Dynamic Lateral Response of Partially Embedded Foundations in Non-Homogeneous Soils

Asheque al Mahbub[1] and Toyoaki Nogami[2]

Abstract

An approximate method is presented which can be used to formulate dynamic stiffness of embedded foundation. Earlier used for homogeneous soil stratum, the method is further extended to non-homogeneous soil stratum herein. Simple closed form solutions are obtained to calculate the stiffnesses for rigid strip foundations under lateral motion. Solutions require iterations. For the cases analyzed, very few iterations are found to be required for the convergence of the computations. Despite considerable simplifications, the developed formulations of the method produce results that are very close to those computed by rigorous methods.

Keywords: Dynamic stiffness; Strip foundation; Embedment; Lateral motion; Non-homogeneous soil

Introduction

Different simplified models for the analysis of dynamic response of foundations have been proposed and their further improvement have been done by various authors (Beredugo & Novak 1972, Novak et. al. 1978, Nogami et al. 1988, 1991a, 1991b, 1992). In the model presented in this paper, the soil medium is treated as an assembly of a number of cells. Each cell (fundamental cell) is idealized as a system of closely spaced one-dimensional strip with distributed mass and the strips are interconnected by distributed springs along their vertical sides as shown in Fig. 1.

In the lateral mode of foundation vibration, the strips deform laterally by shear. The difference in lateral deformation between the two adjacent strips activates the springs to produce the lateral normal forces acting on the side of the strips. On the other hand, in the vertical mode of foundation vibration, the strips deform axially. The difference in axial deformation between the two adjacent strips activates the springs to produce the vertical shear forces along the strip length.

This method was used for foundations in homogeneous soils (Nogami et al. 2001, Nogami & Chen 2002). The present paper extends the approach further to foundations in non-homogeneous soils.

Formulation

Differential Equations for Fundamental Cell

Soil is assumed to be a visco-elastic medium. Its material properties are defined by complex Lamè's properties (G^* and λ^*, where G^* is the complex shear modulus) and unit mass (ρ). A rectangular cell, as shown in Fig. 1, with the Cartesian coordinates ξ and ζ taken respectively in the horizontal and vertical directions, is considered in the soil medium.

The complex moduli of the soil are

$$\left(G(\zeta),\ \lambda(\zeta)\right)=\left(1+C(\zeta-\zeta_0)\right)\left(G(\zeta_a),\ \lambda(\zeta_a)\right)$$

As the moduli are assumed to vary linearly with depth so,

$$\left(G^*(\zeta),\ \lambda^*(\zeta)\right)=(1+2\gamma i)\left(G(\zeta),\ \lambda(\zeta)\right) \tag{1}$$

where $i=\sqrt{-1}$, $G(\zeta)$ and $\lambda(\zeta)$ = Lamè's constants at ζ; ζ_a and ζ_b = upper and lower ends of the cell in the ζ coordinate, respectively and ξ_a and ξ_b = left and right end of the cell in the ξ coordinate, respectively; γ = material damping parameter; C = constant and

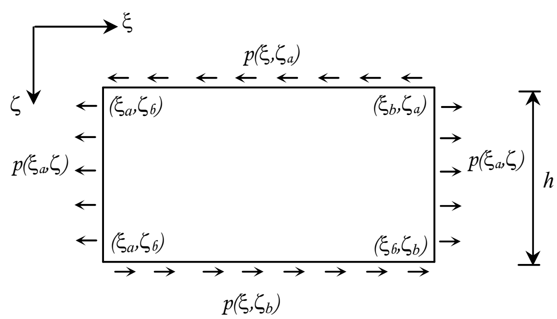
$$\frac{\lambda(\zeta)}{G(\zeta)}=\frac{\lambda(\zeta_a)}{G(\zeta_a)}=\frac{2\nu}{1-2\nu} \tag{2}$$

where ν is the Poisson's ratio.

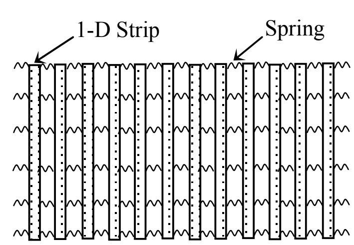
Here, the model is considered for plane strain condition. The displacement is assumed to be limited only in the direction of the load acting on the medium which means horizontal loading only produces horizontal displacements without making any vertical displacement and vice versa. Neglecting the vertical displacement, the equation of horizontal motion of the medium in slightly modified form is written in the frequency domain as

$$\frac{\partial}{\partial x}\left(k_S(z)\frac{\partial u(x,z)}{\partial x}\right)+\frac{\partial}{\partial z}\left(k_C(z)\frac{\partial u(x,z)}{\partial z}\right)+ m_C w^2 u(x,z)=0 \tag{3}$$

where u = displacement amplitude; ω = circular frequency; and denoting m_c = unit mass of soil.



(a) Subjected to horizontal tractions



(b) Composed of 1-D strip and spring

Fig. 1 Fundamental cell

Received 13 February 2004, Accepted 28 November 2004

[1]Lecturer, Dept. of Civil and Environmental Engineering, The University of Asia Pacific, Dhaka
Email: aamahbub@bttb.net.bd

[2]Former Professor, Dept. of Civil Engineering, National University of Singapore, Singapore

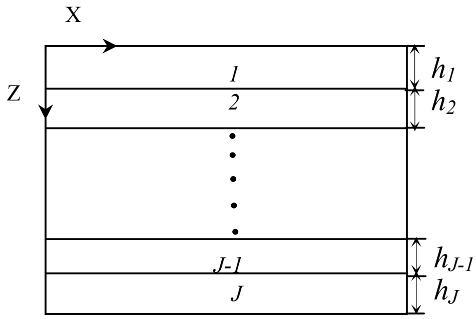


Fig. 2 Secondary cell made of fundamental cells

$$k_c(z) = G^*(z)\bar{k}_c \tag{4a}$$

$$k_s(z) = \left(\lambda^*(z) + 2G^*(z)\right)\bar{k}_s \tag{4b}$$

$$m_c = \rho\,\bar{m}_c \tag{4c}$$

with $\bar{k}_c$, $\bar{k}_s$ and $\bar{m}_c$ are non-dimensional parameters where $\bar{k}_c$ and $\bar{k}_s$ are dependent only on ν (Nogami & Leung 1990) and ρ = density of soil. It is noted that Eq. 3 is also the equation of motion of a strip in the system (as stated earlier) of closely spaced strips that are interconnected by distributed lateral springs along the side (Nogami & Leung 1990). k_c and k_s correspond respectively to the complex column stiffness in shear and the complex spring stiffness, and m_c corresponds to the mass per unit length of the column. These are related uniquely only with the material properties of the original continuous medium through Eqs. 4a~4c.

The displacement is assumed to have the following form

$$u(x,z) = X(x)Z(z) \tag{5}$$

where $X(0) = 1$. Substituting Eq. 5 into Eq. 3 and denoting $\phi(\xi)$ as a weight function, the Galerkin method for weighted residual over ξ in the cell yields

$$\frac{d}{dz}\left\{\left(k_c(z)\int_{x_a}^{x_b} X(x)\phi(x)dx\right)\frac{dZ(z)}{dV}\right\} + \left(k_s(z)\int_{x_a}^{x_b}\frac{d^2X(x)}{dx^2}\phi(x)dx\right)Z(z) + \left(m_c\int_{x_a}^{x_b} X(x)\phi(x)dx\right)Z(z) = 0 \tag{6}$$

Integrating the first term by parts, Eq. 6 results in

$$-\frac{d}{dz}\left(n(z)\frac{dZ(z)}{dz}\right) + \left(k(z) - m\omega^2\right)Z(z) = \phi(x_b)p(x_b,z) - \phi(x_a)p(x_a,z) \tag{7}$$

where ξ_a and ξ_b are left and right ends of the cell in the ξ coordinate, respectively; $p(\xi_{a,b}, \zeta)$ is the traction acting at $(\xi_{a,b}, \zeta)$ expressed as

$$p(x_{a,b},z) = -\left[k_s(z)Z(z)\frac{dX(x)}{dx}\right]_{a,b} \tag{8}$$

and

$$n(z) = k_c(z)\int_{x_a}^{x_b} X(x)\phi(x)dx \tag{9a}$$

$$k(z) = k_s(z)\int_{x_a}^{x_b}\frac{dX(x)}{dx}\frac{d\phi(x)}{dx}dx \tag{9b}$$

$$m = m_c\int_{x_a}^{x_b} X(x)\phi(x)dx \tag{9c}$$

Similarly, substituting Eq. 5 into Eq. 3 and using $\psi(\zeta)$ as a weight function, the Galerkin method for weighted residual over ζ in the cell yields

$$-N(z)\frac{d^2X(x)}{dx^2} + \left(K(z) - M\omega^2\right)X(x) = \psi(z_b)p(x,z_b) - \psi(z_a)p(x,z_a) \tag{10}$$

where $p(\xi, \zeta_{a,b})$ = traction acting at $(\xi, \zeta_{a,b})$ expressed as

$$p(x,z_{a,b}) = -\left[k_C(z)X(x)\frac{dZ(z)}{dz}\right]_{a,b}$$ and

$$N = \int_{z_a}^{z_b} k_s(z)Z(z)\psi(z)dz \tag{11a}$$

$$K = \int_{z_a}^{z_b} k_c(z)\frac{dZ(z)}{dy}\frac{d\psi(z)}{dz}dz \tag{11b}$$

$$M = m_c\int_{z_a}^{z_b} Z(z)\psi(z)dz \tag{11c}$$

Eqs. 7 and 10 are the fundamental differential equations for a cell. The weight functions are selected as $\phi(\xi) = X(\xi)$ and $\psi(\zeta) = Z^*(\zeta)$, in which $Z^*(z)$ is the complex conjugate of $Z(z)$ (Nogami & Chen 2002).

Differential Equations for Secondary Cell

A secondary cell is assumed to contain J non-homogeneous fundamental cells as shown in Fig. 2. The coordinate ξ is assumed to be located at the left end of the secondary cell (i.e. $x = \xi$). The compatibility condition between the j^{th} and $j+1^{th}$ fundamental cells requires

$$X(x)_j = X(x)_{j+1} \tag{12a}$$

$$Z(z_b)_j = Z(z_a)_{j+1} \tag{12b}$$

and the equilibrium condition between the j^{th} and $j+1^{th}$ fundamental cells does

$$p(x,z_b)_j - p(x,z_a)_{j+1} = 0$$ or,

$$\sum_{j=1}^{J}\left\{\psi(z_b)_j p(x,z_b)_j - \psi(z_a)_j p(x,z_a)_j\right\} = \psi(z_b)_J p_b(x,z)_J - \psi(z_a)_1 p_a(x,z)_1 \tag{13}$$

where j = layer number. Thus with Eqs. 12a, 12b and 13, Eqs. 7 and 10 for the fundamental cell lead to the differential equations for the secondary cell as, respectively

$$-\frac{d}{dz}\left(n(z)_j\frac{dZ(z)_j}{dz}\right) + \left(k(z)_j - m\omega^2\right)Z(z)_j = \phi(x_b)p(x_b,z)_j - \phi(x_a)p(x_a,z)_j \qquad j = 1\sim J \tag{14a}$$

$$-\sum_{j=1}^{J} N_j \frac{d^2X(x)}{dx^2} + \sum_{j=1}^{J}\left(K_j - M_j\omega^2\right)X(x) =$$

$$\psi(\zeta_b)_J\, p(x,\zeta_b)_J - \psi(\zeta_a)_1\, p(x,\zeta_a)_1 \qquad (14b)$$

Boundary Value Problem

A partially embedded rigid foundation is considered in the non-homogeneous soil as shown in Fig. 3. Only the shaded area in the figure is considered for formulation. Soil medium around the foundation is divided into three secondary cells (Cells I, II and III) as shown in Fig. 3. The foundation is assumed to undergo the lateral translational motion of amplitude *U*.

Functions X(x) and Z(ζ)

The boundary conditions for each secondary cell are

$$\text{Cell I} \begin{cases} X(x) = 1 \\ Z(\zeta_a)_1 = U \quad \text{and} \quad Z(\zeta_b)_{J_B} = 0 \end{cases} \qquad (15a)$$

$$\text{Cell II} \begin{cases} X(0) = 1 \ \text{and} \ X(\infty) = 0 \\ Z(\zeta_a)_1 = U \quad \text{and} \quad Z(\zeta_b)_{J_B} = 0 \end{cases} \qquad (15b)$$

$$\text{Cell III} \begin{cases} X(0) = 1 \quad \text{and} \quad X(\infty) = 0 \\ Z(\zeta)_j = U \qquad j = 1 \sim J_A \end{cases} \qquad (15c)$$

where J_A = number of fundamental cells in Cell III; and J_B = number of fundamental cells in Cells I and II. In addition, $p(0, \zeta)_j$ for $j = 1 \sim J_B$ in Cell I and $p(x, \zeta_a)$ in Cell III are zero. Adding Eq. 14a for Cell I and Eq. 14a for Cell II together and applying the boundary conditions at the interface between Cells I and II lead to

$$-\frac{d}{d\zeta}\left(n'(\zeta)_j \frac{dZ(\zeta)_j}{d\zeta}\right) + \left(k'(\zeta)_j - m'\omega^2\right)Z(\zeta)_j = 0$$

$$j = 1 \sim J_B \qquad (16a)$$

and Eq. 14b for Cells II and III lead similarly to

$$-\sum_{j=1}^{J_C} N_j \frac{d^2X(x)}{dx^2} + \sum_{j=1}^{J_C}\left(K_j - M_j\omega^2\right)X(x) = 0 \qquad (16b)$$

where $n' = n^I + n^{II}$; $k' = k^I + k^{II}$; $m' = m^I + m^{II}$; and $J_C = J_A + J_B$. With the boundary conditions associated with $X(x)$ for Cells II and III, the solution of Eq. 16b is expressed as

$$X(x)^{II,III} = e^{-bx} \qquad (17a)$$

using the polynomial form, the general solution (Mahbub 2004) of Eq. 16a is

$$Z(\zeta)_j^{I,II} = \left(1 + \sum_{n=2,3,4,\cdots} a_{nj}^{a}\zeta^n\right)a_j + \left(\zeta + \sum_{n=2,3,4,\cdots} a_{nj}^{b}\zeta^n\right)b_j \qquad (17b)$$

where a_j and b_j = unknown constants to be defined later; and

$$b = \sqrt{\left(\sum_{j=1}^{J_C} K_j - \omega^2 M_j\right) \Big/ \left(\sum_{j=1}^{J_C} N_j\right)} \qquad (18a)$$

$$a_{nj}^{a} = \frac{C_j a_{n-3,j}^{a} + D_j a_{n-2,j}^{a} - (n-1)^n A_j a_{n-1,j}^{a}}{n(n-1)B_j}$$

($C_j = 0$ for n = 2~4 and $A_j = 0$ for $n = 2$) (18b)

$$a_{nj}^{b} = \frac{C_j a_{n-3j}^{b} + D_j a_{n-2j}^{b} - (n-1)^n A_j a_{n-1,j}^{b}}{n(n-1)B_j}$$

($C_j = 0$ for $n = 2$ & 3 and $D_j = 0$ for $n = 2$) (18c)

$$A_j = \left\{n'(\zeta_b)_j - n'(\zeta_a)_j\right\}/h_j \qquad (18d)$$

$$B_j = n'(\zeta_a)_j \qquad (18e)$$

$$C_j = \left\{k'(\zeta_b)_j - k'(\zeta_a)_j\right\}/h_j \qquad (18f)$$

$$D_j = k'(\zeta_a)_j - \omega^2 m'_j \qquad (18g)$$

where h_j = thickness of the j^{th} layer. Imposing the compatibility between the soil and foundation to Eqs. 14a for Cell III and to $X(x)$ for Cell I, the rest of the functions for the secondary cells are defined as respectively

$$\left(k(\zeta)_j - m\omega^2\right)Z(\zeta)_j^{III} = -p(0,\zeta)_j^{III} \quad j = 1 \sim J_A \qquad (19a)$$

$$X(x)^I = 1 \qquad (19b)$$

Constants a_j and b_j

The conditions for $Z(\zeta)_j$ in Cells I and II can be stated as

$$Z(\zeta_a)_1 = U \qquad (20a)$$

$$\begin{Bmatrix} Z(\zeta_b) \\ \dot{Z}(\zeta_b) \end{Bmatrix}_j = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n(\zeta_a)_{j+1}}{n(\zeta_b)_j} \end{bmatrix} \begin{Bmatrix} Z(\zeta_a) \\ \dot{Z}(\zeta_a) \end{Bmatrix}_{j+1} \qquad (20b)$$

$$Z(\zeta_b)_{J_B} = 0 \qquad (20c)$$

with $(\zeta_a, \zeta_b)_j = (0, h_j)$ and $(a, b)_j = a_{JB}(a', b')_j$, substituting Eq. 17b into Eqs. 20b and 20c result in, respectively

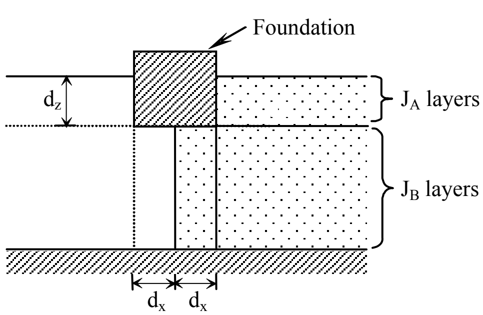


Fig. 3 Soil medium divided into three secondary cells

$$\begin{Bmatrix} a' \\ b' \end{Bmatrix}_j = \frac{1}{D_j}\begin{bmatrix} \left(1+\sum_{n=2} n a_n^b h^{n-1}\right) & -\left(h+\sum_{n=2} a_n^b h^n\right) \\ -\left(\sum_{n=2} n a_n^a h^{n-1}\right) & \left(1+\sum_{n=2} a_n^a h^n\right)\end{bmatrix}_j \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n(0)_{j+1}}{n(h)_j}\end{bmatrix}\begin{Bmatrix} a' \\ b'\end{Bmatrix}_{j+1} \qquad (21a)$$

$$\begin{Bmatrix} a' \\ b' \end{Bmatrix}_{J_B} = \begin{Bmatrix} 1 \\ -\left(1+\sum_{n=2} a_n^a h^n\right) \Big/ \ h\left(1+\sum_{n=2} a_n^b h^{n-1}\right) \end{Bmatrix}_{J_B} \qquad (21b)$$

where, $$D_j = \left(1+\sum_{n=2} a_n^a h^n\right)_j \left(1+\sum_{n=2} n a_n^b h^n\right)_j - \left(h+\sum_{n=2} a_n^b h^n\right)_j \left(\sum_{n=2} n a_n^a h^{n-1}\right)_j \qquad (22)$$

Therefore, starting with $(a', b')_{JB}$ given by Eq. 21b, $(a', b')_j$ can be computed from $j = J_B - 1$ through 1 successively by Eq. 21a. After $(a', b')_1$ is computed, a_{JB} is obtained to satisfy Eq. 20a at the top (j = 1) in Cells I and II. Then, $(a', b')_j$ is computed from $(a, b)_j = a_{JB}(a', b')_j$.

Dynamic Stiffness for Partially Embedded Foundation

When the force P is applied to the foundation, the equilibrium condition at the foundation is stated as

$$P = -2\left\{ \left(n(z)_I \frac{\partial u(d_x,z)_I}{\partial z}\right)^{I}\Bigg|_{z=0} + \left(n(z)_I \frac{\partial u(0,z)_I}{\partial z}\right)^{II}\Bigg|_{z=0} + \left(\sum_{j=1}^{J_A}\int_0^{h_j} p(0,z)_j\,dz\right)^{III}\right\}$$

$$= -2\left\{ \left(n(z)_I \frac{dZ(z)_I}{dz}\right)^{I}\Bigg|_{z=0} + \left(n(z)_I \frac{dZ(z)_I}{dz}\right)^{II}\Bigg|_{z=0} + \left(\sum_{j=1}^{J_A}\int_0^{h_j} \left(k(z)_j - \omega^2 m\right) Z(z)_j\,dz\right)^{III}\right\} \qquad (23)$$

Evaluating the above expression with Eqs. 9a~9c and U = 1, the dynamic soil stiffness for this foundation (K_f) is

$$K_f = -2\left[n'(z_a)_I\right]^{I,II} - 2\left[\sum_{j=1}^{J_B}\left\{k(z_a)_j + k(z_b)_j\right\}h_j - \omega^2 m h_j\right]^{III}$$

$$= -2\left[k_c(0)_I\left(d_x + \frac{1}{\beta}\right)\right]^{I,II} - \left[\sum_{j=1}^{J_A}\beta\left\{k_s(0)_j + k_s(h)_j\right\}h_j - \omega^2\frac{m_c}{\beta}h_j\right]^{III}$$

where d_x = half-width of the foundation.

In non-dimensional form,

$$\bar{K}_f = -2\left[\bar{k}_c(0)_I\left(1 + \frac{1}{\bar{\beta}}\right)\right]^{I,II} - \left[\sum_{j=1}^{J_A}\bar{\beta}\left\{\bar{k}_s(0)_j + \bar{k}_s(h)_j\right\}\bar{h}_j - a_0^2\frac{\bar{h}_j}{\bar{\beta}}\right]^{III} \qquad (24)$$

where,

$\bar{K}_f = \dfrac{K_f}{G(0)}$, $\bar{k}_{c,s}(0) = \dfrac{k_{c,s}(0)}{G(0)}.d_x$,

$\bar{k}_s(h) = \dfrac{k_s(h)}{G(0)}.d_x$, $\bar{\beta} = \beta.d_x$, $\bar{h}_j = \dfrac{h_j}{d_x}$, and

non-dimensional frequency, $a_0 = \dfrac{a\omega}{v_s}$ with shear wave velocity of soil $v_s = \sqrt{\dfrac{G(0)}{m_c}}$

Computational Procedure

1. For any frequency ω, a value for the complex parameter β (e.g., [0.1, 0.1]) is first assumed.
2. Using β, the parameters n, k, m are calculated using the expressions shown in Eqs. 9a~9c, Eq. 17a.
3. The series of parameters of a_{nj}^a and a_{nj}^b for each layer of soil are calculated using the Eqs. 18b~18g.
4. The parameters N, K, M are calculated using the expressions shown in Eqs. 11a~11c, Eqs. 21a~21c and Eq. 17b for each layer of soil. Each parameter is then summed up respectively for all the layers of soil
5. New β using Eq. 18a is computed.
6. Newly computed β is compared with the previous assumed β, if the difference is beyond the tolerance (a tolerance of 1% is found sufficient) then steps 2 to 7 are continued until the difference is within the tolerance.
7. Once the β is within the tolerance then it is normalized by multiplying with half-width of the foundation and normalized stiffness are calculated using Eq. 24.

Computed Results

The dynamic soil stiffnesses for rigid strip foundations are computed by the expression given by Eq. 24. All parameters in the expression except β, are provided as inputs. The parameter β to define $X(x)$ and the constants $(a, b)_j$ to define $Z(\zeta)_j$ are mutually coupled in the formulation. Thus they are calculated iteratively in the process mentioned above. In the computations carried out below, the convergence in the iterations was achieved generally within 8 iterations for tolerance of 1%.

A foundation of d_x = 4m is assumed to rest on the surface of soil underlain by a rigid base at depth $2d_x$. The conditions considered for the soil are: $v_s(z) = v_s(0)(1+1.5z/d_x)$ with v_s = shear wave velocity of soil (or $G(z) = G(0)(1+1.5(z/d_x)^2)$; $\nu = 1/3$; and $\gamma = 0.08$. The soil is divided equally into 8 or 24 homogeneous layers as shown in Fig. 4. It is also divided equally into eight non-homogeneous layers in which *G(z)* varies linearly with *z* within a layer. *G(z)* in the latter distribution is nearly equal to the original *G(z)*. Soil stiffnesses computed for these three cases are shown in Fig. 5. It is seen in the figure that even dividing soil stratum in significantly very high number (24 layers) of homogeneous layers is not sufficient to get stiffnesses closed to that divided in non-homogenous layers (8 layers). The accuracy of the method for non-homogenous layers is verified in Fig. 6 in which $\nu = 0.25$; and $\gamma = 0.05$. It is found that the stiffnesses (real) computed using this simplified method shows good proximity with those computed by accurate and elaborate method (Gazetas 1980).

Then, in the same non-homogeneous soil profile, stiffnesses for the foundation without or with embedment (d_z = 0 or $0.5d_x$) is computed and plotted in Fig. 7. It is clear from the figures that not only the magnitude of the soil stiffnesses is affected by the foundation embedment but also the way of its frequency dependency is also affected.

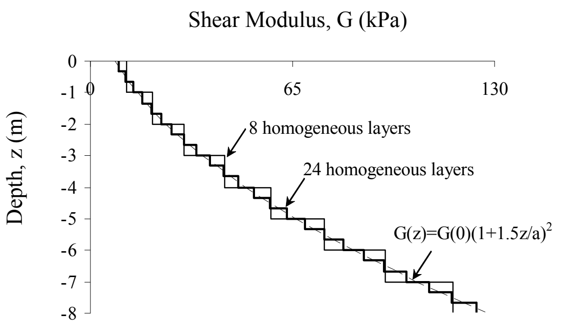


Fig. 4 Distribution of shear modulus with depth replaced by 8 and 24 homogeneous layers

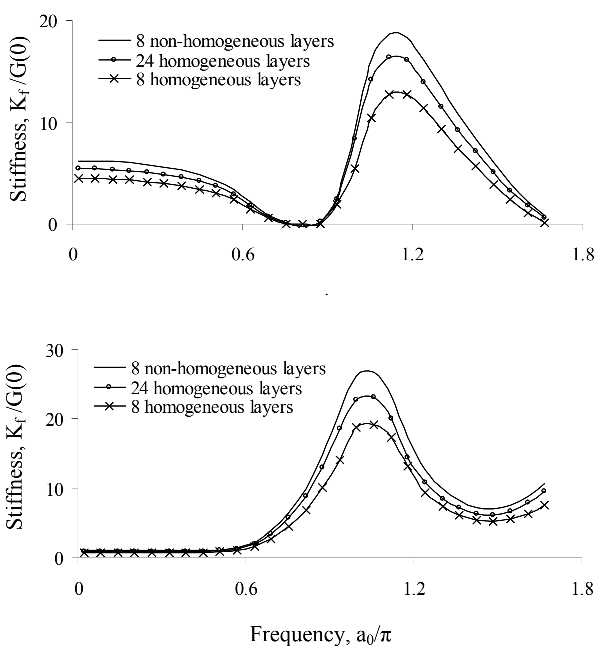


Fig. 5 Dynamic soil stiffnesses computed for three cases

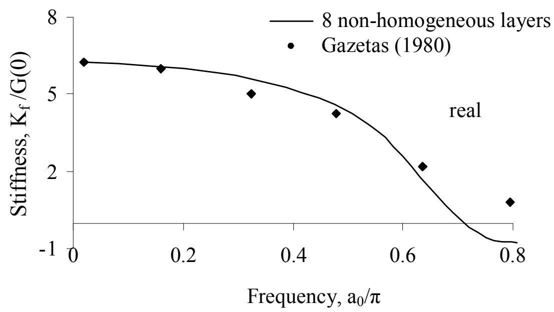


Fig. 6 Accuracy of simplified method

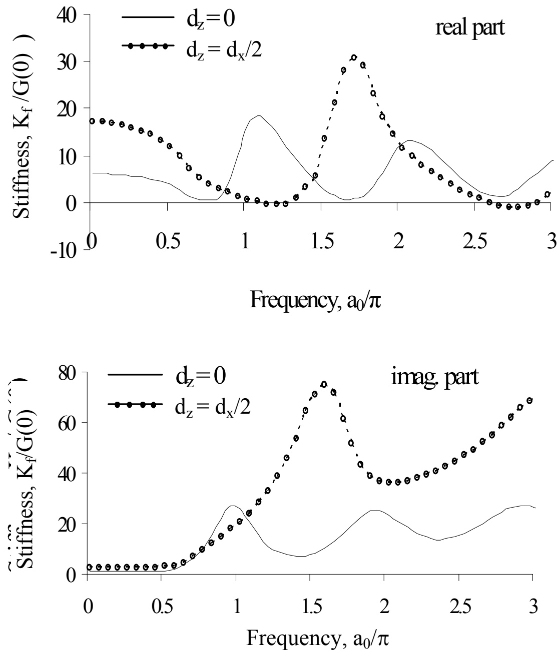


Fig. 7 Effect of embedment on dynamic stiffness

Conclusions

The approximate method enables us to formulate the soil stiffness under lateral loads for partially embedded strip foundations in non-homogeneous soils in a simple closed form. It requires iterations in computation. Sufficient convergence is generally observed within a very small number of iterations for the cases computed herein. The developed formulation can calculate the dynamic stiffness very close to that computed by far more rigorous methods.

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