/EI = – 5PL^3/48EI$



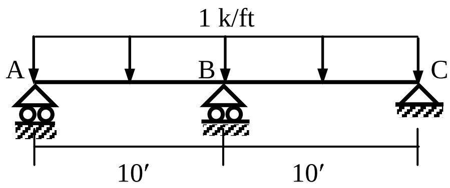
$\Delta_{1,1} = \int m_1\, m_1\, dS/EI$
$= L/3\ (L)\ (L)/EI = L^3/3EI$

$\therefore$ (i) $\Rightarrow – 5PL^3/48EI + L^3/3EI = 0 \Rightarrow R_A = 5P/16$



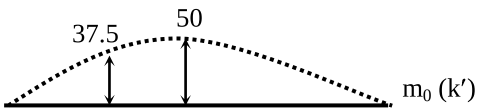
$\therefore M = m_0 + R_A\, m_1 = m_0 + (5P/16)\, m_1$
$M_A = 0,\ M_B = 5PL/32,\ M_C = –PL/2 + 5PL/16 = –3PL/16$

Example 2

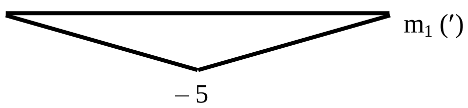


EI = constant
dosi = 3 × 2 + 4 – 3 × 3 = 1
Take R_B as the redundant

$\therefore \Delta_{1,0} + R_B \Delta_{1,1} = \Delta_B = 0$ ………….…(i)

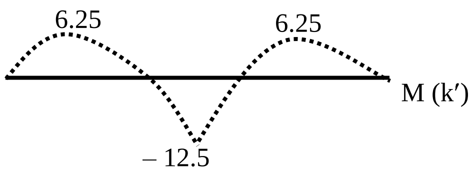


$\Delta_{1,0} = 2 \times [2 \times 37.5 + 50]\ (–5 \times 10/6)/EI = –2083.33/EI$



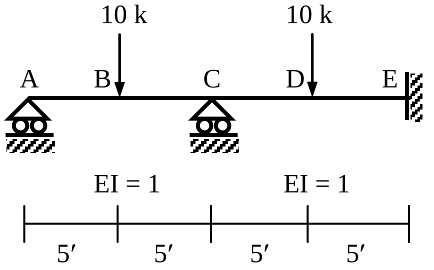
$\Delta_{1,1} = \int m_1\, m_1\, dS/EI$
$= 2 \times 10/3\ (–5)\ (–5)/EI = 166.67/EI$

$\therefore$ (i) $\Rightarrow –2083.33/EI + 166.67\ R_B/EI = 0$
$\Rightarrow R_B = 12.5$ k



$\therefore M = m_0 + R_B\, m_1 = m_0 + 12.5\, m_1$
$M_A = 0,\ M_B = 50 – 62.5 = – 12.5$ k′, $M_C = 0$

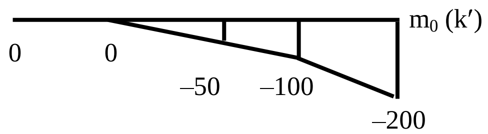
Example 3



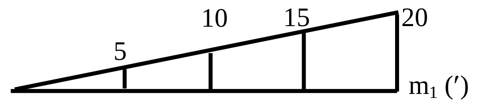
dosi = 3×2 + 5 – 3×3 = 2

$\Delta_{1,0} + R_A\,\Delta_{1,1} + R_C\,\Delta_{1,2} = \Delta_A = 0$ ……(i)

$\Delta_{2,0} + R_A\,\Delta_{2,1} + R_C\,\Delta_{2,2} = \Delta_C = 0$ ……(ii)



$\Delta_{1,0} = \int m_1\, m_0\, dS/EI$
= {10/6 (–100) (30+5)
+ 5/6[(–100)(30+20) + (–200)(40+15)]}/EI
= –19166.67/EI



$\Delta_{2,0} = \int m_2\, m_0\, dS/EI$
= {5/6 (5) (–200–50)
+ 5/6 [(5)(–200–200)+(10)(–400–100)]}/EI = – 6875/EI

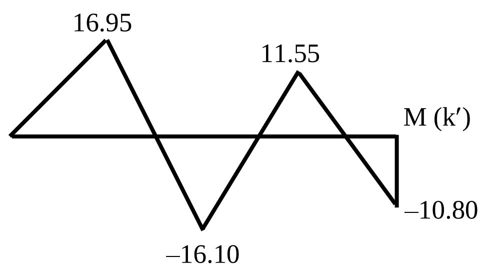
$\Delta_{1,1} = \int m_1\, m_1\, dS/EI$ = 20/3 (20) (20)/EI
= 2666.67/EI

$\Delta_{1,2} = \Delta_{2,1} = \int m_1\, m_2\, dS/EI$
= {5/6(5)(30+10)
+5/6 [(5)(30+20)+(10)(40+15)]}/EI
= 833.33/EI

$\Delta_{2,2} = \int m_2\, m_2 dS/EI$ = 10/3 (10) (10)/EI
= 333.33/EI

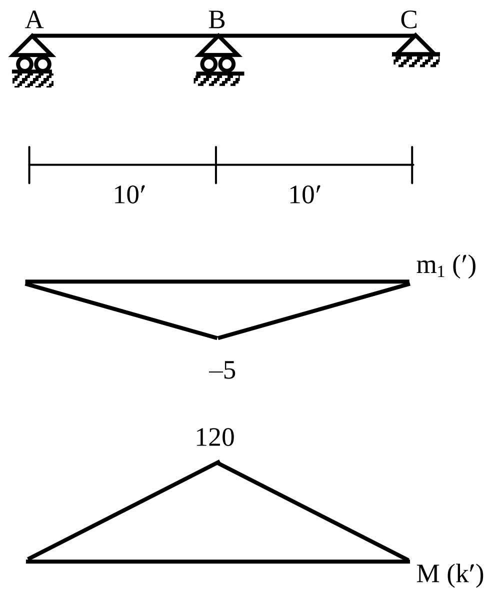
$\therefore$ Avoiding the factors EI
(i) $\Rightarrow$ 2666.67 R_A + 833.33 R_C = 19166.67
(ii) $\Rightarrow$ 833.33 R_A + 333.33 R_C = 6875
$\Rightarrow R_A = [19166.67 \times 333.33 - 833.33 \times 6875]/[2666.67 \times 333.33 - 833.33^2] = 3.39$ k
and $R_C = [2666.67 \times 6875 - 833.33 \times 19166.67]/[2666.67 \times 333.33 - 833.33^2] = 12.14$ k

$\therefore\ M = m_0 + R_A\, m_1 + R_C\, m_2 = m_0 + 3.39\, m_1 + 12.14\, m_2$
$M_A = 0$, $M_B = 0 + 3.39 \times 5 + 0 \times 12.14 = 16.95$ k′, $M_C = -50 + 3.39 \times 10 + 0 \times 12.14 = -16.10$ k′,
$M_D = -100 + 3.39 \times 15 + 5 \times 12.14 = 11.55$ k′, $M_E = -200 + 3.39 \times 20 + 10 \times 12.14 = -10.80$ k′



Analysis for Support Settlement

Example 4



Support B settles 0.10′
EI = 40×10^3 k-ft^2
dosi = $3 \times 2 + 4 - 3 \times 3 = 1$

$\Delta_{1,0} + R_B \Delta_{1,1} = \Delta_B = -0.10$ ……(i)

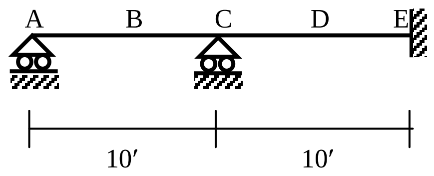
$\Delta_{1,0} = 0$

$\Delta_{1,1} = \int m_1 m_1 \, dS/EI$
$= 2 \times 10/3 \, (-5) \, (-5)/EI = 166.67/40 \times 10^3$

$\therefore$ (i) $\Rightarrow 166.67 \, R_B/40 \times 10^3 + 0 = -0.10$
$\Rightarrow R_B = -4000/166.67 = -24$ k

$\therefore$ $M = m_0 + R_B m_1 = -24 \, m_1$ [in k′]

Example 5



Support C settles 0.10′
EI = 40×10^3 k-ft^2
dosi = 2

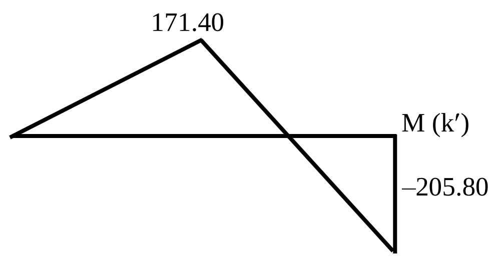
$\Delta_{1,0} + R_A \Delta_{1,1} + R_C \Delta_{1,2} = \Delta_A = 0$……..….(i)

$\Delta_{2,0} + R_A \Delta_{2,1} + R_C \Delta_{2,2} = \Delta_C = -0.10$.…(ii)

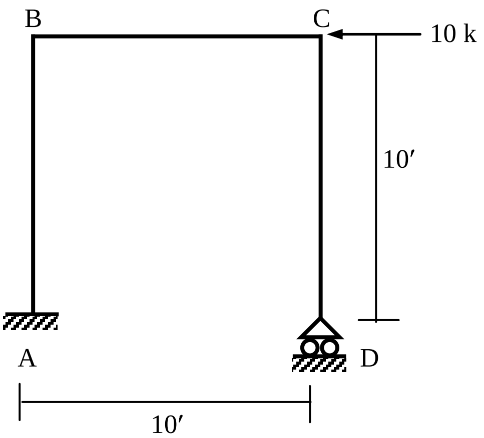
$\Delta_{1,0} = 0$, $\Delta_{2,0} = 0$
$\Delta_{1,1} = \int m_1 m_1 \, dS/EI = 2666.67/EI$, $\Delta_{1,2} = \Delta_{2,1} = 833.33/EI$, $\Delta_{2,2} = 333.33/EI$

$\therefore$ (i) $\Rightarrow$ $(2666.67/EI) \, R_A + (833.33/EI) \, R_C = 0$
(ii) $\Rightarrow$ $(833.33/EI) \, R_A + (333.33/EI) \, R_C = -0.10$
$\Rightarrow R_A = 17.14$ k, and $R_C = -54.86$ k

$\therefore$ $M = m_0 + R_A m_1 + R_C m_2 = 17.14 \, m_1 - 54.86 \, m_2$ [in k′]
$M_A = 0$, $M_B = 17.14 \times 5 - 0 \times 54.86 = 85.70$, $M_C = 17.14 \times 10 - 0 \times 54.86 = 171.40$,
$M_D = 17.14 \times 15 - 5 \times 54.86 = -17.20$, $M_E = 17.14 \times 20 - 10 \times 54.86 = -205.80$

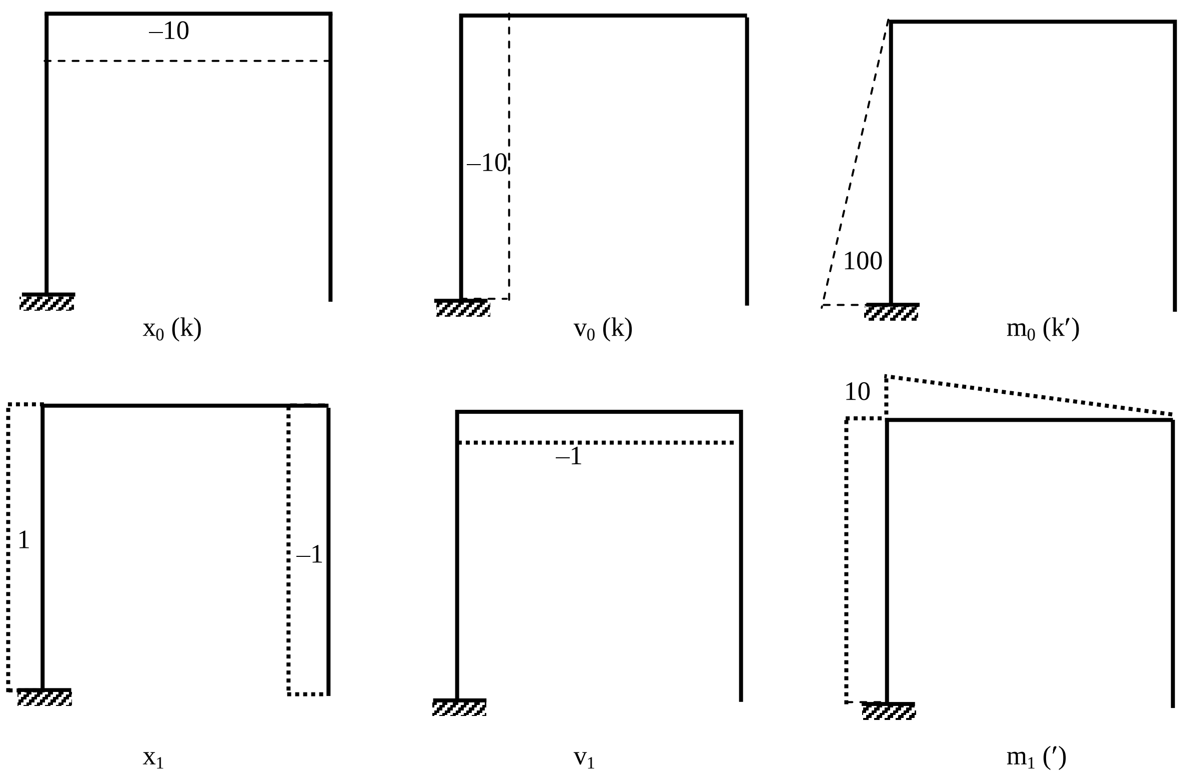


Combined Flexural, Shear and Axial Deformations



$EA = 400\times10^3$ k, $GA^* = 125\times10^3$ k
$EI = 40\times10^3$ k-ft^2
dosi = $3\times3 + 4 - 3\times4 = 1$

The vertical reaction at D (V_D) is taken as the redundant.

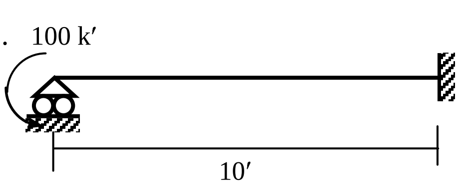


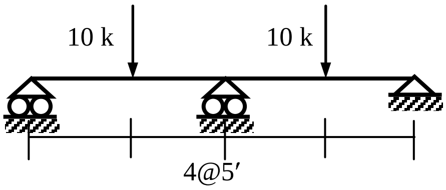
$\Delta_{1,0} = \int (x_1\, x_0/EA)\, dS + \int (v_1\, v_0\, /GA^*)\, dS + \int (m_1\, m_0\, /EI)\, dS$
$\quad = 0 + 0 + 10/2\ (100)(10)/(40\times10^3) = 0.125$ ft
$\Delta_{1,1} = \int (x_1\, x_1/EA)\, dS + \int (v_1\, v_1\, /GA^*)\, dS + \int (m_1\, m_1\, /EI)\, dS$
$\quad = 2\times10\times(1\times1)/(400\times10^3)+10\times(1\times1)/(125\times10^3)+[10\times(10\times10)+10\times(10\times10)/3]/(40\times10^3)$
$\quad = 0.05\times10^{-3} + 0.08\times10^{-3} +33.33\times10^{-3} = 33.46\times10^{-3}$
$\therefore V_D = -0.125/33.46\times10^{-3} = -3.74$ k

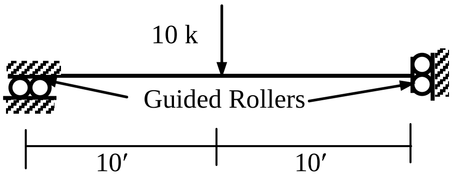
Problems on Flexibility Method (Beams/Frames)

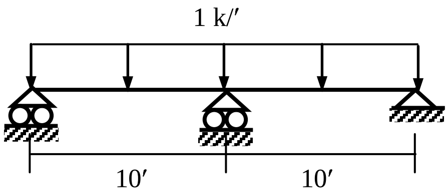
Assume EA = 400×10^3 k, GA* = 125×10^3 k, EI = 40×10^3 k-ft^2

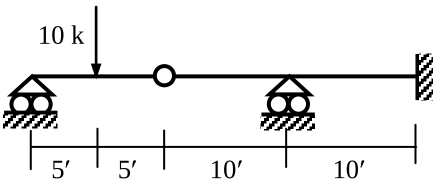
Beams

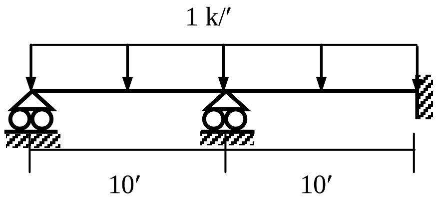
1.


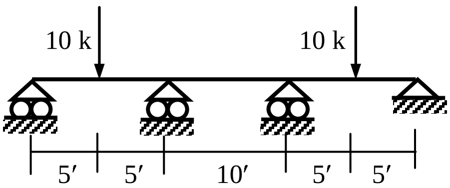
2.


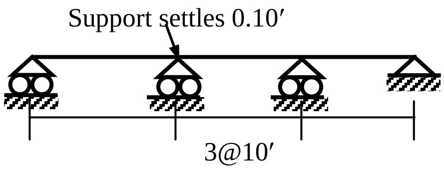
3.


4.


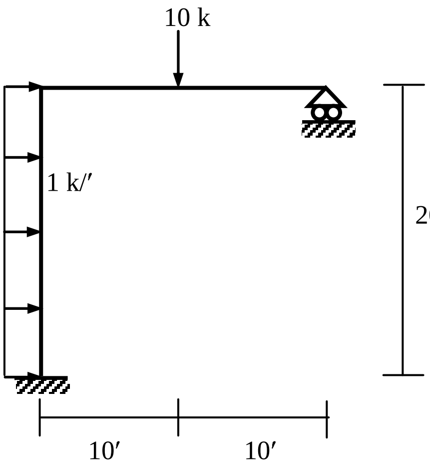
5.


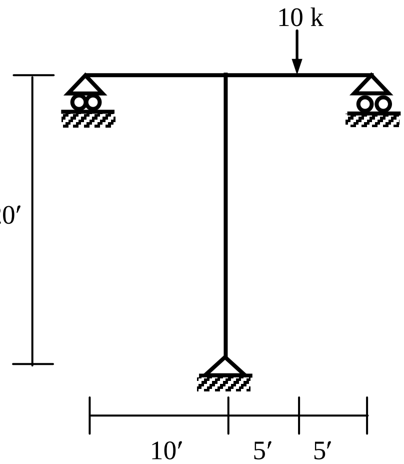
6.


7.


8.


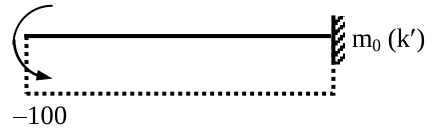
Frames

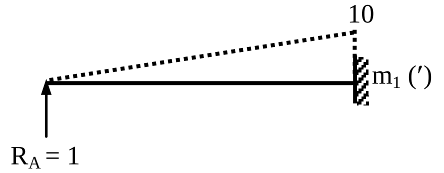
9.


10.


Solution of Problems on Flexibility Method (Beams/Frames)

1. dosi = 3 + 4 – 6 = 1; i.e., assume R_A as the redundant

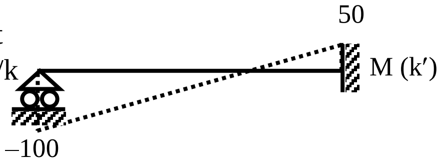




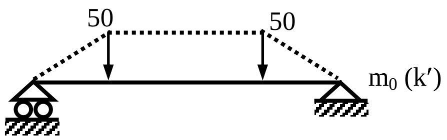
$\Delta_{1,0} = \int (m_0\, m_1/EI)\, dS = 10/2\ (–100)\ (10)/(40\times10^3) = –0.125$ ft

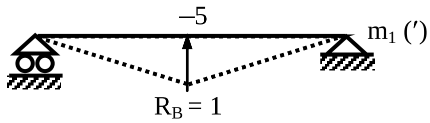
$\Delta_{1,1} = \int (m_1\, m_1/EI)\, dS = 10/3\ (10\times10)/(40\times10^3) = 8.33\times10^{–3}$ ft/k

$\therefore R_A = 0.125/(8.33\times10^{–3}) = 15$ k



2. dosi = 6 + 4 – 9 = 1; i.e., assume R_B as the redundant

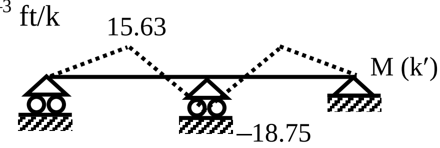




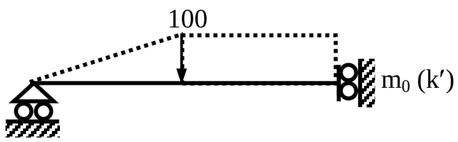
$\Delta_{1,0} = \int (m_0\, m_1/EI)\, dS = 2\ \{5/3\ (50)\ (–2.5) + 5/2\ (50)\ (–2.5\ –5)\}/(40\times10^3) = –0.0573$ ft

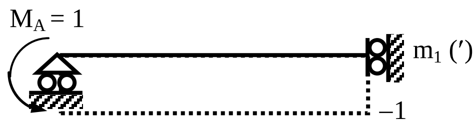
$\Delta_{1,1} = \int (m_1\, m_1/EI)\, dS = 2\ \{10/3\ (–5)\ (–5)\}/(40\times10^3) = 4.17\times10^{–3}$ ft/k

$\therefore R_B = 0.0573/(4.17\times10^{–3}) = 13.75$ k



3. dosi = 3 + 4 – 6 = 1; i.e., assume M_A as the redundant

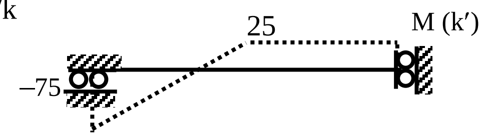




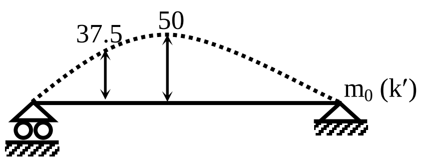
$\Delta_{1,0} = \int (m_0\, m_1/EI)\, dS = \{10/2\ (100)\ (–1) + 10\ (100)\ (–1)\}/(40\times10^3) = –0.0375$ ft

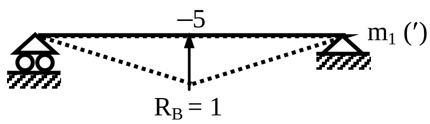
$\Delta_{1,1} = \int (m_1\, m_1/EI)\, dS = 20\ (–1)\ (–1)/(40\times10^3) = 0.5\times10^{–3}$ ft/k

$\therefore M_A = 0.0375/(0.5\times10^{–3}) = 75$ k-ft



4. dosi = 6 + 4 – 9 = 1; i.e., assume R_B as the redundant

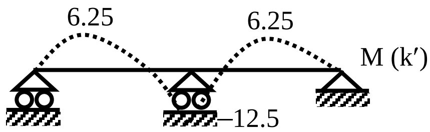




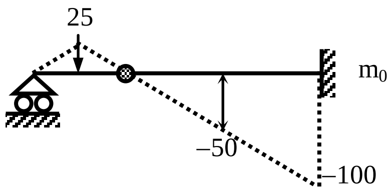
$\Delta_{1,0} = \int (m_0\, m_1/EI)\, dS = 2\ \{10/6\ (2 \times 37.5 + 50)\ (–5)\}/(40\times10^3) = –0.0521$ ft

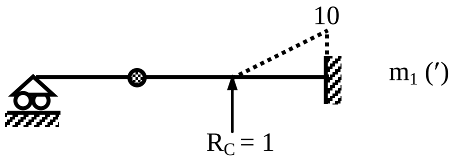
$\Delta_{1,1} = 4.17\times10^{–3}$ ft/k (as in Problem 2)

$\therefore R_B = 0.0521/(4.17\times10^{–3}) = 12.5$ k



5. dosi = 9 + 5 – 12 – 1 = 1; i.e., assume R_C as the redundant

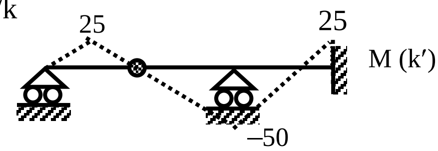




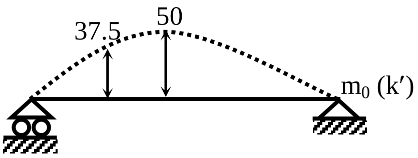
$\Delta_{1,0} = \int (m_0\, m_1/EI)\, dS = 10/6\ (–50–2\times100)\ (10)/(40\times10^3) = –0.1042$ ft

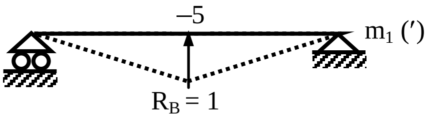
$\Delta_{1,1} = \int (m_1\, m_1/EI)\, dS = 10/3\ (10)\times(10)/(40\times10^3) = 8.33\times10^{–3}$ ft/k

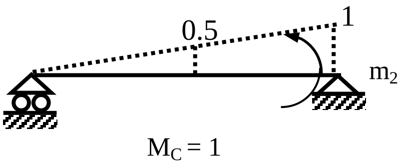
$\therefore R_B = 0.1042/(8.33\times10^{–3}) = 12.5$ k



6. dosi = 6 + 5 – 9 = 2; i.e., assume R_B and M_C as the redundants







$\Delta_{1,0} = –0.0521$ ft, $\Delta_{1,1} = 4.17\times10^{–3}$ ft/k (as in Problem 4)

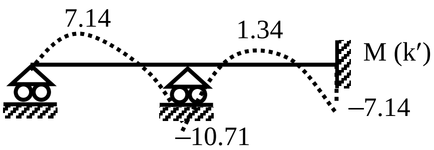
$\Delta_{2,0} = \int (m_0\, m_2/EI)\, dS = 20/6\ (2\times50+0)\ (1)/(40\times10^3) = 8.33\times10^{–3}$ rad

$\Delta_{1,2} = \Delta_{2,1} = \int (m_1\, m_2/EI)\, dS = \{10/3\ (–5)(0.5) +10/6\ (–5)(1+2\times0.5)\}/(40\times10^3) = –0.625\times10^{-3}$ rad/k

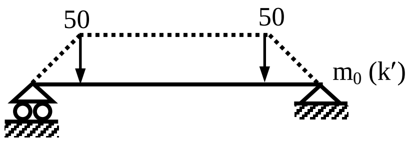
$\Delta_{2,2} = \int (m_2\, m_2/EI)\, dS = 20/3\ (1)(1)/(40\times10^3) = 0.167\times10^{-3}$ rad/k-ft

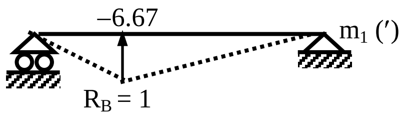
$\therefore 4.17\ R_B – 0.625\ M_C = 52.1;$ and $–0.625\ R_B + 0.167\ M_C = –8.33$

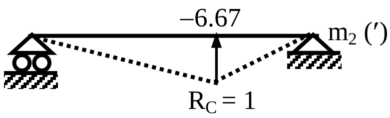
$\Rightarrow R_B = 11.43$ k, $M_C = –7.14$ k-ft



7. dosi = 6 + 5 – 9 = 2; i.e., assume R_B and R_C as the redundants







$\Delta_{1,0} = \int (m_0\, m_1/EI)\, dS$

$=–\{5/3(50)(3.33)+5/2(50)(3.33+6.67)+15/2(50)(6.67+1.67)+5/3(50)(1.67)\}/(40\times10^3)= –119.8\times10^{–3}$ ft

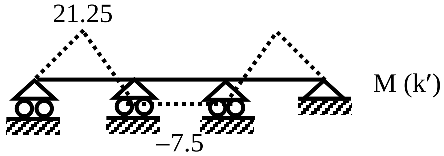
$\Delta_{1,1} = \int (m_1\, m_1/EI)\, dS = \{10/3\ (–6.67)\ (–6.67) + 20/3\ (–6.67)\ (–6.67)\}/(40\times10^3) = 11.11\times10^{–3}$ ft/k

$\Delta_{1,2} = \Delta_{2,1} = \int (m_1\, m_2/EI)\, dS = [10/3(–6.67)(–3.33) + 10/6\{(–6.67)(–2\times3.33–6.67)$

$+(–3.33)(–2\times6.67–3.33)\}+10/3(–6.67)(–3.33)]/(40\times10^3) = 9.72\times10^{–3}$ ft/k

$\Delta_{2,0} = \Delta_{1,0} = –119.8\times10^{–3}$ ft, $\Delta_{2,2} = \Delta_{1,1} = 11.11\times10^{–3}$ ft/k

$\therefore R_B = R_C = 5.75$ k (i.e., upward)



8. dosi = 6 + 5 – 9 = 2; i.e., assume R_B and R_C as the redundants

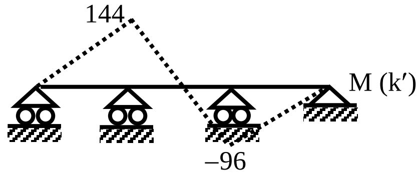
m_0 is zero here, but m_1 and m_2 are same as in Problem 7.

$\Delta_{1,0} = \int (m_0\, m_1/EI)\, dS = 0$, $\Delta_{2,0} = \int (m_0\, m_2/EI)\, dS = 0$

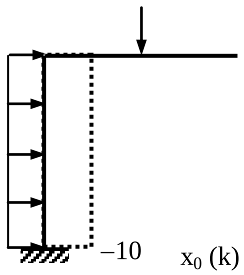
$\Delta_{1,1} = 11.11\times10^{-3}$ ft/k, $\Delta_{1,2} = \Delta_{2,1} = 9.72\times10^{-3}$ ft/k, $\Delta_{2,2} = \Delta_{1,1} = 11.11\times10^{-3}$ ft/k

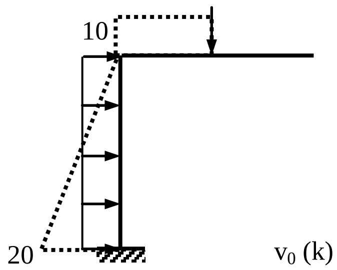
$\therefore 11.11\times10^{-3}\, R_B + 9.72\times10^{-3}\, R_C = -0.10$

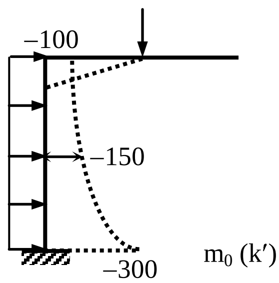
$9.72\times10^{-3}\, R_B + 11.11\times10^{-3}\, R_C = 0 \Rightarrow R_B = -38.4$ k (i.e., downward), $R_C = 33.6$ k (i.e., upward)

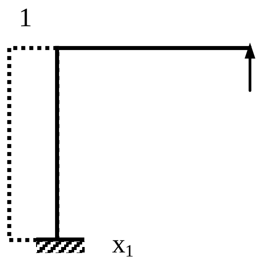


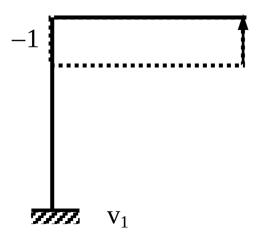
9. dosi = 6 + 4 – 9 = 1; i.e., assume R_C as the redundant

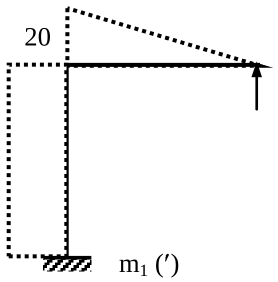












$\Delta_{1,0} = \int(x_0\, x_1/EA)\, dS + \int(v_0\, v_1/GA^*)\, dS + \int(m_0\, m_1/EI)\, dS$

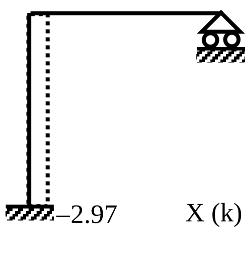
$= 20(-10)(1)/(400\times10^3) + 10(10)(-1)/(125\times10^3)$

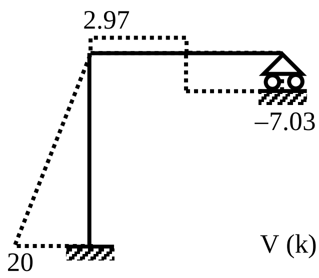
$-\{10/6(100)(2\times20+10)+20/6(100+4\times150+300)(20)\}/(40\times10^3) = (-0.5-0.8-1875)\times10^{-3} = -1.876$ ft

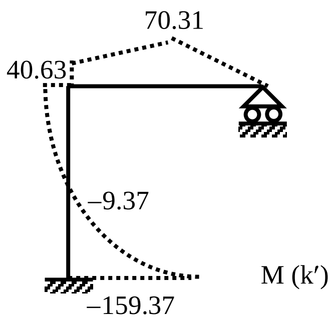
$\Delta_{1,1} = \int(x_1\, x_1/EA)dS + \int(v_1\, v_1/GA^*)dS + \int(m_1\, m_1/EI)dS = 20(1)(1)/(400\times10^3) + 20(-1)(-1)/(125\times10^3) +$

$\{20/3\ (20)\ (20) + 20\ (20)\ (20)\}/(40\times10^3) = (0.05 + 0.16 + 266.67)\times10^{-3} = 0.2669$ ft/k

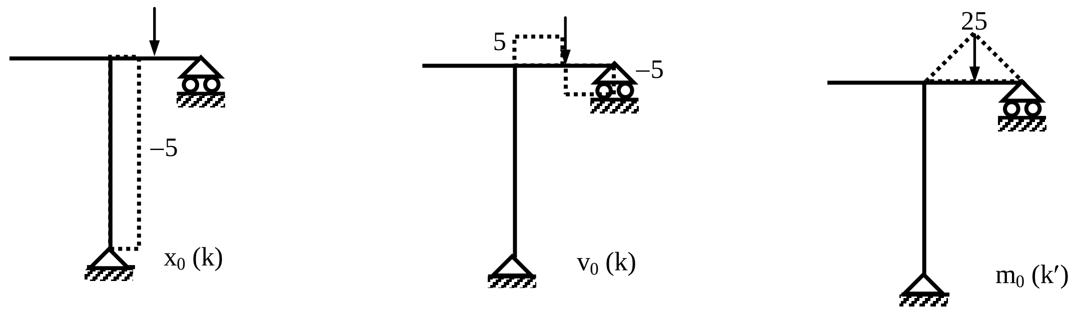
$\therefore R_B = 1.876/(0.2669) = 7.03$ k

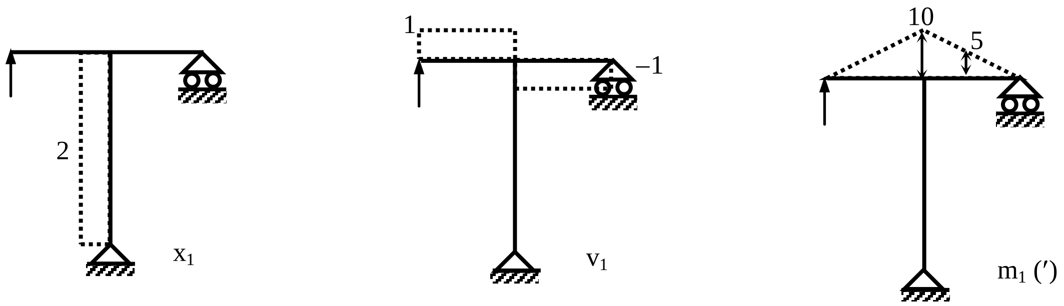






10. dosi = 6 + 4 – 9 = 1; i.e., assume R_A as the redundant





$\Delta_{1,0} = \int(x_0\, x_1/EA)\, dS + \int(v_0\, v_1/GA^*)\, dS + \int(m_0\, m_1/EI)\, dS$

$= 20(-5)(2)/(400\times10^3) + 5\{(5)(-1)+(-5)(-1)\}/(125\times10^3) + \{5/6(25)(2\times5+10)+5/3(25)(5)\}/(40\times10^3)$

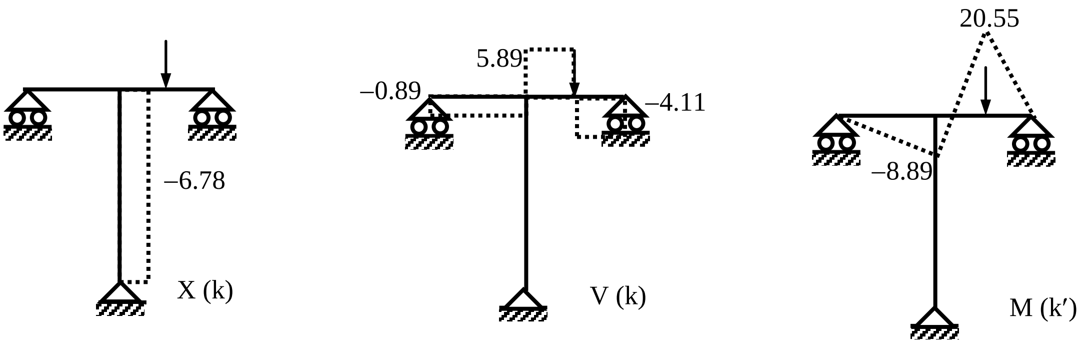
$= (-0.5 + 0 + 15.63)\times10^{-3} = 15.13\times10^{-3}$ ft

$\Delta_{1,1} = \int(x_1\, x_1/EA)\, dS + \int(v_1\, v_1/GA^*)\, dS + \int(m_1\, m_1/EI)\, dS$

$= 20(2)(2)/(400\times10^3) + \{10(1)(1)+10(-1)(-1)\}/(125\times10^3) + 2\{10/3(10)(10)\}/(40\times10^3)$

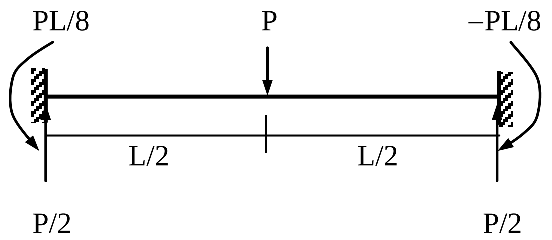
$= (0.2 + 0.16 + 16.67)\times10^{-3} = 17.03\times10^{-3}$ ft/k

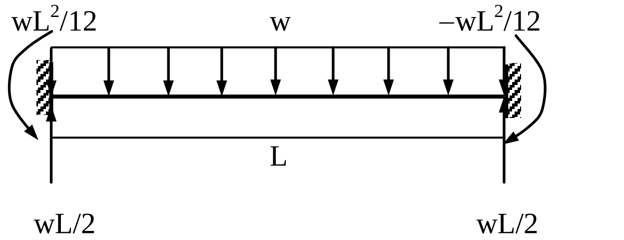
$\therefore R_A = -15.13\times10^{-3}/(17.03\times10^{-3}) = -0.889$ k

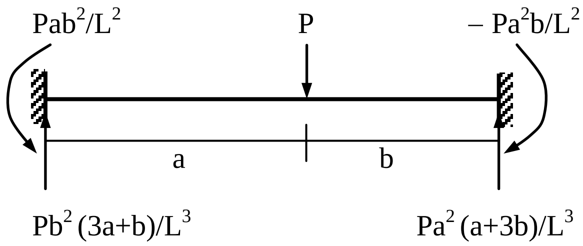


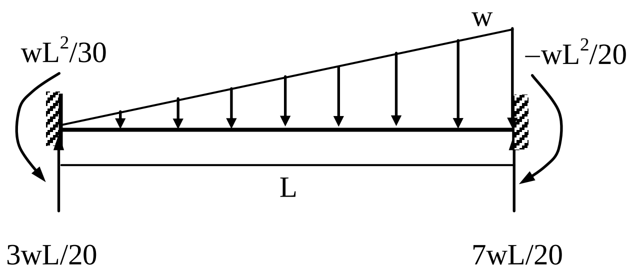
The Moment Distribution Method

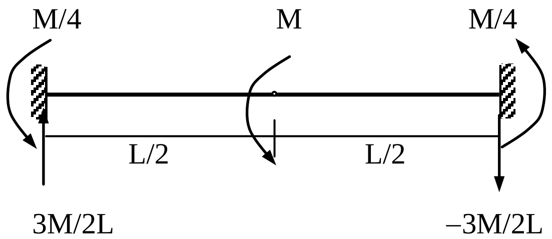
Fixed End Reactions for One-dimensional Prismatic Members under Typical Loadings

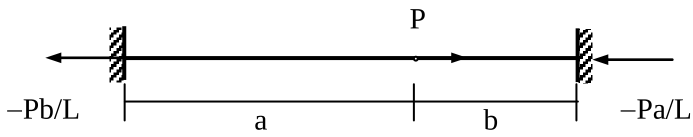


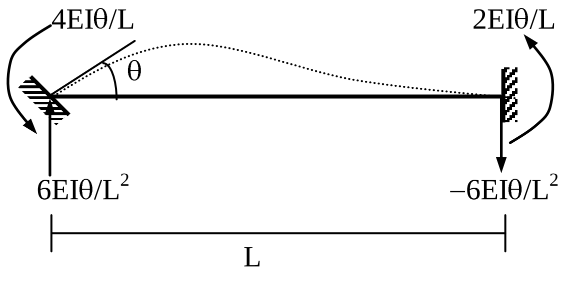


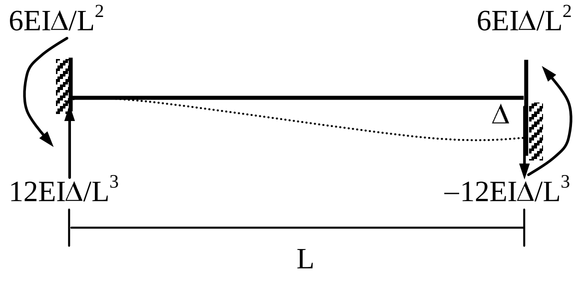




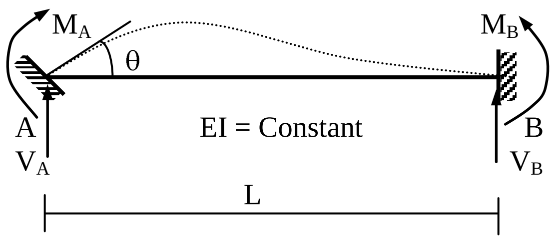


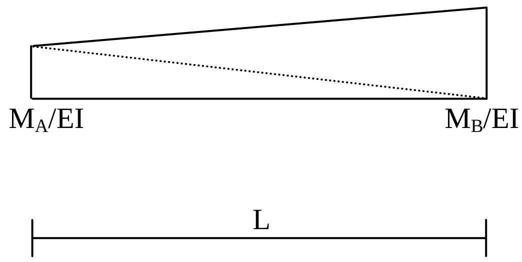






End Rotation and Rotational Stiffness of Fixed Ended Prismatic Members





Using the Moment Area Theorems between A and B

1st Theorem $\Rightarrow (M_A/EI + M_B/EI) \times L/2 = -\theta$

$\Rightarrow M_A + M_B = -\,2\ EI\theta/L$ ……………………..……...(1)

2nd Theorem $\Rightarrow M_A/EI \times L/2 \times L/3 + M_B/EI \times L/2 \times 2L/3 = 0 \Rightarrow (M_A/6 + M_B/3)\ L^2 = 0$

$\Rightarrow M_B = -\ M_A/2$ ..…………………..………(2)

$\therefore$(1) $\Rightarrow M_A/2 = -\,2\ EI\theta/L \Rightarrow M_A = -\,4\ EI\theta/L$ …………………..………..(3)

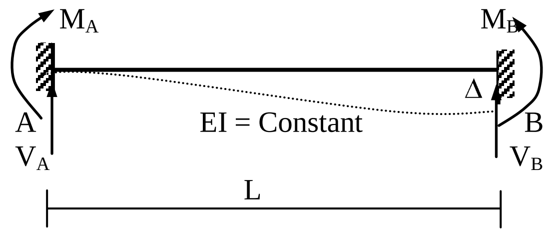
and (2) $\Rightarrow M_B = 2\ EI\theta/L$ …………………..………..(4)

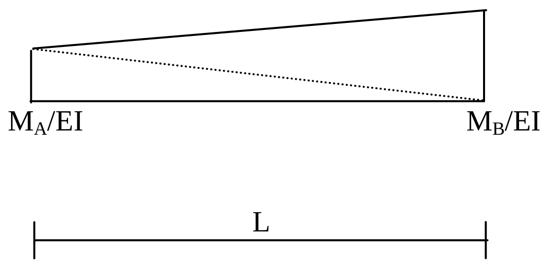
<u>The term 4EI/L is called the rotational stiffness and the ratio ($-M_B/M_A$ =) 0.5 the carry over factor</u> of the member AB.

Taking $\sum M_B = 0$ and $\sum M_A = 0$, V_A and V_B can be derived to be $6EI/L^2$ and $-6EI/L^2$.

Note that for anti-clockwise rotation θ, the moments M_A and M_B are both anti-clockwise but have different signs in the BMD.

End Deflection and Shear Stiffness of Fixed Ended Prismatic Members





Using the Moment Area Theorems between A and B

1st Theorem $\Rightarrow (M_A/EI + M_B/EI) \times L/2 = 0 \ \Rightarrow M_B = -\ M_A$ ……………………..……...(1)

2nd Theorem $\Rightarrow M_A/EI \times L/2 \times L/3 + M_B/EI \times L/2 \times 2L/3 = \Delta \Rightarrow M_A + 2M_B = 6EI\Delta/\ L^2$ ...………(2)

$\therefore$(1), (2) $\Rightarrow -M_A = 6EI\Delta/\ L^2 \Rightarrow M_A = -6EI\Delta/\ L^2$ …………………..………..(3)

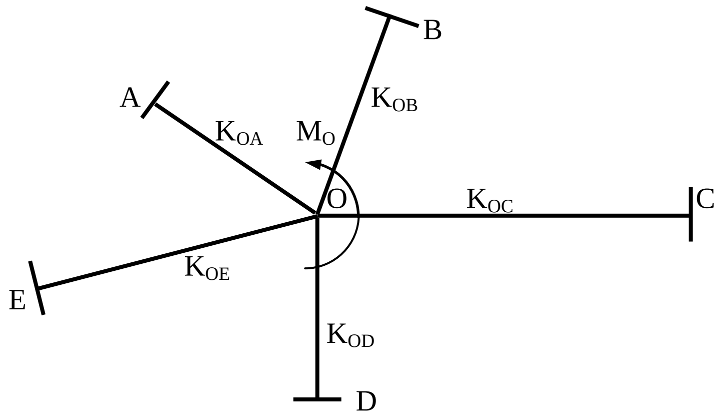
and (2) $\Rightarrow M_B = 6EI\Delta/\ L^2$ …………………..………..(4)

Taking $\sum M_B = 0$ and $\sum M_A = 0$, V_A and V_B can be derived to be $12EI\Delta/L^3$ and $-12EI\Delta/L^3$.

<u>The term $12EI/L^3$ is called the shear stiffness</u> of the member AB.

Note that M_A and M_B are both anti-clockwise here but have different signs in the BMD.

Rotation of a Joint and Moment Distribution Factors (MDF)



Flexural members OA, OB, OC…...are joined at joint O and have rotational stiffnesses of K_{OA}, K_{OB}, K_{OC}…….respectively; i.e., for unit rotation of the joint O they require moments K_{OA}, K_{OB}, K_{OC}……. respectively to be applied at O.

If a moment M_O applied at joint O causes it to rotate by an angle θ, the following moments are needed to rotate members OA, OB, OC…...

$$M_{OA} = K_{OA}\,\theta \qquad \text{………..(1)}$$
$$M_{OB} = K_{OB}\,\theta \qquad \text{………..(2)}$$
$$M_{OC} = K_{OC}\,\theta \qquad \text{………..(3)}$$
………………………

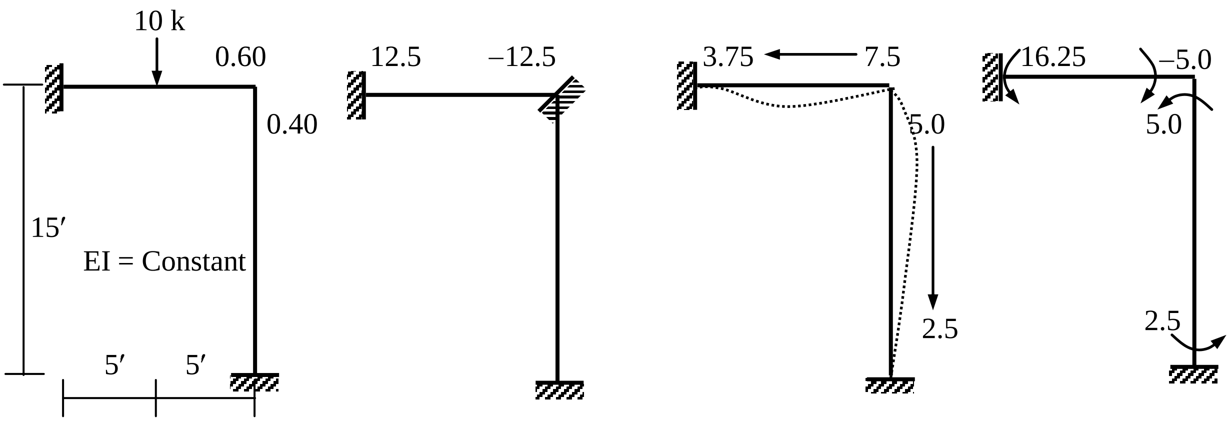
Adding (1), (2), (3)…. $\Rightarrow M_{OA} + M_{OB} + M_{OC} + \ldots\ldots = K_{OA}\,\theta + K_{OB}\,\theta + K_{OC}\,\theta + \ldots\ldots$ ………..(4)

Since $M_O = M_{OA} + M_{OB} + M_{OC} + \ldots\ldots\ldots$

$$M_O = (K_{OA} + K_{OB} + K_{OC} + \ldots\ldots)\,\theta = K_O\,\theta \qquad [K_O = K_{OA} + K_{OB} + K_{OC} + \ldots\ldots]$$
$$\Rightarrow \theta = M_O/(K_O) \qquad \text{………..(5)}$$
$$\therefore (1) \Rightarrow M_{OA} = [K_{OA}/K_O]\,M_O \qquad \text{………..(6)}$$
$$(2) \Rightarrow M_{OB} = [K_{OB}/K_O]\,M_O \qquad \text{………..(7)}$$
$$(3) \Rightarrow M_{OC} = [K_{OC}/K_O]\,M_O \qquad \text{………..(8)}$$
………………………

The factors $[K_{OA}/K_O]$, $[K_{OB}/K_O]$, $[K_{OC}/K_O]$……….are the moment distribution factors (MDF) of members OA, OB, OC……..respectively. Therefore the distributed moments in members are proportional to their respective MDFs.

<u>Example</u>



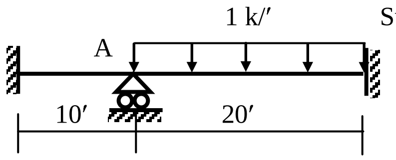
[Load and MDF] [FEM] [Dist M] [Total end M]

Problems on Moment Distribution

Assume EI = constant = 40×10^3 k-ft^2

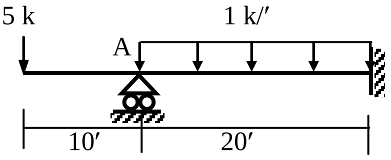
<u>Beams</u>

1.

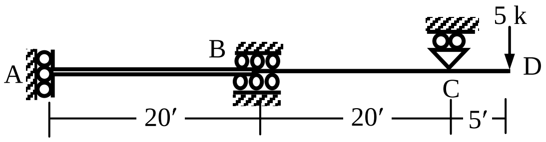


Support A settles 0.05′

2.

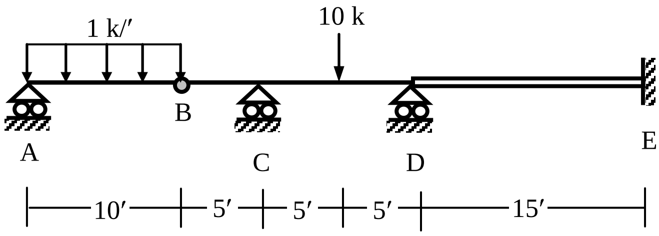


3.



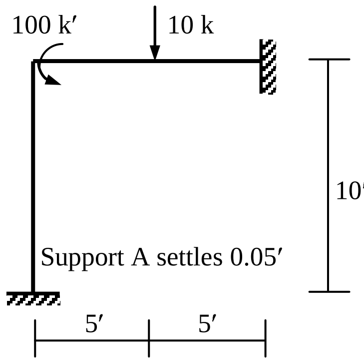
A and B are guided roller supports
EI_{AB} = 2 EI

4.



B in an Internal Hinge
EI_{DE} = 2 EI

5.



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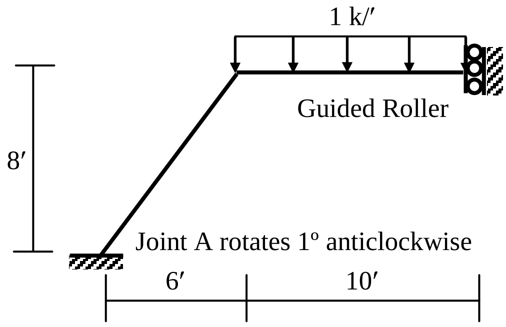

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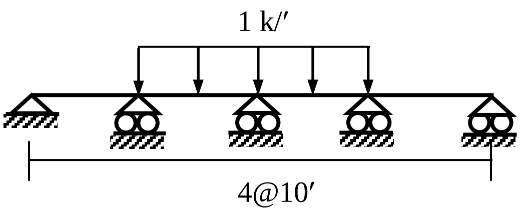
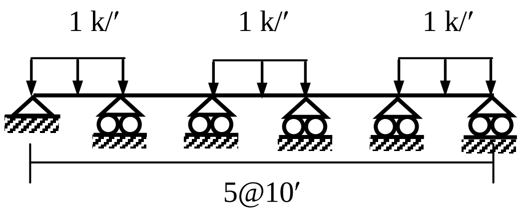
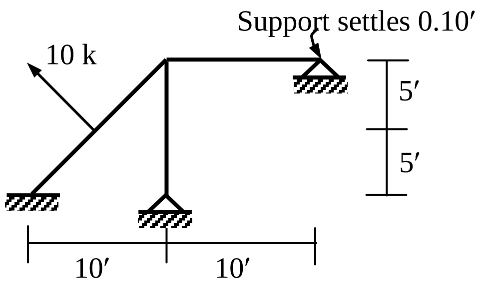


8.

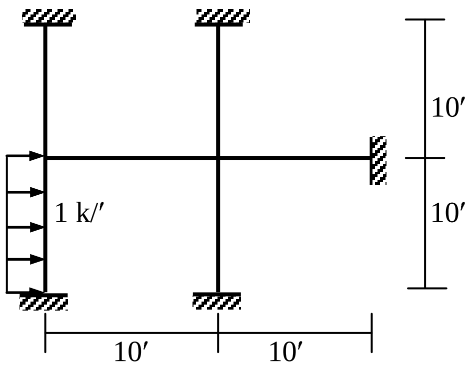


<u>Frames</u>

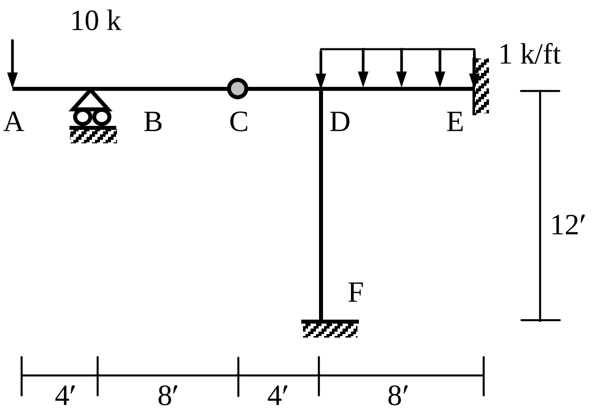
9.



10.

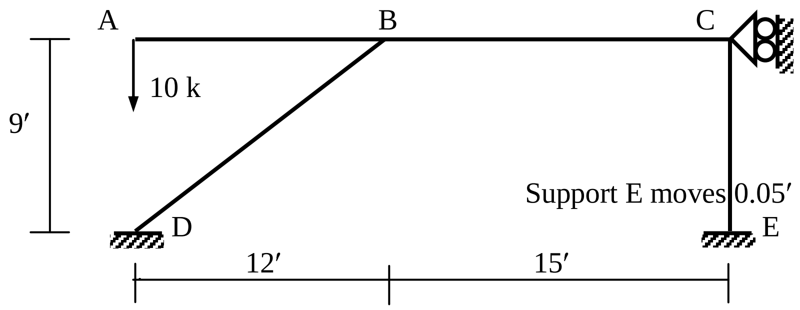


1.



EI = Constant
C is an Internal Hinge

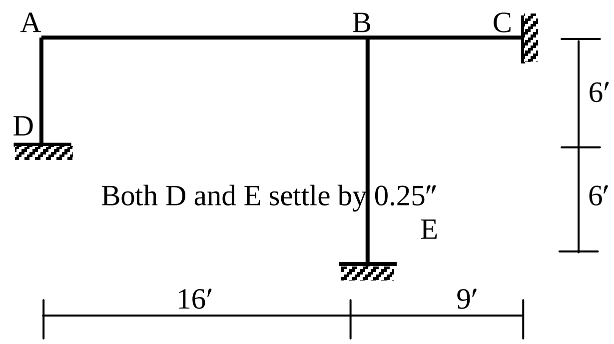
2.



$EI = 40{,}000$ k-ft^2

Support E moves 0.05′ downwards

3.



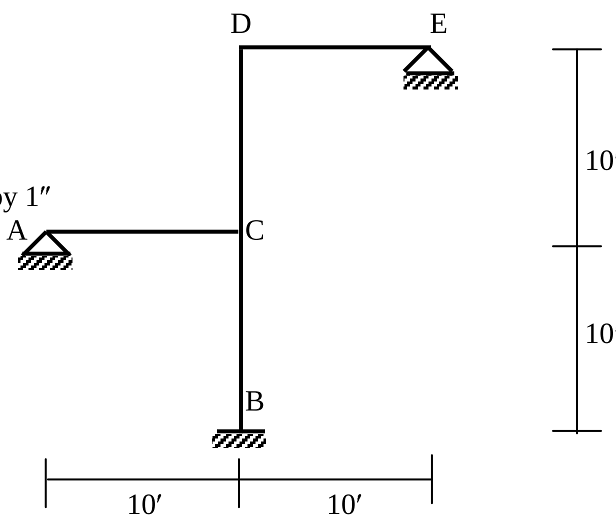
$EI = 40{,}000$ k-ft^2

Both D and E settle by 0.25″

4.

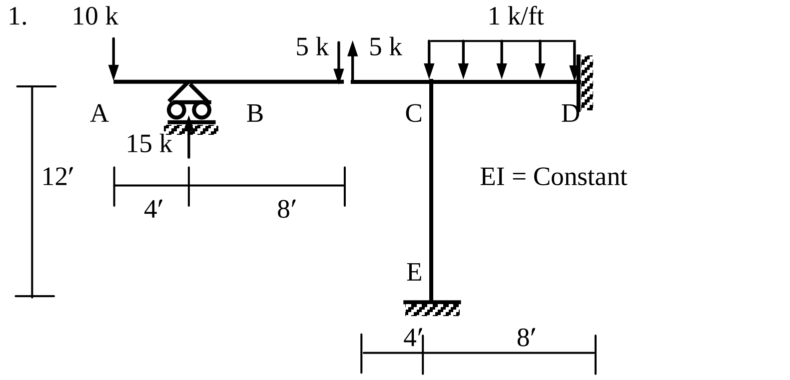
$EI = 40{,}000$ k-ft^2

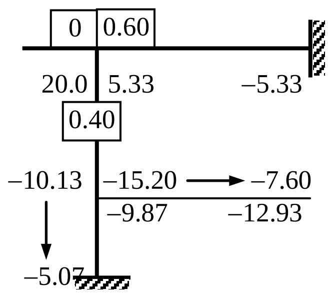
Both A and B settle by 1″

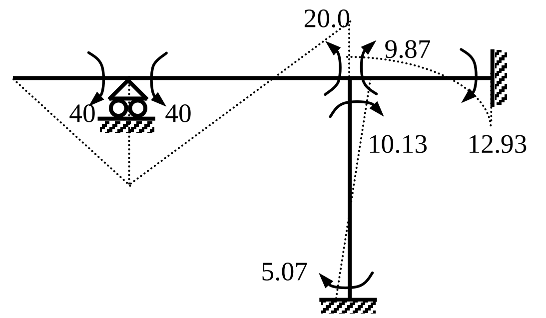


Moment Distribution for Frames (from past exams)

1.

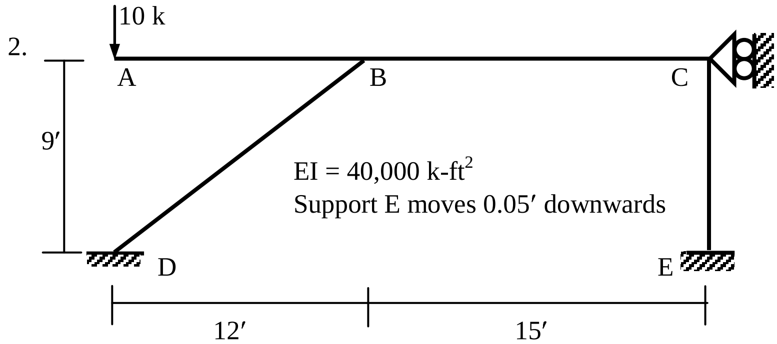




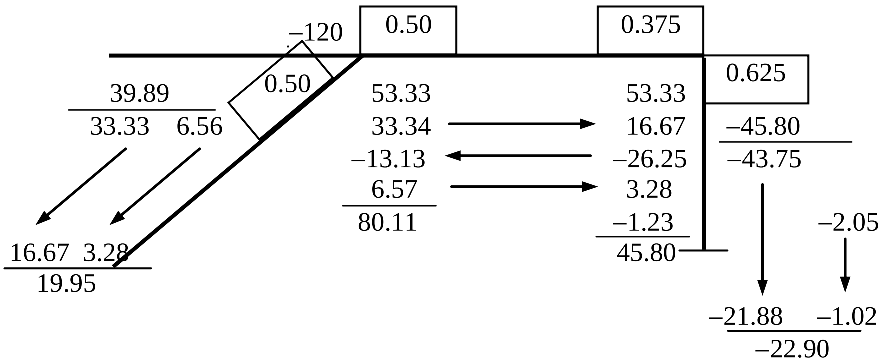


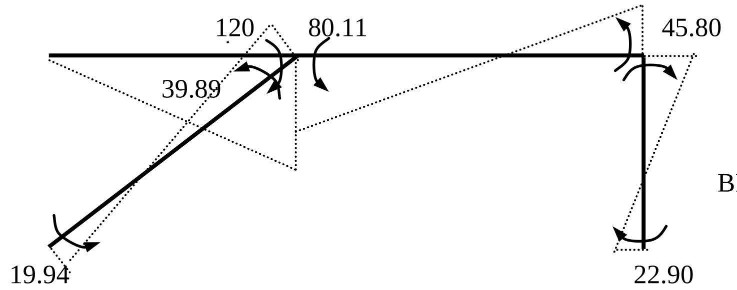
BMD (k′)

2.



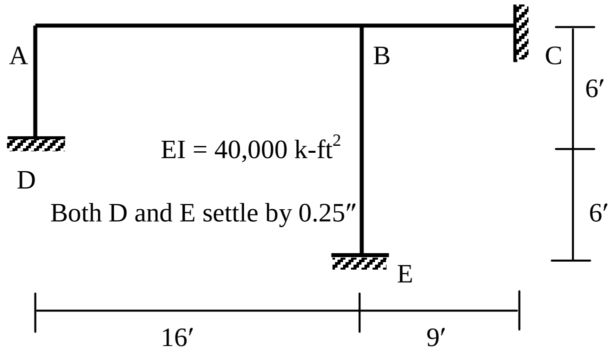
$\therefore 6EI\Delta/L^2$
$= 6\times40000\times0.05/15^2 = 53.33$ k′

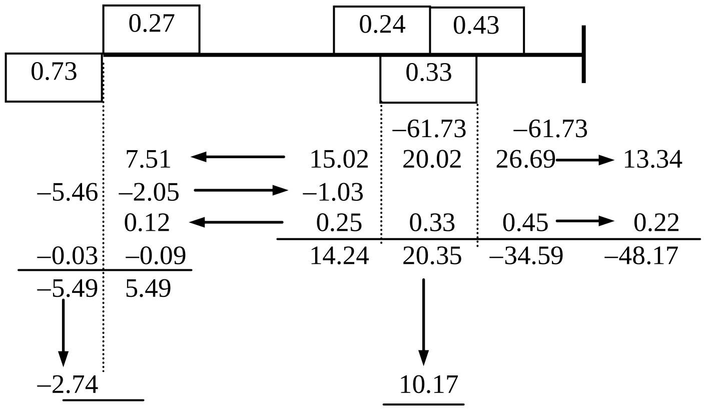


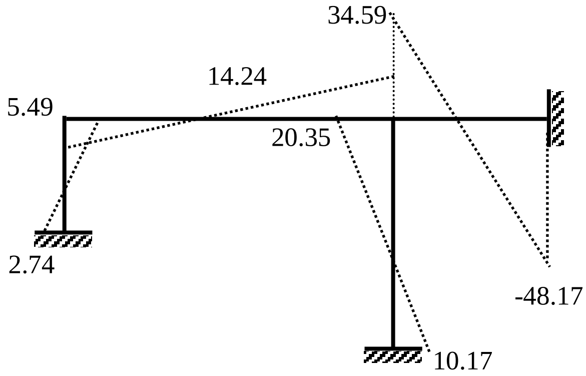


BMD (k′)

3.

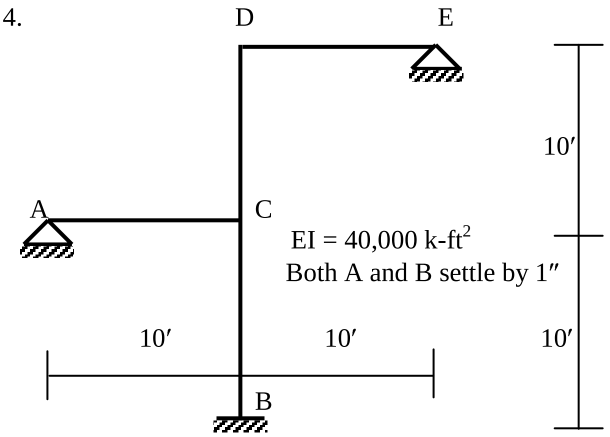

A
B
C
D
E
EI = 40,000 k-ft²
Both D and E settle by 0.25″
6′
6′
16′
9′

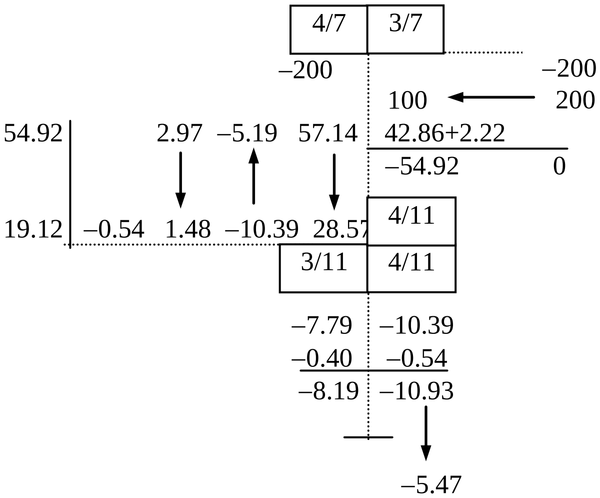

0.27
0.24
0.43
0.73
0.33
−61.73
−61.73
7.51
15.02
20.02
26.69
13.34
−5.46
−2.05
−1.03
0.12
0.25
0.33
0.45
0.22
−0.03
−0.09
14.24
20.35
−34.59
−48.17
−5.49
5.49
−2.74
10.17

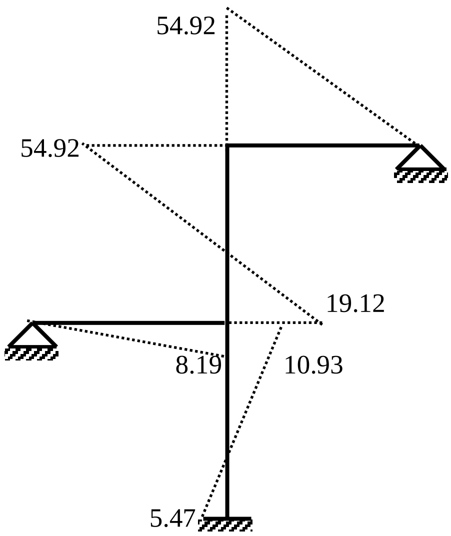

34.59
14.24
5.49
20.35
2.74
-48.17
10.17

BMD (k′)

4.


D
E
A
C
B
EI = 40,000 k-ft²
Both A and B settle by 1″
10′
10′
10′
10′

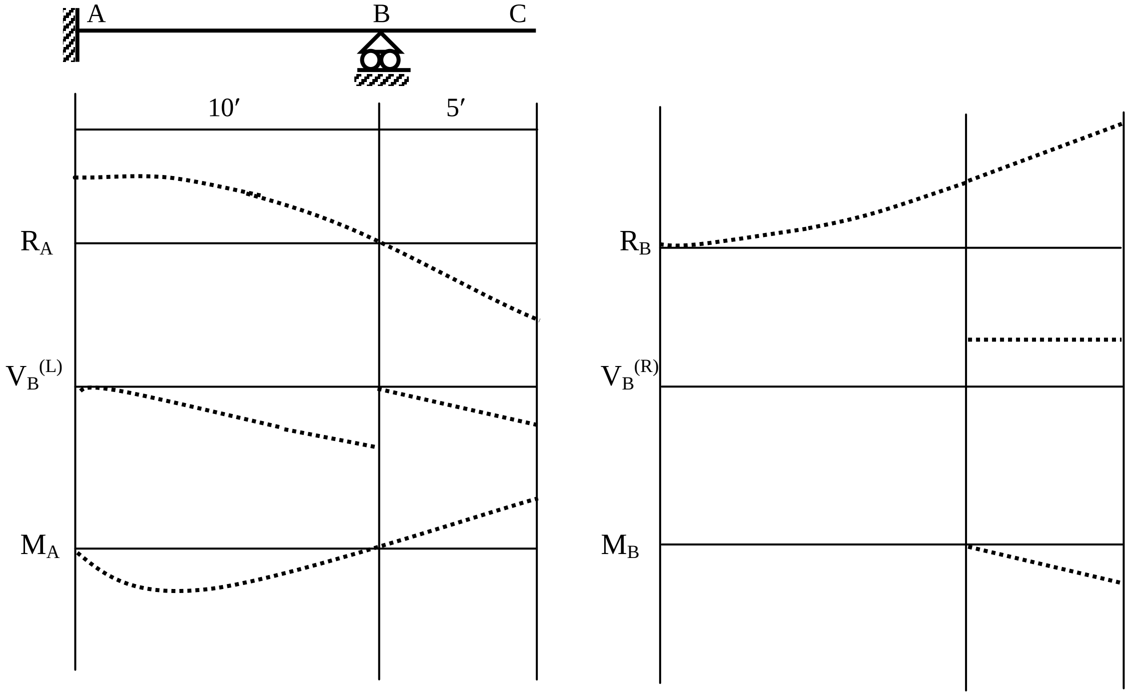

4/7
3/7
−200
−200
100
200
54.92
2.97
−5.19
57.14
42.86+2.22
−54.92
0
19.12
−0.54
1.48
−10.39
28.57
4/11
3/11
4/11
−7.79
−10.39
−0.40
−0.54
−8.19
−10.93
−5.47


54.92
54.92
19.12
8.19
10.93
5.47

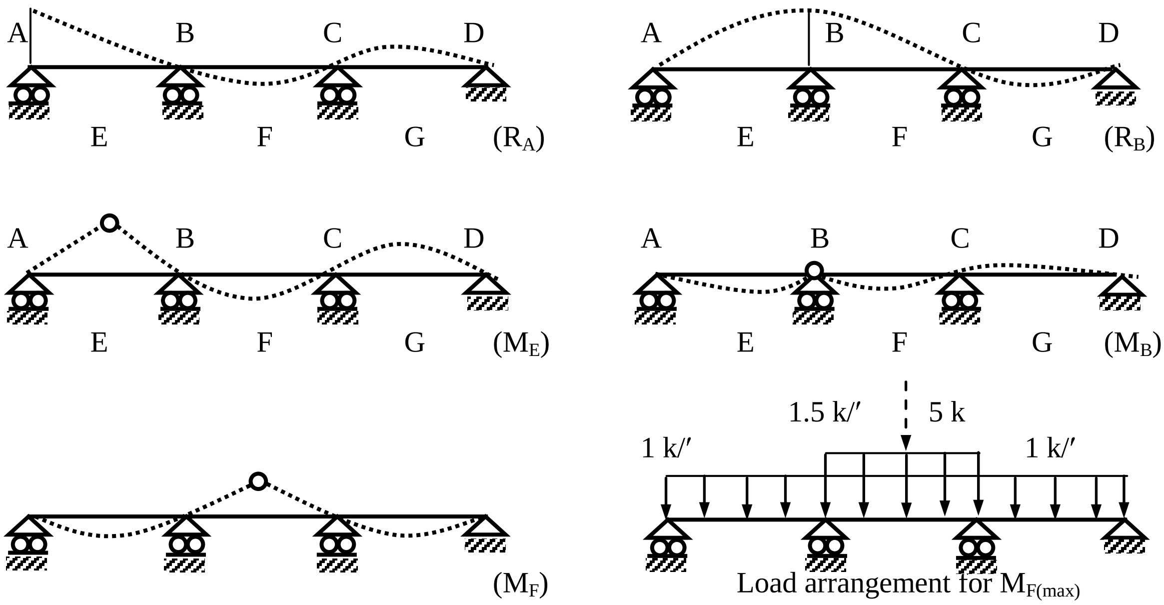
BMD (k′)

Qualitative Influence Lines and Maximum Forces

1. For the beam shown below, draw the influence lines of R_A, R_B, $V_B^{(L)}$, $V_B^{(R)}$, M_A, M_B.



2. For the beam shown below, DL = 1 k/′, moving LL = 0.5 k/′ (UDL), 5 k (concentrated). Calculate the maximum values of R_A, R_B, M_E, M_B and M_F [Each span is 10′ long].

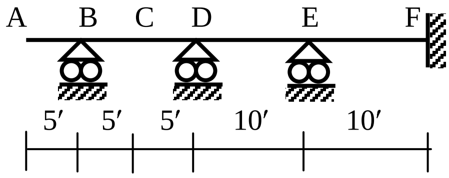


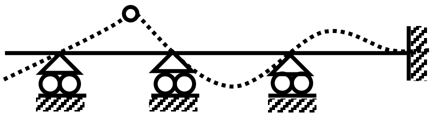
Final end moments | 0 −16.25 | 16.25

$\therefore\ M_{F(max)} = -16.25 + 1.5 \times 10^2/8 + 5 \times 10/4 = 15$ k′

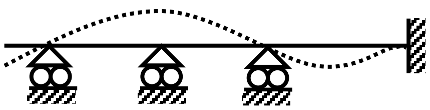
3. For the beam shown below, draw the qualitative influence lines for
   (i) Bending moments M_C, M_D, M_E, M_F
   (ii) Support reactions R_B, R_D, R_E, R_F
   (iii) Shear forces $V_B^{(R)}$, $V_D^{(L)}$, $V_D^{(R)}$, $V_E^{(L)}$, $V_E^{(R)}$, V_F

   If the beam is subjected to a uniformly distributed DL = 1.5 k/ft and moving LL = 0.5 k/ft (uniformly distributed) and 5 k (concentrated), calculate the maximum values of
   (i) positive M_C, (ii) positive R_D and (iii) positive $V_E^{(R)}$ [Given: EI = constant].

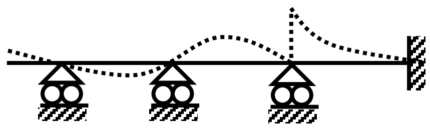




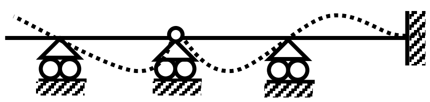
IL of M_C



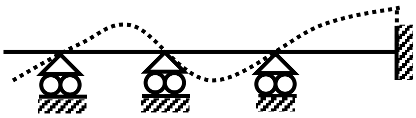
IL of R_D



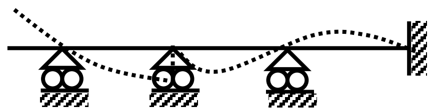
IL of $V_E^{(R)}$



IL of M_D

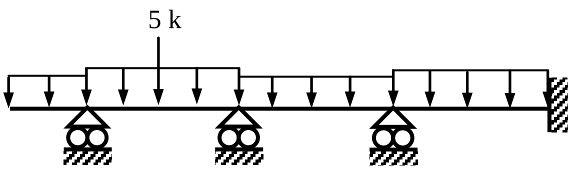


IL of R_F



IL of $V_D^{(L)}$

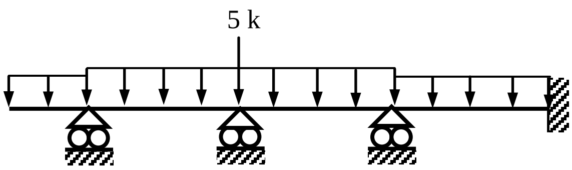
(i) Maximum positive value of M_C:



1.5 k/′	2 k/′		1.5 k/′		2 k/′		UDL
0	1	3/7	4/7	0.5	0.5		DF
−18.75	22.92	−22.92	12.5	−12.5	16.67	−16.67	FEM
	(−4.17)						
		−2.08					
		(5.36)	(7.14)	3.57			
			−1.93	(−3.87)	(−3.87)	−1.93	
		(0.83)	(1.10)	0.55			
			−0.14	(−0.27)	(−0.28)	−0.14	
		(0.06)	(0.08)				
−18.75	18.75	−18.75	18.75	−12.52	12.52	−18.74	

∴ Maximum value of M_C
$= -18.75 + 2\times10^2/8 + 5\times10/4 = 18.75$ k′

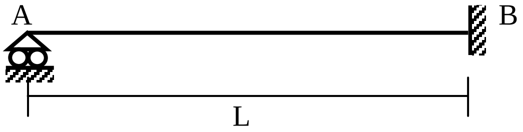
(ii) Maximum positive value of R_D:



1.5 k/′	2 k/′		2 k/′		1.5 k/′	
0	1	3/7	4/7	0.5	0.5	
−18.75	16.67	−16.67	16.67	−16.66	12.5	−12.5
	(2.08)			(2.08)	(2.08)	
		1.04	1.04			1.04
		(−0.89)	(−1.19)	−0.60		
			0.15	(0.30)	(0.30)	0.15
		(−0.06)	(−0.09)			
−18.75	18.75	−16.58	16.58	−14.88	14.88	−11.31

$V_D^{(L)} = -2\times10/2+(18.75-16.58)/10 = -9.78$ k
$V_D^{(R)} = 2\times10/2+(16.58-14.88)/10 = 10.17$ k
∴ Maximum $R_D = V_D^{(R)} - V_D^{(L)} + 5 = 24.95$ k

Quantitative Influence Lines for Indeterminate Structures



EI = Constant = 1 (assume)

From Moment-Curvature Relationship, EI $d^2v/dx^2 = M(x) = R_A x$

$\therefore$ In this case, $d^2v/dx^2 = M(x) = R_A\, x$ ……………………..(1)

$\Rightarrow dv/dx = \theta(x) = R_A\, x^2/2 + C_1$ ……………………..(2)

$\Rightarrow v(x) = R_A x^3/6 + C_1 x + C_2$ ……………………..(3)

There are three unknowns in these equations; i.e., R_A, C_1 and C_2

For the given beam, there are three known boundary conditions from which these three unknowns can be calculated.

The boundary conditions are, $v(0) = 1$, $v(L) = 0$ and $\theta(L) = 0$ ……………………..(4)

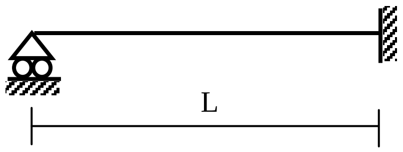
Using $v(0) = 1$ in (3) $\Rightarrow 1 = 0 + 0 + C_2 \Rightarrow C_2 = 1$ ……………………..(5)

$\therefore$ Using $v(L) = 0$ in (3) $\Rightarrow 0 = R_A L^3/6 + C_1 L + 1 \Rightarrow R_A L^3/6 + C_1 L = -1$ ……………………..(6)

$\therefore$ Using $\theta(L) = 0$ in (2) $\Rightarrow 0 = R_A L^2/2 + C_1 \Rightarrow R_A L^2/2 + C_1 = 0$ ……………………..(7)

Solving (6) and (7), $R_A = 3/L^3$ and $C_1 = -3/2L$ ……………………..(8)

$\therefore\ v(x) = (x/L)^3/2 - 3(x/L)/2 + 1$ ……………………..(9)



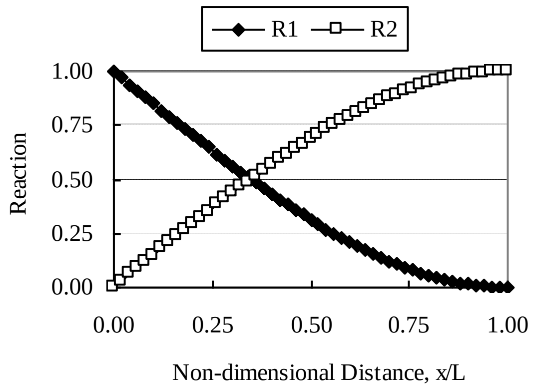


Fig. 1: Influence Lines for Reactions

Once the equation of IL for R_A is determined, the equations of IL for shear force and bending moment at any section can also be derived.

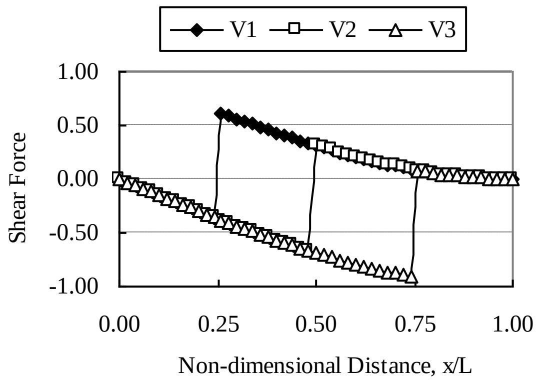


Fig. 2: Influence Lines for Shear Forces

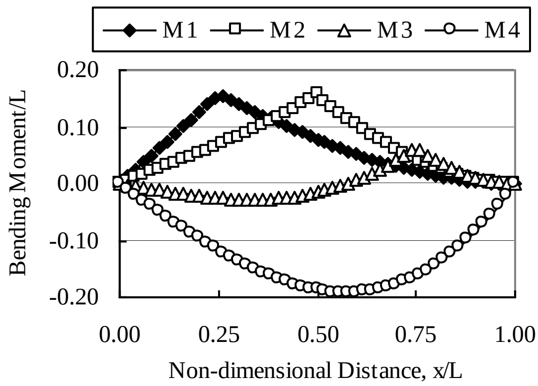


Fig. 3: Influence Lines for Bending Moments

Short Questions and Explanations

Flexibility Method for 2D Trusses vs. 2D Frames

1. Unknowns: Forces only vs. Forces + Moments
2. No. of Unknowns: *dosi* = m + r – 2j vs. *dosi* = 3m + r – 3j
3. Member Properties: E, A vs. E, G, A, A∗, I
4. Deformations considered: Axial vs. Axial, Shear, Flexural
5. Forces Calculated: Member Axial Forces vs. Member Axial, Shear Forces, BM's
6. Structural Displacements: Deflections vs. Deflections + Rotations

Also learn

- Lateral Load Analysis by Portal vs. Cantilever Method
- Vertical Load Analysis by ACI Coefficients vs. Approximate Hinge locations
- Difference between approximate Methods for Truss Analysis
- Flexibility Method vs. Moment Distribution Method

Briefly explain why

- it is often useful to perform approximate analysis of statically indeterminate structures
- *dosi* of 3D truss = m + r – 3j and *dosi* of 3D frame = 6m + r – 6j – h
- axial deformations are sometimes neglected for structural analysis of beams/frames but not trusses
- support settlement is to be considered/avoided in designing statically indeterminate structures
- unit load method is often used in the structural analysis by Flexibility Method
- a guided roller can be used in modeling one-half of a symmetric structure
- the terms moment distribution factor and carry over factor in the Moment Distribution Method
- the influence lines of statically determinate structures are straight while the influence lines of statically indeterminate structures are curved

Comment on

- two basic characteristics of the Flexibility Matrix of a structure
- the main advantage and limitation of the Moment Distribution Method
- advantage of using modified stiffness in the Moment Distribution Method
- the applicability of 'qualitative' and 'quantitative' influence lines

Non-coplanar Forces and Analysis of Space Truss

Non-coplanar Force

A vector in space may be defined or located by any three mutually perpendicular reference axes Ox, Oy and Oz (Fig. 1). This vector may be resolved into three components parallel to the three reference axes.

If the force OC (of magnitude F) makes angles α, β and γ with the three reference axes Ox, Oy and Oz, then the components of the force along these axes are given by

$F_x = F \cos \alpha$ ……………………………(i)

$F_y = F \cos \beta$ ……………………………(ii)

$F_z = F \cos \gamma$ ……………………………(iii)

$\therefore [(i)^2 + (ii)^2 + (iii)^2] \Rightarrow F = \sqrt{[F_x^{\ 2} + F_y^{\ 2} + F_z^{\ 2}]}$ ……………(iv)

$\therefore$ (i) $\Rightarrow \cos \alpha = F_x/\sqrt{[F_x^{\ 2} + F_y^{\ 2} + F_z^{\ 2}]}$ ………………(v)

(ii) $\Rightarrow \cos \beta = F_y/\sqrt{[F_x^{\ 2} + F_y^{\ 2} + F_z^{\ 2}]}$ ………………(vi)

(iii) $\Rightarrow \cos \gamma = F_z/\sqrt{[F_x^{\ 2} + F_y^{\ 2} + F_z^{\ 2}]}$ ………………(vii)

The values of cos α, cos β and cos γ given by Eqs. (v), (vi) and (vii) are called the *direction cosines* of the vector F.

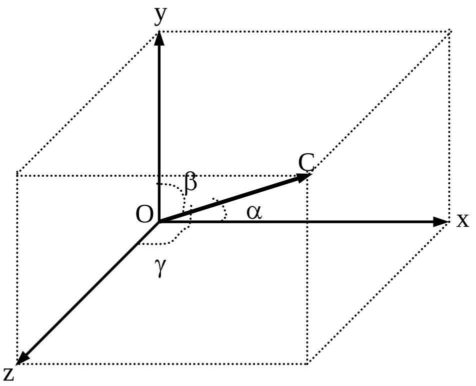


Fig. 1: Non-coplanar Force and Components

Space Truss

Although simplified two-dimensional structural models are quite common, all civil engineering structures are actually three-dimensional. Among them, electric towers, offshore rigs, rooftops of large open spaces like industries or auditoriums are common examples of three-dimensional or space truss. The members of a space truss are non-coplanar and therefore their axial forces can be modeled as non-coplanar forces.

Since there is only one force per member and three equilibrium equations per joint of a space truss, the degree of statical indeterminacy (dosi) of such a structure is given by

dosi = m + r – 3j ……………………………(viii)

The three equilibrium equations per joint of a space truss are related to forces in the three perpendicular axes x, y and z

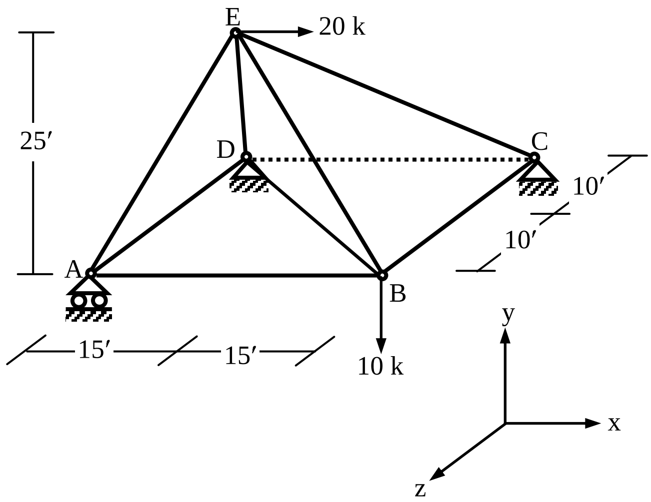
$\sum F_x = 0$, $\sum F_y = 0$ and $\sum F_z = 0$ ……………………………(ix)

However the other three equilibrium equations related to moments; i.e.,

$\sum M_x = 0$, $\sum M_y = 0$ and $\sum M_z = 0$ ……………………………(x)

may also be needed to calculate the support reactions of the truss. Here, it is pertinent to mention that the moment of a force about an axis is zero if the force is parallel to the axis (when it does not produce any rotational tendency about that axis) or intersects it (when the perpendicular distance from the axis is zero).

Example: Calculate the support reactions and member forces of the truss shown below.



Ignoring the zero force member CD
dosi = m + r – 3j = 8 + 7 – 3 × 5 = 0
∴ The structure is statically determinate.

Member	L_x	L_y	L_z	C_x	C_y	C_z
AB	30	0	0	1.00	0.00	0.00
BC	0	0	–20	0.00	0.00	–1.00
BD	–30	0	–20	–0.83	0.00	–0.56
AD	0	0	–20	0.00	0.00	–1.00
AE	15	25	–10	0.49	0.81	–0.32
BE	–15	25	–10	–0.49	0.81	–0.32
CE	–15	25	10	–0.49	0.81	0.32
DE	15	25	10	0.49	0.81	0.32

$\sum M_{CD} = 0 \Rightarrow Y_A \times 20 - 10 \times 20 = 0 \Rightarrow Y_A = 10$ k ……………(1)
$\sum M_{BC} = 0 \Rightarrow Y_A \times 30 + Y_D \times 30 + 20 \times 25 = 0 \Rightarrow Y_D = -26.67$ k ……………(2)
$\sum M_{y(D)} = 0 \Rightarrow -20 \times 10 + Z_C \times 30 = 0 \Rightarrow Z_C = 6.67$ k ……………(3)
$\sum F_y = 0 \Rightarrow Y_A + Y_C + Y_D - 10 = 0 \Rightarrow Y_C = 26.67$ k ……………(4)
$\sum F_x = 0 \Rightarrow X_D + X_C + 20 = 0$ ……………(5)
$\sum F_z = 0 \Rightarrow Z_D + Z_C = 0$
$\Rightarrow Z_D = -Z_C = -6.67$ k [using (3)] ……………(6)

Equilibrium of Joint A (unknowns F_{AB}, F_{AD} and F_{AE}):
$\sum F_x = 0 \Rightarrow F_{AB} + 0.49\, F_{AE} = 0$ ……………(7)
$\sum F_y = 0 \Rightarrow 0.81\, F_{AE} + 10 = 0 \Rightarrow F_{AE} = -12.33$ k ……………(8)
$\Rightarrow F_{AB} = -0.49\, F_{AE} = 6.00$ k [using (7)] ……………(9)
$\sum F_z = 0 \Rightarrow -F_{AD} - 0.32\, F_{AE} = 0 \Rightarrow F_{AD} = 4.00$ k [using (8)] ……………(10)

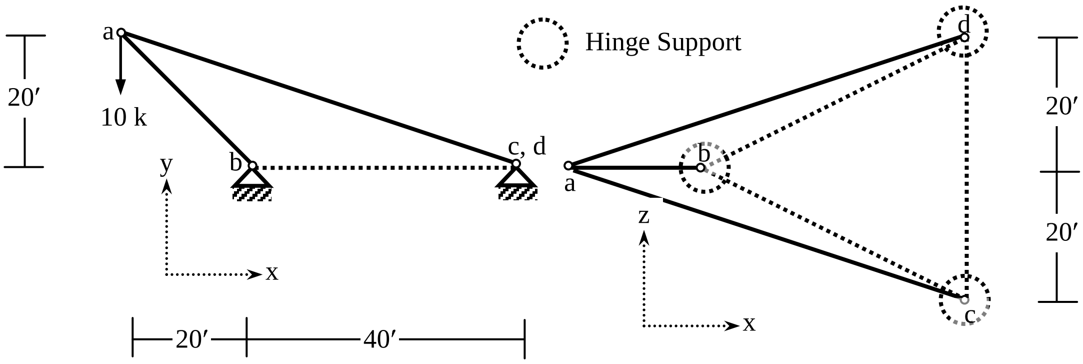
Equilibrium of Joint B (unknowns F_{BC}, F_{BD} and F_{BE}):
$\sum F_x = 0 \Rightarrow -F_{BA} - 0.83\, F_{BD} - 0.49\, F_{BE} = 0$ ……………(11)
$\sum F_y = 0 \Rightarrow 0.81\, F_{BE} - 10 = 0 \Rightarrow F_{BE} = 12.33$ k ……………(12)
$\Rightarrow F_{BD} = (-F_{BA} - 0.49\, F_{BE})/0.83 = -14.42$ k [using (11)] ……………(13)
$\sum F_z = 0 \Rightarrow -F_{BC} - 0.56\, F_{BD} - 0.32\, F_{BE} = 0 \Rightarrow F_{BC} = 4.00$ k [using (12), (13)] ……………(14)

Equilibrium of Joint C (unknowns X_C and F_{CE}):
$\sum F_x = 0 \Rightarrow X_C - 0.49\, F_{CE} = 0$ ……………(15)
$\sum F_y = 0 \Rightarrow 26.67 + 0.81\, F_{CE} = 0 \Rightarrow F_{CE} = -32.88$ k ……………(16)
$\Rightarrow X_C = 0.49\, F_{CE} = -16$ k [using (16)] ……………(17)
$\sum F_z = 0 \Rightarrow 6.67 + 0.32\, F_{CE} + F_{CB} = 0 \Rightarrow F_{CB} = 4.00$ k [verified]

Equilibrium of Joint D (unknowns X_D and F_{DE}):
$\sum F_x = 0 \Rightarrow X_D + 0.49\, F_{DE} + 0.83\, F_{DB} = 0$ ……………(18)
$\sum F_y = 0 \Rightarrow -26.67 + 0.81\, F_{DE} = 0 \Rightarrow F_{DE} = 32.88$ k ……………(19)
$\Rightarrow X_D = -4.00$ [using (13), (19)] ……………(20)
$\Rightarrow X_C = -20 - X_D = -16.00$ [using (5)] ……………(21)
$\sum F_z = 0 \Rightarrow -6.67 + 0.32\, F_{DE} + F_{DA} + 0.56\, F_{DB} = 0 \Rightarrow -6.67 + 10.67 + 4.00 - 8.00 = 0$
$\Rightarrow 0 = 0$ [verified]

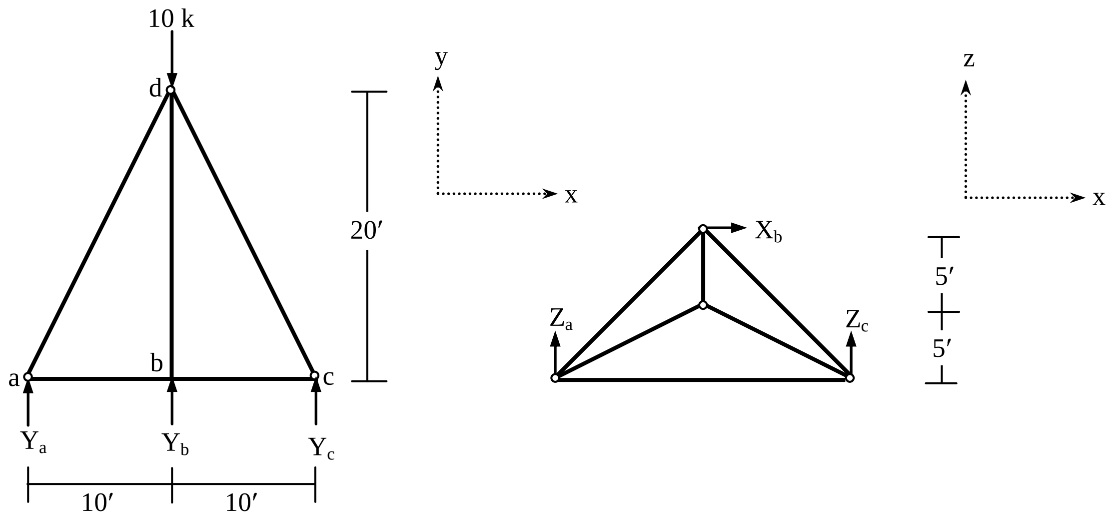
Problems on the Analysis of Space Trusses

1. Calculate the member forces of the space truss loaded as shown below.

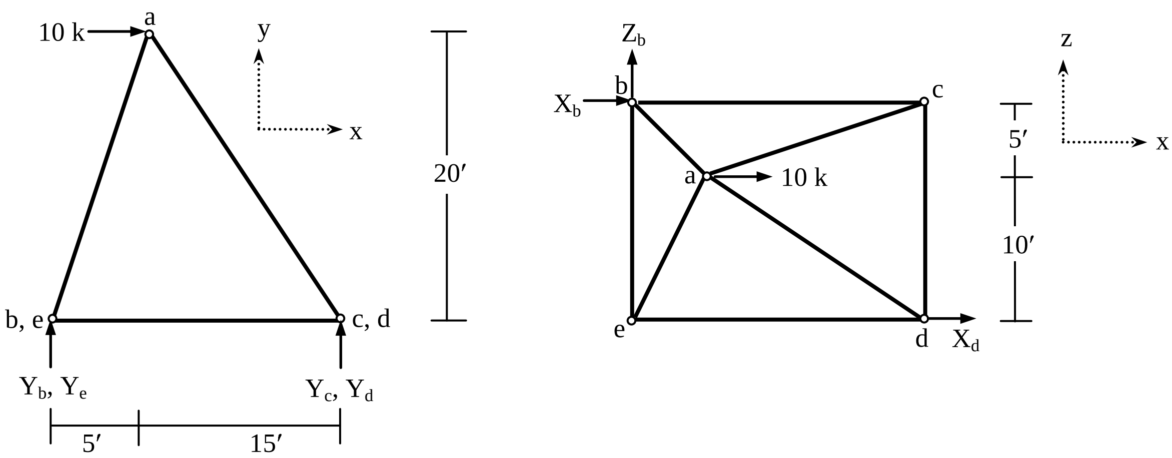


2. Calculate the horizontal (along x axis) deflection of joint E and vertical (along y axis) deflection of joint B of the space truss analyzed in class [Given: EA/L = constant = 500 k/ft].

3. Calculate the support reactions and member forces of the space truss loaded as shown below. Also calculate the vertical (along y axis) deflection of the joint d [Given: EA/L = constant = 500 k/ft].



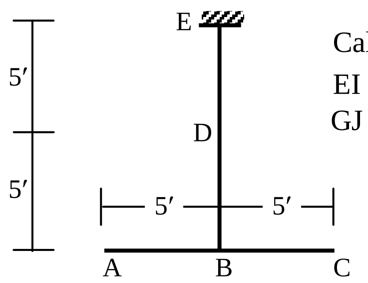
4. Calculate the support reactions, member forces and also the horizontal (along x axis) deflection of the joint a of the space truss loaded as shown below [Given: EA/L = constant = 500 k/ft].



Deflection of Grids due to Combined Flexural and Torsional Deformations

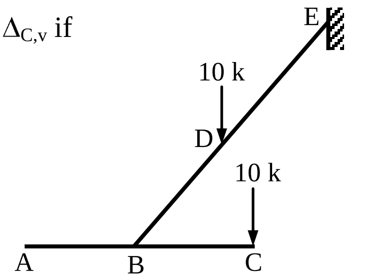
Grids are 2-dimensional (coplanar) structures with one deflection and two rotations at each node.
If the structure is in the x-z plane, the deflection is out-of-plane (along the y axis) while the rotations are about two in-plane axes (x and z axis). Grids are loaded perpendicular to the structural plane and have three forces per member; i.e., shear force, bending moment and torsion.

Example

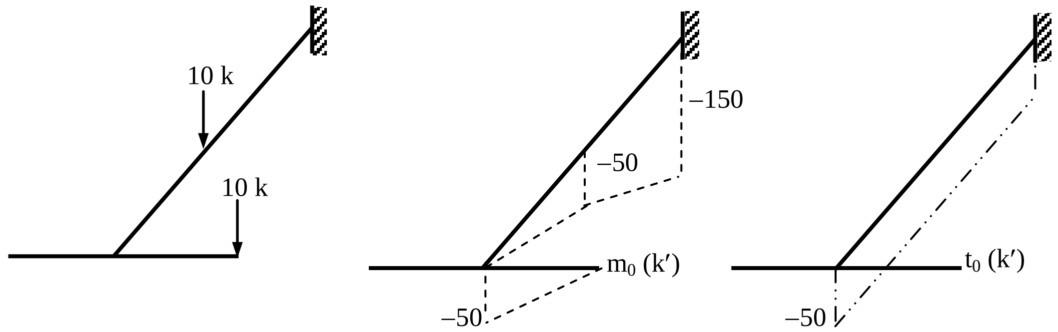


Top View

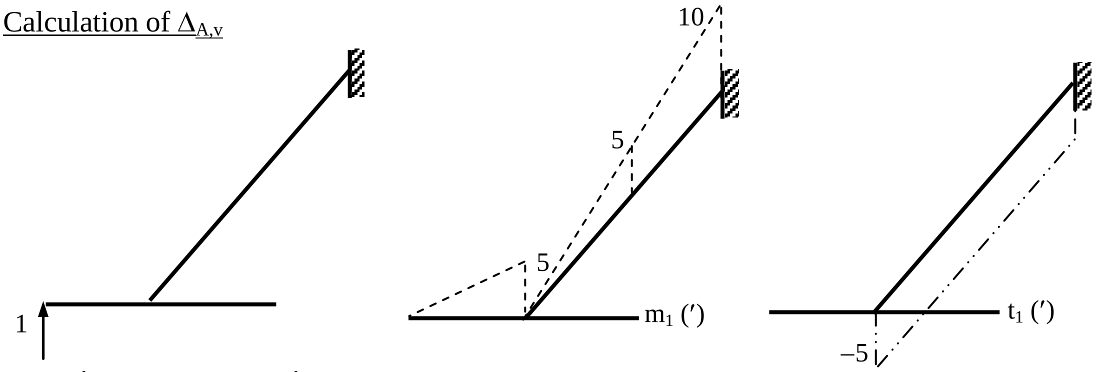
Calculate $\Delta_{A,v}$ and $\Delta_{C,v}$ if
EI = 40×10^3 k-ft^2
GJ = 30×10^3 k-ft^2



Isometric View

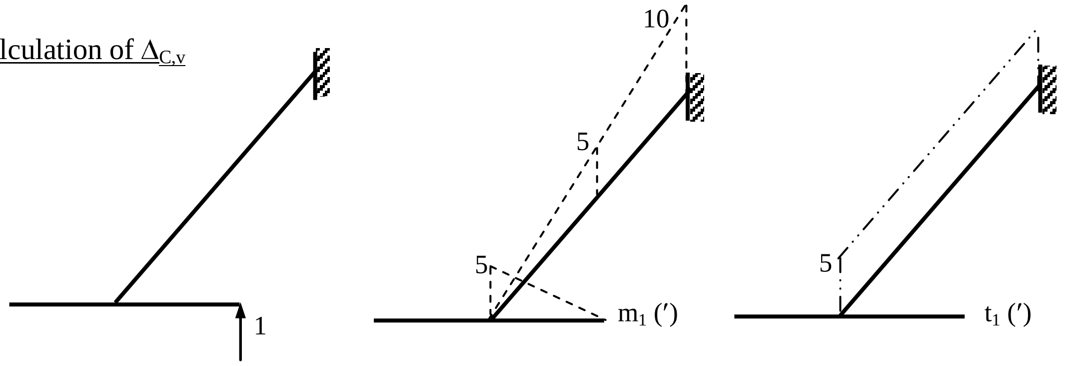


Calculation of $\Delta_{A,v}$



$\Delta_{A,v} = \int (m_1\, m_0/EI)\, dS + \int (t_1\, t_0/GJ)\, dS$
$= \{[5\times5\times(-50)/3] + 5\times[5\times(-250)+10\times(-350)]/6\}/(40\times10^3) + \{10\times(-5)\times(-50)\}/(30\times10^3)$
$= -109.38\times10^{-3} + 83.33\times10^{-3} = -26.05\times10^{-3}$ ft

Calculation of $\Delta_{C,v}$

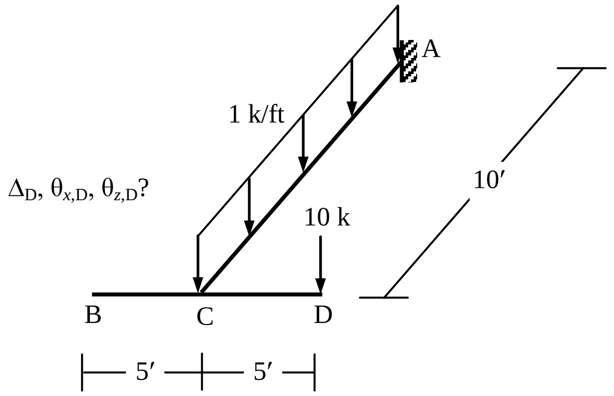


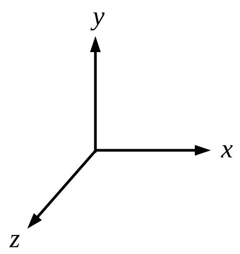
$\Delta_{C,v} = \{[5\times5\times(-50)/3]+[5\times5\times(-50)/3]+5\times[5\times(-250)+10\times(-350)]/6\}/(40\times10^3) + \{10\times(5)\times(-50)\}/(30\times10^3)$
$= -120.80\times10^{-3} - 83.33\times10^{-3} = -204.13\times10^{-3}$ ft

Problems on Deflection of Grids using Method of Virtual Work (Unit Load Method)

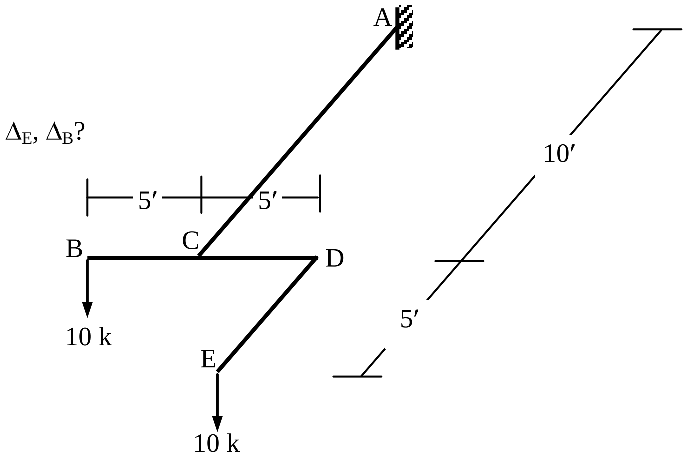
Assume EI = 40×10^3 k-ft^2, GJ = 30×10^3 k-ft^2

1.

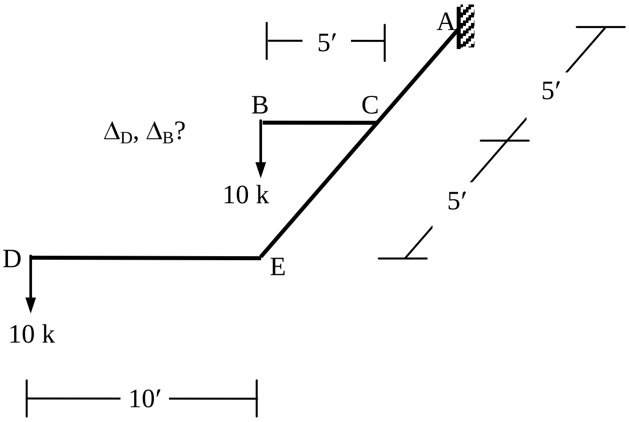




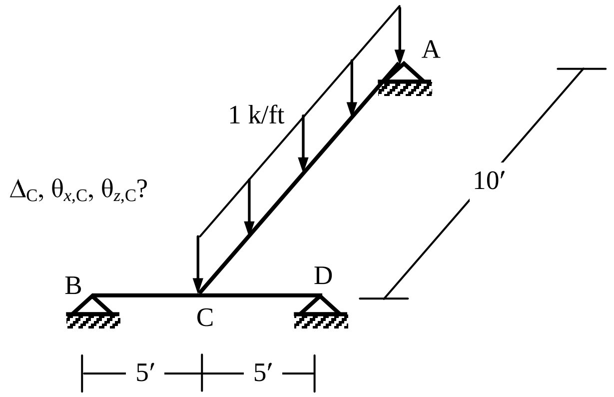
2.



3.

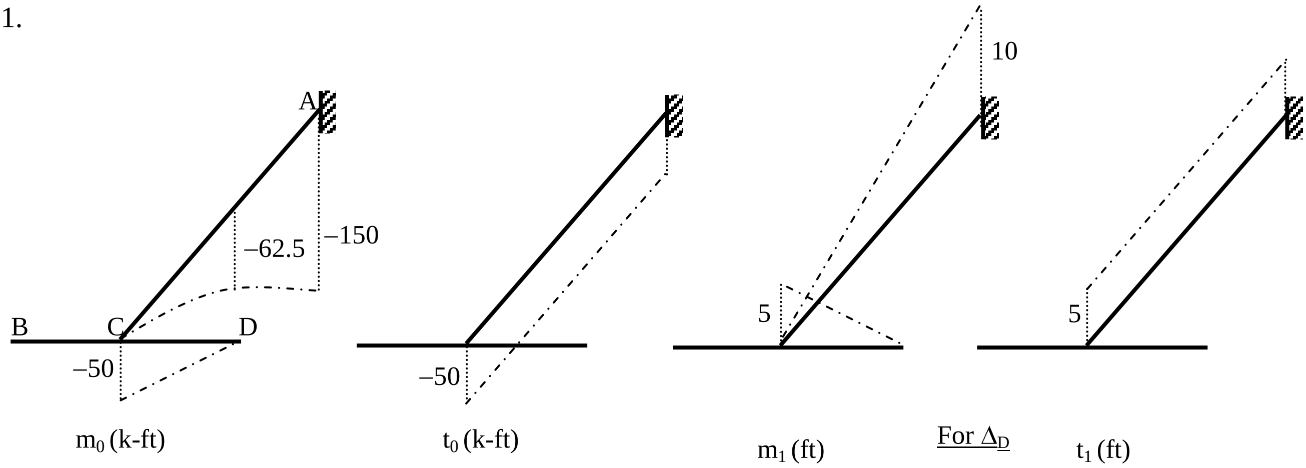


4.



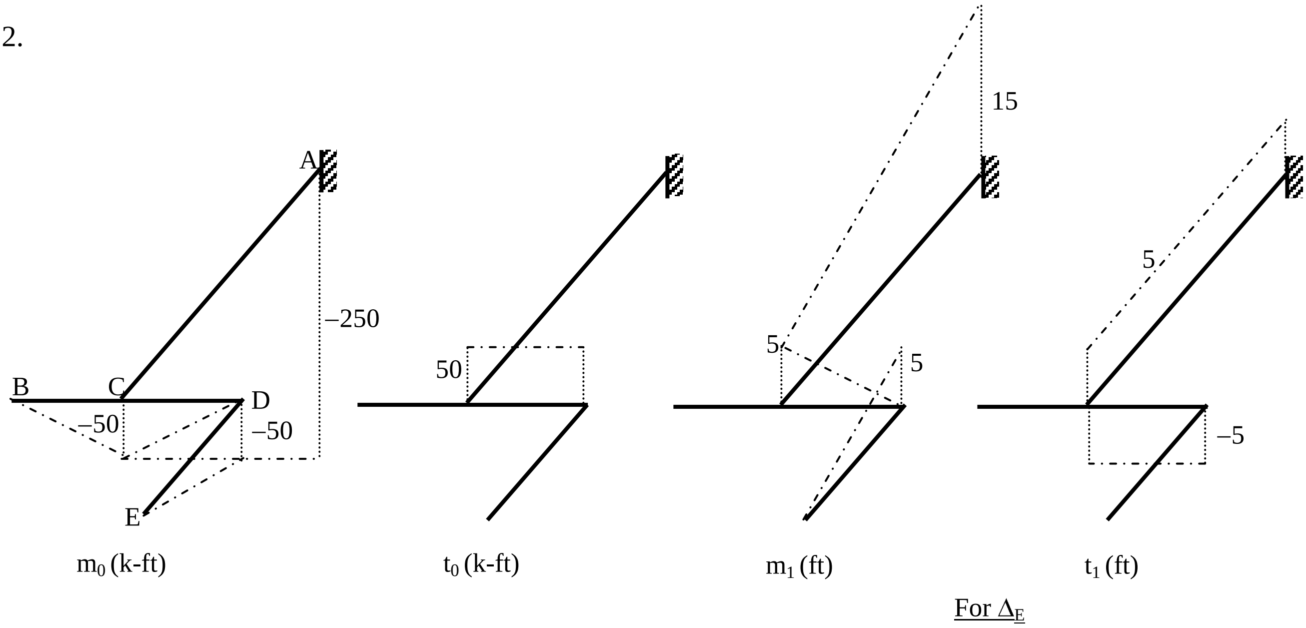
Deflection of Grids using Method of Virtual Work

1.



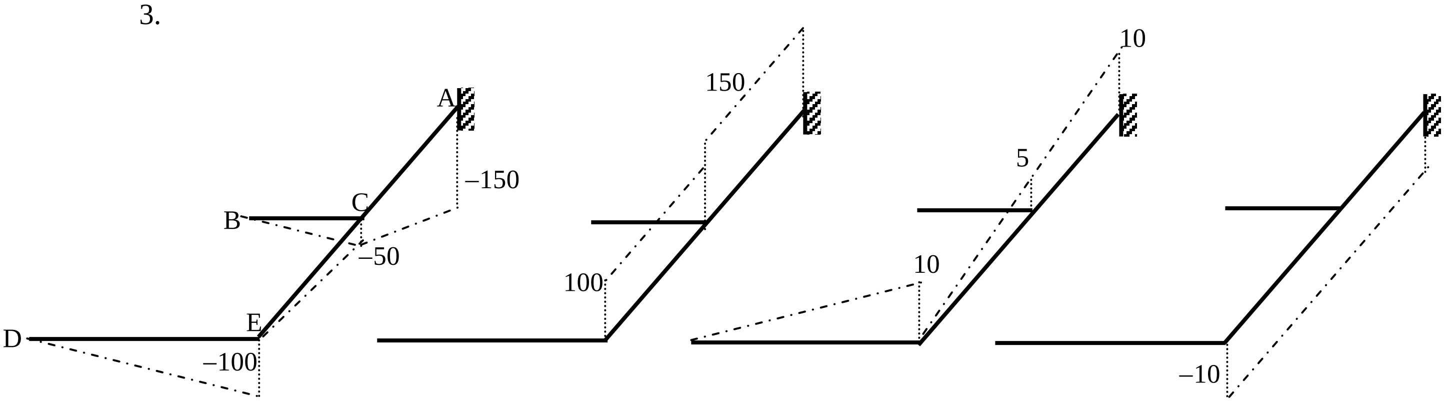
$\Delta_D = [(-50)(5)5/3 + \{2(-62.5) + (-150)\}(10)(10)/6]/(40 \times 10^3) + [(-50)(5)10]/(30 \times 10^3)$
$= -0.125 - 0.0833 = -0.2083$ ft

2.



$\Delta_E = [(-50)(5)5/3 + (-50)(5)5/3 + \{(-50)\ (25) + (-250)\ (35)\}(10)/6]/(40 \times 10^3) + [(50)(-5)5]/(30 \times 10^3)$
$= -0.4375 - 0.0417 = -0.4792$ ft

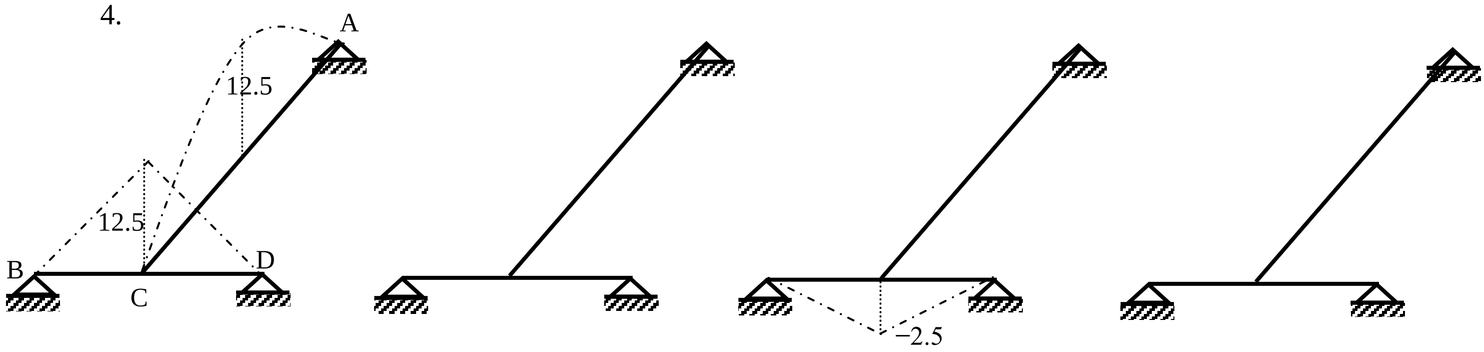
3.



m_0 (k-ft) t_0 (k-ft) m_1 (ft) <u>For Δ_D</u> t_1 (ft)

$$\Delta_D = [(-100)(10)10/3 + (-50)(5)5/3 + \{(-50)\,(20) + (-150)\,(25)\}(5)/6]/(40 \times 10^3) + [(100 + 150)(-10)5]/(30 \times 10^3)$$
$$= -0.1927 - 0.4167 = -0.6094 \text{ ft}$$

4.



m_0 (k-ft) t_0 (k-ft) m_1 (ft) t_1 (ft)

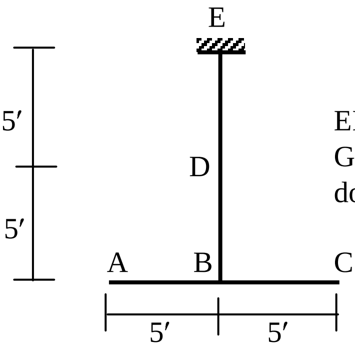
<u>For Δ_C</u>

$$\Delta_D = [(12.5)(-2.5)\ 5/3 \times 2]/(40 \times 10^3)$$
$$= -26.04 \times 10^{-3} \text{ ft}$$

Flexibility Method for Grids (Combined Flexural and Torsional Deformations)

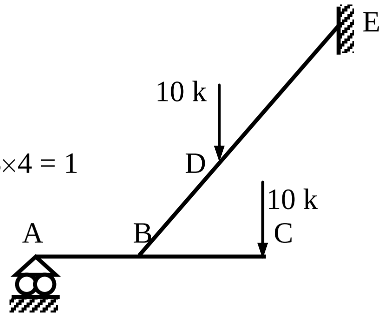
Grids are 2-dimensional (coplanar) structures with one deflection and two rotations at each node.
If the structure is in the x-z plane, the deflection is out-of-plane (along the y axis) while the rotations are about two in-plane axes (x and z axis).
Grids are loaded perpendicular to the structural plane and have three forces per member; i.e., shear force, bending moment and torsion.

Example

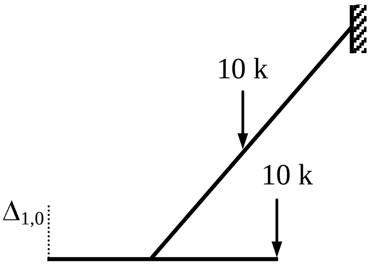


Top View

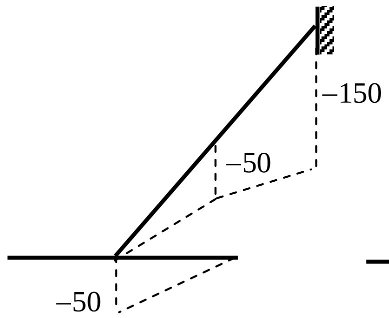
$EI = 40\times10^3$ k-ft^2
$GJ = 30\times10^3$ k-ft^2
dosi = 3×3 + 4 – 3×4 = 1



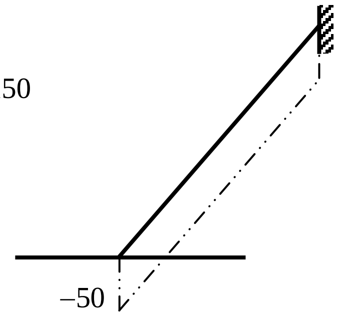
Isometric View



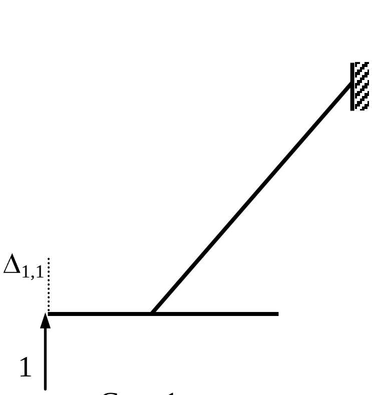
Case 0



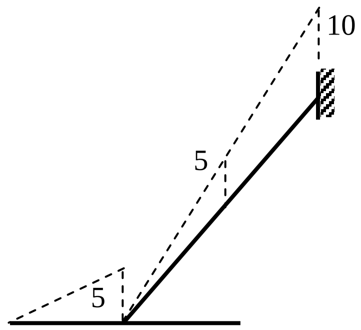
m_0 (k′)



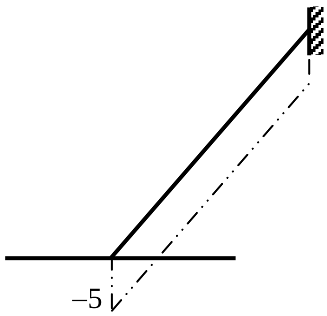
t_0 (k′)



Case 1



m_1 (′)

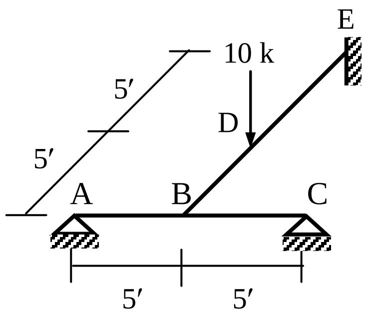


t_1 (′)

$\Delta_{1,0} = \int (m_1 m_0/EI)\,dS + \int (t_1 t_0/GJ)\,dS$
$= \{[5\times5\times(-50)/3] + 5\times[5\times(-250)+10\times(-350)]/6\}/(40\times10^3) + \{10\times(-5)\times(-50)\}/(30\times10^3)$
$= -109.38\times10^{-3} + 83.33\times10^{-3} = -26.05\times10^{-3}$ft
$\Delta_{1,1} = \int (m_1 m_1/EI)\,dS + \int (t_1 t_1/GJ)\,dS$
$=\{5\times(5\times5)/3+10\times(10\times10)/3\}/(40\times10^3) + \{10\times(-5)\times(-5)\}/(30\times10^3)$
$= 9.38\times10^{-3} + 8.33\times10^{-3} = 17.71\times10^{-3}$ ft/k
$\therefore V_A = 26.05\times10^{-3}/17.71\times10^{-3} = 1.47$ k

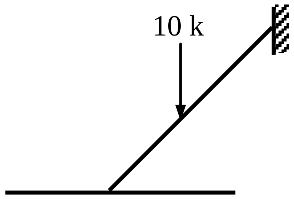
Problems on Flexibility Method for Grids

1.

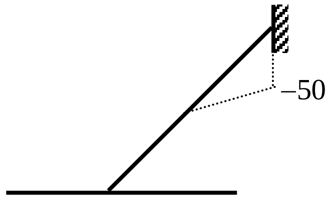


Assume (for all members)
$EI = 40\times10^3$ k-ft^2
$GJ = 30\times10^3$ k-ft^2
dosi = 3×3 + 5 – 3×4 = 2

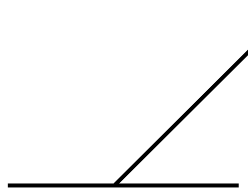
It will be convenient to take Y_A and Y_C as the 2 redundants



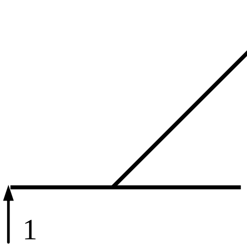
Case 0 (Y_A=0, Y_C=0)



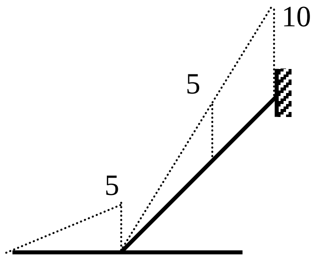
m_0 (k′)



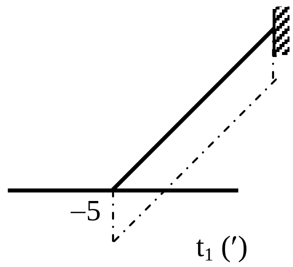
t_0 (k′)



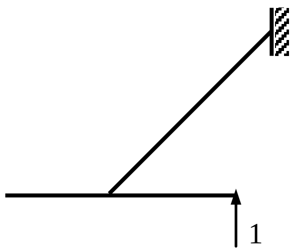
Case 1 (Y_A= 1, Y_C= 0)



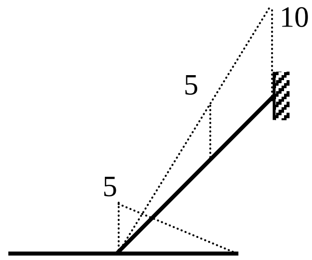
m_1 (′)



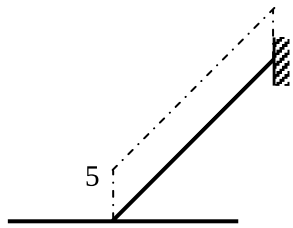
t_1 (′)



Case 2 (Y_A= 0, Y_C= 1)



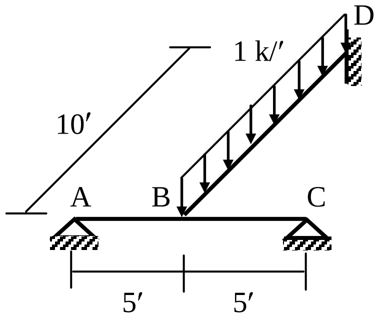
m_2 (′)

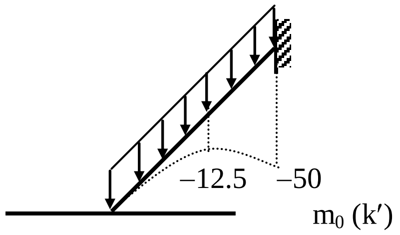


t_2 (′)

$\Delta_{1,0} = \int (m_1\, m_0/EI)\, dS + \int (t_1\, t_0/GJ)\, dS = \{5\times(-50)(20+5)/6\}/(40\times10^3) + 0 = -26.04\times10^{-3}$ ft $= \Delta_{2,0}$
$\Delta_{1,1} = \int (m_1^2/EI)\, dS + \int (t_1^2/GJ)\, dS = \{5\times(5\times5)/3+10\times(10\times10)/3\}/(40\times10^3) + \{10\times(-5)^2\}/(30\times10^3)$
$= 9.38\times10^{-3} + 8.33\times10^{-3} = 17.71\times10^{-3}$ ft/k $= \Delta_{2,2}$
$\Delta_{1,2} = \Delta_{2,1} = \int (m_1\, m_2/EI)\, dS + \int (t_1\, t_2/GJ)\, dS = \{10\times(10\times10)/3\}/(40\times10^3) - 8.33\times10^{-3} = 0$
$\therefore Y_A = Y_C = 26.04\times10^{-3}/17.71\times10^{-3} = 1.47$ k
$\Rightarrow Y_E = 10 - Y_A - Y_C = 7.06$ k

2.





This problem is very similar to Problem 1, the main difference being m_0 shown above.

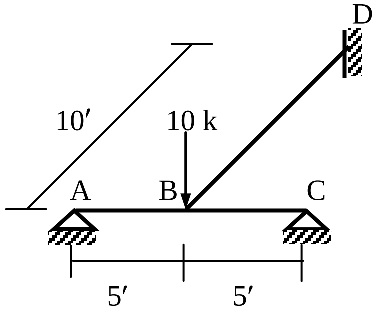
$\Delta_{1,0} = \int (m_1\, m_0/EI)\, dS + \int (t_1\, t_0/GJ)\, dS$

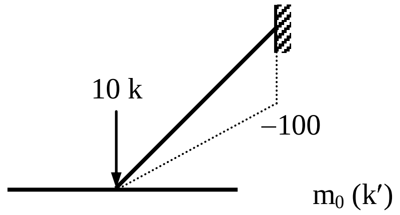
$= \{10\times (0\times0–4\times12.5\times5–50\times10)/6\}/(40\times10^3) +0 = –31.25\times10^{–3}$ ft $= \Delta_{2,0}$

$\Delta_{1,1}$, $\Delta_{1,2}$, $\Delta_{2,1}$, $\Delta_{2,2}$ remaining the same, $Y_A = Y_C = 31.25\times10^{–3}/17.71\times10^{–3} = 1.76$ k

$\Rightarrow Y_D = 10 – Y_A – Y_C = 6.48$ k

3.



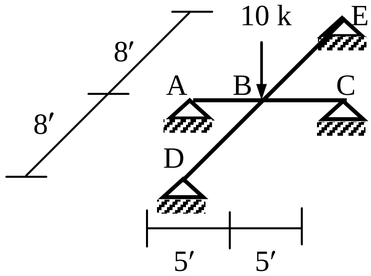


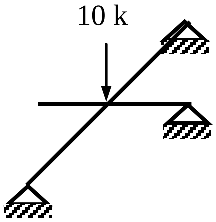
Here, $\Delta_{1,0} = \int (m_1\, m_0/EI)\, dS + \int (t_1\, t_0/GJ)\, dS$

$= \{10\times (–100\times10)/3\}/(40\times10^3) +0 = –83.33\times10^{–3}$ ft $= \Delta_{2,0}$

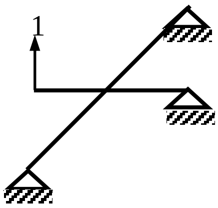
$\Delta_{1,1}$, $\Delta_{1,2}$, $\Delta_{2,1}$, $\Delta_{2,2}$ remaining the same, $Y_A = Y_C = 83.33\times10^{–3}/17.71\times10^{–3} = 4.71$ k $\Rightarrow Y_D = 0.58$ k

4.

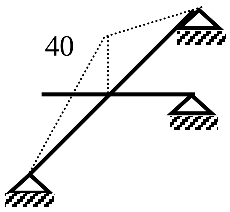




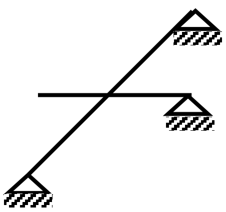
Case 0 (Y_A= 0)



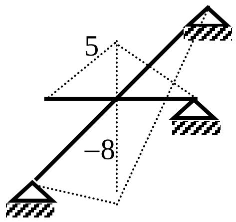
Case 1 (Y_A= 1)



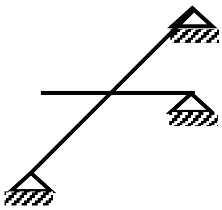
m_0 (k′)



t_0 (k′)



m_1 (k′)



t_1 (k′)

Here, $\Delta_{1,0} = \int(m_1m_0/EI)\, dS+\int(t_1t_0/GJ)dS = 2\times\{8\times40\times(–8)/3\}/(40\times10^3) +0 = – 42.67\times10^{–3}$ ft

$\Delta_{1,1} = \int(m_1^2/EI)\, dS+\int(t_1^2/GJ)dS = 2\times\{5\times(5)^2/3+8\times(–8)^2/3\}/(40\times10^3) +0 = 10.62\times10^{–3}$ ft

$\therefore Y_A = 42.67\times10^{–3}/10.62\times10^{–3} = 4.02$ k, $Y_C = 4.02$ k, $Y_D = 0.98$ k, $Y_E = 0.98$ k