Joint Displacements and Forces

1. Coordinate Systems

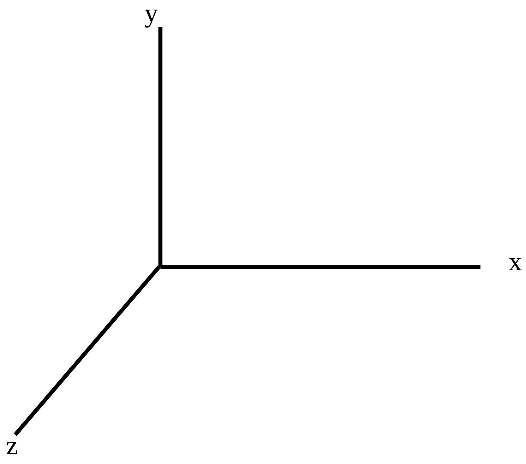


Fig. 1: Coordinate System1

(widely used and also applied in this course)

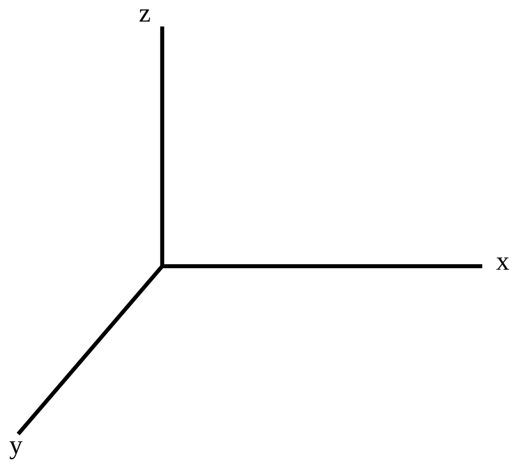


Fig. 2: Coordinate System2

(used in some formulations)

2. Sign Convention for Joint Displacements and Forces

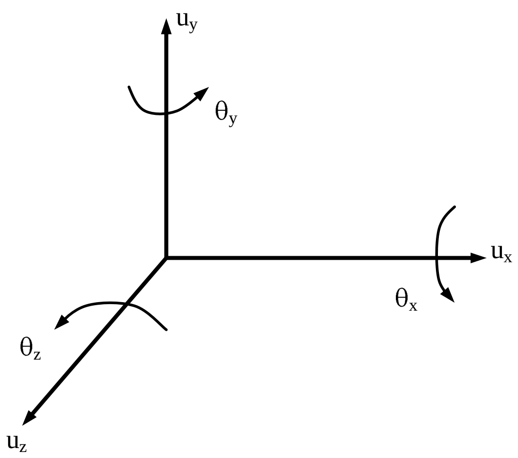


Fig. 3: Sign convention for Displacements

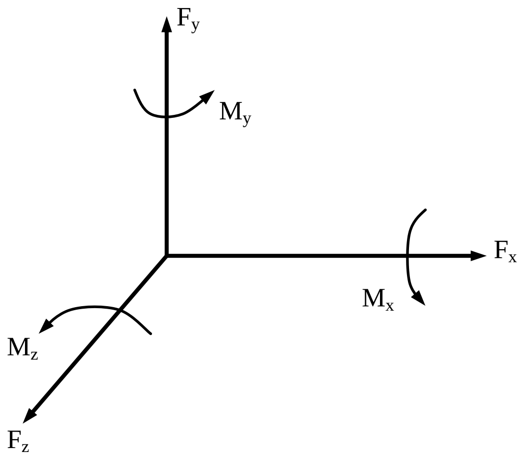


Fig. 4: Sign convention for Forces

3. Sign Convention for Two-dimensional Problems

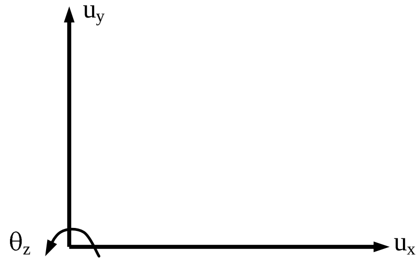


Fig. 5: Two-Dimensional Displacements

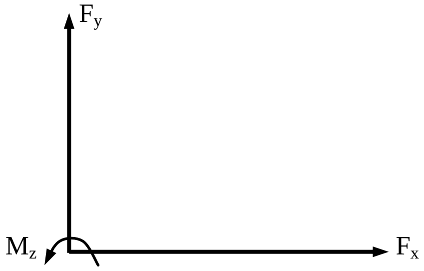
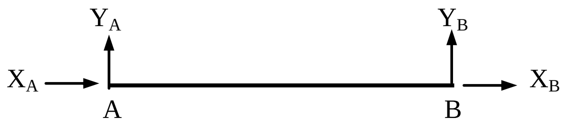


Fig. 6: Two-Dimensional Forces

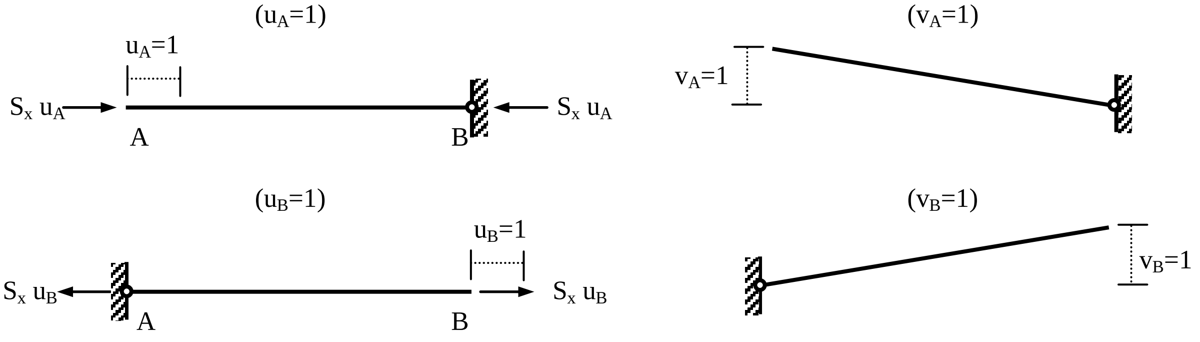
Stiffness Matrix for Truss Members in the Local Axes System

Consider a truss member AB subjected to forces (X_A, Y_A) and (X_B, Y_B) at joints A and B.



Assume that the length of the member is L, its modulus of elasticity is E and cross-sectional area A.
$\therefore$ The axial stiffness of the member, S_x = Load to produce unit deflection = EA/L
Also assume that the member has no flexural or shear stiffness.

If the displacements of joints A and B are (u_A, v_A) and (u_B, v_B), the effect of the external forces may result in the following cases.



Equilibrium equations:

$$\sum F_{x(A)} = 0 \Rightarrow X_A = S_x\, u_A + 0 - S_x\, u_B + 0 \qquad \text{......(1)}$$

$$\sum F_{y(A)} = 0 \Rightarrow Y_A = 0 + 0 + 0 + 0 \qquad \text{......(2)}$$

$$\sum F_{x(B)} = 0 \Rightarrow X_B = -S_x\, u_A + 0 + S_x\, u_B + 0 \qquad \text{......(3)}$$

$$\sum F_{y(B)} = 0 \Rightarrow Y_B = 0 + 0 + 0 + 0 \qquad \text{......(4)}$$

Eqs. (1)~(4) can be summarized in matrix form as

$$\begin{pmatrix} S_x & 0 & -S_x & 0 \\ 0 & 0 & 0 & 0 \\ -S_x & 0 & S_x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} u_A \\ v_A \\ u_B \\ v_B \end{Bmatrix} = \begin{Bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \end{Bmatrix}$$

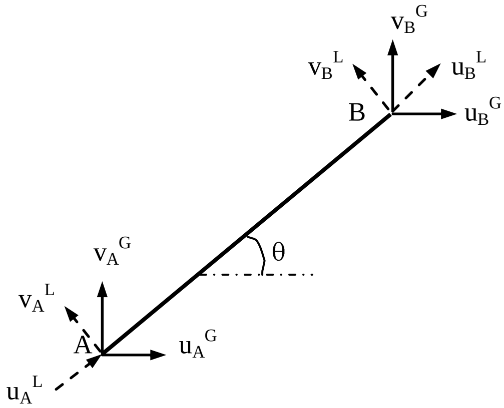
$$\Rightarrow \mathbf{K_m}^{\mathbf{L}}\, \mathbf{u_m}^{\mathbf{L}} = \mathbf{p_m}^{\mathbf{L}} \qquad \text{......(5)}$$

where $\mathbf{K_m}^{\mathbf{L}}$ = The stiffness matrix of member AB in the local axis system,
$\mathbf{u_m}^{\mathbf{L}}$ = The displacement vector of the member in the local axis system, and
$\mathbf{p_m}^{\mathbf{L}}$ = The force vector of the member in the local axis system

Transformation of Stiffness Matrix from Local to Global Axes

The member matrices formed in the local axes system can be transformed into the global axes system by considering the angles they make with the horizontal.

The local vectors and global vectors are related by the following equations.



Local and global joint displacements of a truss member

$$u_A^L = u_A^G \cos\theta + v_A^G \sin\theta \quad \text{.....................(6)}$$

$$v_A^L = -u_A^G \sin\theta + v_A^G \cos\theta \quad \text{.....................(7)}$$

$$u_B^L = u_B^G \cos\theta + v_B^G \sin\theta \quad \text{.....................(8)}$$

$$v_B^L = -u_B^G \sin\theta + v_B^G \cos\theta \quad \text{.....................(9)}$$

In matrix form

$$\begin{Bmatrix} u_A^L \\ v_A^L \\ u_B^L \\ v_B^L \end{Bmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{Bmatrix} u_A^G \\ v_A^G \\ u_B^G \\ v_B^G \end{Bmatrix}$$

$$\Rightarrow \mathbf{u_m^L} = \mathbf{T_m}\, \mathbf{u_m^G} \quad \text{.....................(10)}$$

where $\mathbf{T_m}$ is called the transformation matrix for member AB, which connects the displacement vector $\mathbf{u_m^L}$ in the local axes of AB with the displacement vector $\mathbf{u_m^G}$ in the global axes.

A similar expression can be obtained for the force vectors $\mathbf{p_m^L}$ and $\mathbf{p_m^G}$; i.e.,

$$\Rightarrow \mathbf{p_m^L} = \mathbf{T_m}\, \mathbf{p_m^G} \quad \text{.....................(11)}$$

$\therefore$ Eq. (5) can be rewritten as $\Rightarrow \mathbf{K_m^L}\, \mathbf{T_m}\, \mathbf{u_m^G} = \mathbf{T_m}\, \mathbf{p_m^G}$ ....(12)

$$\Rightarrow (\mathbf{T_m^{-1}}\, \mathbf{K_m^L}\, \mathbf{T_m})\, \mathbf{u_m^G} = \mathbf{p_m^G}$$

$$\Rightarrow (\mathbf{T_m^{T}}\, \mathbf{K_m^L}\, \mathbf{T_m})\, \mathbf{u_m^G} = \mathbf{p_m^G} \quad \text{.....................(13)}$$

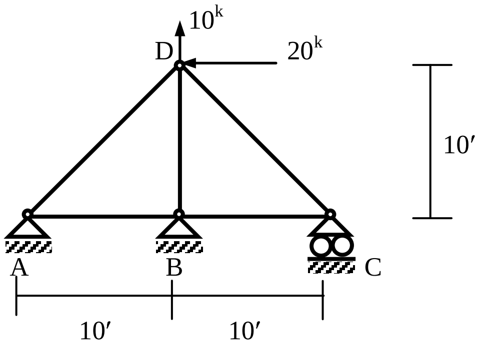
where $\mathbf{T_m^T}$ is the transpose of the transformation matrix $\mathbf{T_m}$, which is also = $\mathbf{T_m^{-1}}$

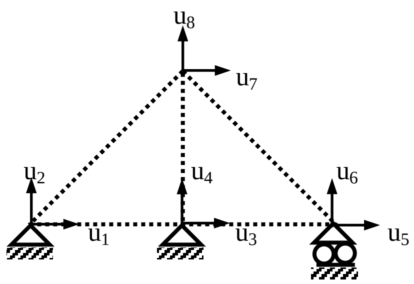
If $(\mathbf{T_m^T}\, \mathbf{K_m^L}\, \mathbf{T_m})$ is written as $\mathbf{K_m^G}$, the member stiffness matrix in the global axis system, then

$$\mathbf{K_m^G} = S_x \begin{pmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{pmatrix} \quad [\text{where } C = \cos\theta,\ S = \sin\theta]$$

Assembly of Stiffness Matrix and Load Vector of a Truss

Assemble the global stiffness matrix and write the global load vector of the truss shown below. Also write the boundary conditions [EA/L = Constant = 500 kip/ft].





Member AB: (C = 1, S = 0)

$$\mathbf{K_{AB}}^{\mathbf{G}} = 500\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Member BC: (C = 1, S = 0)

$$\mathbf{K_{BC}}^{\mathbf{G}} = 500\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Member BD: (C = 0, S = 1)

$$\mathbf{K_{BD}}^{\mathbf{G}} = 500\begin{array}{c} \begin{array}{cccc} 3 & 4 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 7 \\ 8 \end{array}$$

Member AD: (C = 1/√2, S = 1/√2)

$$\mathbf{K_{AD}}^{\mathbf{G}} = 500\begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \end{array}$$

Member CD: (C = –1/√2, S = 1/√2)

$$\mathbf{K_{CD}}^{\mathbf{G}} = 500\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

$$\mathbf{K}^{\mathbf{G}} = 500\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1+0.5 & 0+0.5 & -1 & 0 & & & -0.5 & -0.5 \\ 0+0.5 & 0+0.5 & 0 & 0 & & & -0.5 & -0.5 \\ -1 & 0 & 1+1+0 & 0+0+0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0+0+0 & 0+0+1 & 0 & 0 & 0 & -1 \\ & & -1 & 0 & 1+0.5 & 0-0.5 & -0.5 & 0.5 \\ & & 0 & 0 & 0-0.5 & 0+0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & 0+0.5+0.5 & 0+0.5-0.5 \\ -0.5 & -0.5 & 0 & -1 & 0.5 & -0.5 & 0+0.5-0.5 & 1+0.5+0.5 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \qquad \mathbf{p}^{\mathbf{G}} = \begin{Bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \\ 0 \\ Y_C \\ -20 \\ 10 \end{Bmatrix}$$

Boundary Conditions: $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$, $u_6 = 0$

Boundary Conditions, Support Reactions and Member Forces

After assembly of the member stiffness matrices, the equilibrium equations were

$$500\begin{pmatrix} 1.5 & 0.5 & -1 & 0 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & -0.5 & -0.5 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1.5 & -0.5 & -0.5 & 0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & 1 & 0 \\ -0.5 & -0.5 & 0 & -1 & 0.5 & -0.5 & 0 & 2 \end{pmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \\ 0 \\ Y_C \\ -20 \\ 10 \end{Bmatrix}$$

Applying the boundary conditions ($u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$, $u_6 = 0$), the equations are modified to

$$500\begin{pmatrix} 1.5 & -0.5 & 0.5 \\ -0.5 & 1 & 0 \\ 0.5 & 0 & 2 \end{pmatrix}\begin{Bmatrix} u_5 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -20 \\ 10 \end{Bmatrix} \Rightarrow \begin{cases} u_5 = -22.22 \times 10^{-3} \text{ ft} \\ u_7 = -51.11 \times 10^{-3} \text{ ft} \\ u_8 = 15.56 \times 10^{-3} \text{ ft} \end{cases}$$

Once displacements are known, support reactions can be calculated from equilibrium equations; i.e.,

$X_A = 750\,u_1 + 250\,u_2 - 500\,u_3 + 0\,u_4 + 0\,u_5 + 0\,u_6 - 250\,u_7 - 250\,u_8 = 0 + 0 + 0 + 0 + 0 + 0 + 12.78 - 3.89 = 8.89^k$

Similarly, $Y_A = 12.78 - 3.89 = 8.89^k$, $X_B = 11.11^k$, $Y_B = -7.78^k$, $Y_C = 5.56 - 12.78 - 3.89 = -11.11^k$

The bar forces can be calculated from the equation $P_{AB} = (EA/L)\,\{(u_B^G - u_A^G)\cos\theta + (v_B^G - v_A^G)\sin\theta\}$

$\therefore P_{AB} = 500\{(u_3 - u_1)\cos 0° + (u_4 - u_2)\sin 0°\} = 0$, $P_{BC} = 500\,\{(u_5 - u_3)\cos 0° + (u_6 - u_4)\sin 0°\} = -11.11^k$,

$P_{BD} = 500\{(u_7 - u_3)\cos 90° + (u_8 - u_4)\sin 90°\} = 7.78^k$, $P_{AD} = 500\{(u_7 - u_1)\cos 45° + (u_8 - u_2)\sin 45°\} = -12.57^k$, $P_{CD} = 500\,\{(u_7 - u_5)\cos 135° + (u_8 - u_6)\sin 135°\} = 15.71^k$

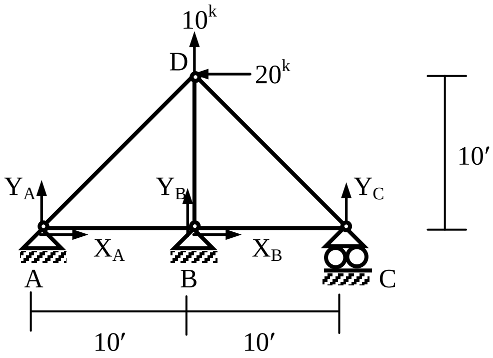
In addition to the externally applied forces if the support C settles 0.10′, then $u_6 = -0.10'$ is known

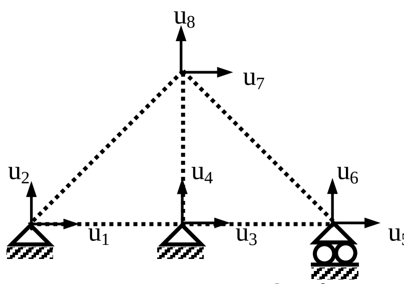
$\therefore$ Applying boundary conditions ($u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$, $u_6 = -0.10'$), the equations become

$$500\begin{pmatrix} 1.5 & -0.5 & 0.5 \\ -0.5 & 1 & 0 \\ 0.5 & 0 & 2 \end{pmatrix}\begin{Bmatrix} u_5 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0+250\,u_6 \\ -20-250\,u_6 \\ 10+250\,u_6 \end{Bmatrix} = \begin{Bmatrix} -25 \\ 5 \\ -15 \end{Bmatrix} \Rightarrow \begin{cases} u_5 = -33.33 \times 10^{-3} \text{ ft} \\ u_7 = -6.67 \times 10^{-3} \text{ ft} \\ u_8 = -6.67 \times 10^{-3} \text{ ft} \end{cases}$$

$\therefore P_{AB} = 0$, $P_{BC} = -16.67^k$, $P_{BD} = -3.33^k$, $P_{AD} = -4.71^k$, $P_{CD} = 23.57^k$

Stiffness Formulation using Equilibrium of Joints





EA/L = Constant = 500 kip/ft

For the truss ABCD, the equilibrium equations of joints A, B, C and D take the following forms when the equation for member force [i.e., $P_{AB} = (EA/L)\ \{(u_B^G - u_A^G)\cos\theta + (v_B^G - v_A^G)\sin\theta\}$] is applied

$\sum F_{xA}=0 \Rightarrow X_A+P_{AB}+P_{AD}\cos 45° = 0$

$\Rightarrow X_A + 500(u_3-u_1) + 500\{(u_7-u_1)\cos 45° + (u_8-u_2)\sin 45°\}\cos 45° = 0$

$\Rightarrow 500\ \{(1.0 + 0.5)\ u_1 + 0.5\ u_2 - 1.0\ u_3 - 0.5\ u_7 - 0.5\ u_8\} = X_A$ …………………(1)

$\sum F_{yA} = 0 \Rightarrow Y_A + P_{AD}\sin 45° = 0 \Rightarrow Y_A + 500\ \{(u_7-u_1)\cos 45° + (u_8-u_2)\sin 45°\}\sin 45° = 0$

$\Rightarrow 500\ \{0.5\ u_1 + 0.5\ u_2 - 0.5\ u_7 - 0.5\ u_8\} = Y_A$ …………………(2)

$\sum F_{xB} = 0 \Rightarrow X_B - P_{AB} + P_{BC} = 0 \Rightarrow X_B - 500\ (u_3-u_1) + 500\ (u_5-u_3) = 0$

$\Rightarrow 500\ \{-1.0\ u_1 + 2.0\ u_3 - 1.0\ u_5\} = X_B$ …………………(3)

$\sum F_{yB} = 0 \Rightarrow Y_B + P_{BD} = 0 \Rightarrow Y_B + 500\ (u_8-u_4) = 0$

$\Rightarrow 500\ \{1.0\ u_4 - 1.0\ u_8\} = Y_B$ …………………(4)

$\sum F_{xC} = 0 \Rightarrow -P_{BC} - P_{CD}\cos 45° = 0$

$\Rightarrow -500\ (u_5-u_3) - 500\{(u_7-u_5)\cos 135° + (u_8-u_6)\sin 135°\}\cos 45° = 0$

$\Rightarrow 500\ \{-1.0\ u_3 + (1.0 + 0.5)\ u_5 - 0.5\ u_6 - 0.5\ u_7 + 0.5\ u_8\} = 0$ …………………(5)

$\sum F_{yC} = 0 \Rightarrow Y_C + P_{CD}\sin 45° = 0 \Rightarrow Y_C + 500\ \{(u_7-u_5)\cos 135° + (u_8-u_6)\sin 135°\}\sin 45° = 0$

$\Rightarrow 500\ \{-0.5\ u_5 + 0.5\ u_6 + 0.5\ u_7 - 0.5\ u_8\} = Y_C$ …………………(6)

$\sum F_{xD} = 0 \Rightarrow -20 - P_{AD}\cos 45° + P_{CD}\cos 45° = 0 \Rightarrow -20 - 500\ \{(u_7-u_1)\cos 45° + (u_8-u_2)\sin 45°\}\cos 45° + 500\ \{(u_7-u_5)\cos 135° + (u_8-u_6)\sin 135°\}\cos 45° = 0$

$\Rightarrow 500\ \{-0.5\ u_1 - 0.5\ u_2 - 0.5\ u_5 + 0.5\ u_6 + (0.5 + 0.5)\ u_7 + (0.5 - 0.5)\ u_8\} = -20$ ……………(7)

$\sum F_{yD} = 0 \Rightarrow 10 - P_{BD} - P_{AD}\sin 45° - P_{CD}\sin 45° = 0$

$\Rightarrow 10 - 500\ (u_8-u_4) - 500\ \{(u_7-u_1)\cos 45° + (u_8-u_2)\sin 45°\}\sin 45°$

$- 500\ \{(u_7-u_5)\cos 135° + (u_8-u_6)\sin 135°\}\sin 45° = 0$

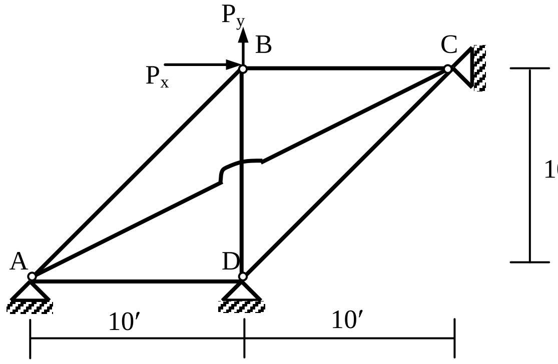
$\Rightarrow 500\{-0.5\ u_1 - 0.5\ u_2 - 1.0\ u_4 + 0.5\ u_5 - 0.5\ u_6 + (0.5 - 0.5)\ u_7 + (1.0 + 0.5 + 0.5)\ u_8\} = 10$ ...…….(8)

Eqs. (1)~(8) are the same equations given by the Stiffness Matrix assembled earlier.

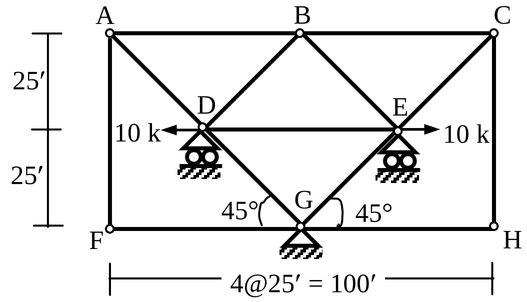
After applying boundary conditions for the known displacements u_1, u_2, u_3, u_4 and u_6, Eqs. (5), (7) and (8) can be solved for the three unknown displacements u_5, u_7 and u_8, whereupon Eqs. (1)~(4) and (6) can be used to calculate the support reactions X_A, Y_A, X_B, Y_B and Y_C.

Problems on Stiffness Method for Trusses

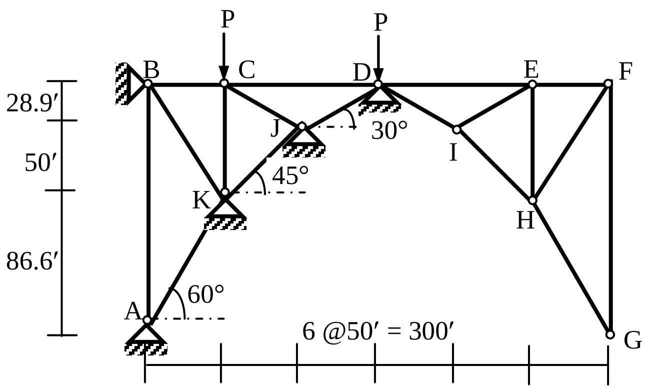
1. Assemble the global stiffness matrix and write the global load vector of the truss shown below (do not apply boundary conditions) [EA/L = Constant = 1000 kip/ft].



2. In the truss shown below, ignore the zero-force members and formulate the stiffness matrix, load vector and write down the boundary conditions [Given: EA/L = constant = 1000 kip/ft].



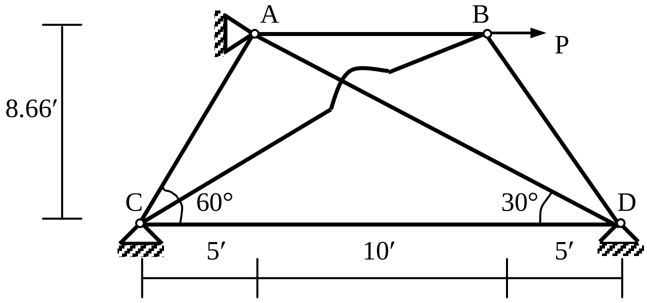
3. In the truss shown below, ignore the zero-force members and formulate the stiffness matrix, load vector and write down the boundary conditions [Given: EA/L = constant = 1000 kip/ft].



4. In the truss described in Question 1, the forces in members BC and BD are both 10 kips (tensile). Calculate the other member forces and the applied loads P_x and P_y.

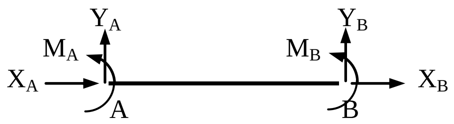
5. For the truss described in Question 2, the force in member DE is 8 k (tension). Calculate the forces in the other members of the truss and deflections of joints D and E.

6. In the truss shown below, the joint B moves 0.05′ horizontally (i.e., no vertical movement) due to the applied force P. Calculate the forces in all the members of the truss [EA/L = Constant = 500 kip/ft].



Stiffness Matrix for 2-Dimensional Frame Members in the Local Axes System

Consider a frame member AB subjected to forces (X_A, Y_A, M_A) and (X_B, Y_B, M_B) at joints A and B.

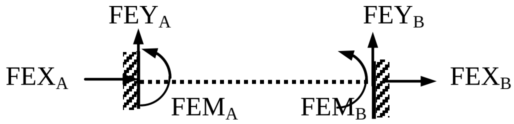


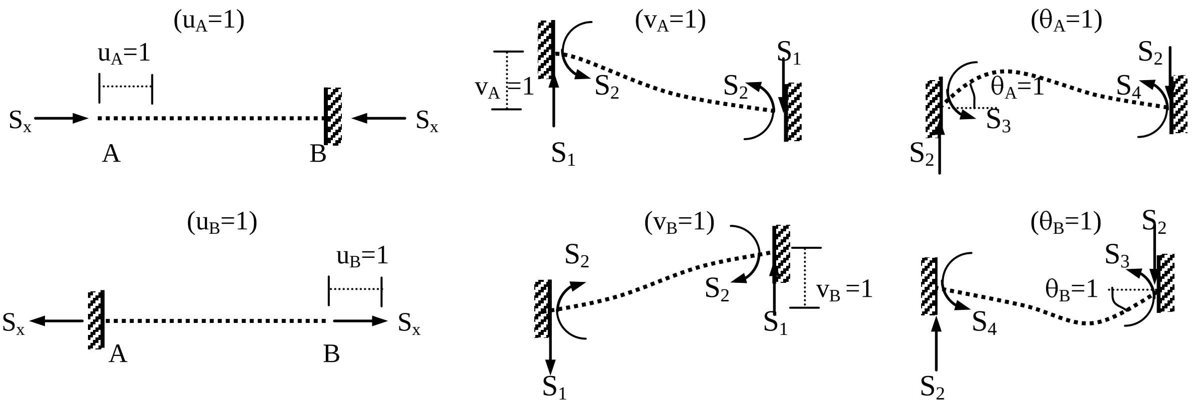
Assume that the length of the member = L, its modulus of elasticity = E, cross-sectional area = A and moment of inertia about z-axis = I.

$\therefore$ The axial stiffness of the member, S_x = Load to produce unit deflection = EA/L

Also assume S_1 = shear stiffness = $12EI/L^3$, $S_2 = 6EI/L^2$, S_3 = flexural stiffness = $4EI/L$, $S_4 = 2EI/L$

If the displacements and rotations of joints A and B are (u_A, v_A, θ_A), (u_B, v_B, θ_B) and the fixed-end reactions are denoted by 'FE', the external forces may result in the following cases.





Equilibrium equations:

$\sum F_{x(A)} = 0 \Rightarrow X_A = FEX_A + S_x u_A + 0 + 0 - S_x u_B + 0 + 0$ ……………….(1)

$\sum F_{y(A)} = 0 \Rightarrow Y_A = FEY_A + 0 + S_1 v_A + S_2 \theta_A + 0 - S_1 v_B + S_2 \theta_B$ ……………….(2)

$\sum M_{z(A)} = 0 \Rightarrow M_A = FEM_A + 0 + S_2 v_A + S_3 \theta_A + 0 - S_2 v_B + S_4 \theta_B$ ……………….(3)

$\sum F_{x(B)} = 0 \Rightarrow X_B = FEX_A - S_x u_A + 0 + 0 + S_x u_B + 0 + 0$ ……………….(4)

$\sum F_{y(B)} = 0 \Rightarrow Y_B = FEY_B + 0 - S_1 v_A - S_2 \theta_A + 0 + S_1 v_B - S_2 \theta_B$ ……………….(5)

$\sum M_{z(B)} = 0 \Rightarrow M_B = FEM_B + 0 + S_2 v_A + S_4 \theta_A + 0 - S_2 v_B + S_3 \theta_B$ ……………….(6)

$$\Rightarrow \begin{pmatrix} S_x & 0 & 0 & -S_x & 0 & 0 \\ 0 & S_1 & S_2 & 0 & -S_1 & S_2 \\ 0 & S_2 & S_3 & 0 & -S_2 & S_4 \\ -S_x & 0 & 0 & S_x & 0 & 0 \\ 0 & -S_1 & -S_2 & 0 & S_1 & -S_2 \\ 0 & S_2 & S_4 & 0 & -S_2 & S_3 \end{pmatrix} \begin{Bmatrix} u_A \\ v_A \\ \theta_A \\ u_B \\ v_B \\ \theta_B \end{Bmatrix} = \begin{Bmatrix} X_A \\ Y_A \\ M_A \\ X_B \\ Y_B \\ M_B \end{Bmatrix} - \begin{Bmatrix} FEX_A \\ FEY_A \\ FEM_A \\ FEX_B \\ FEY_B \\ FEM_B \end{Bmatrix}$$

$$\Rightarrow \mathbf{K_m^L\, u_m^L = q_m^L - f_m^L = p_m^L} \qquad \text{……………….(7)}$$

where $\mathbf{K_m^L}$ = The stiffness matrix of member AB in the local axis system,

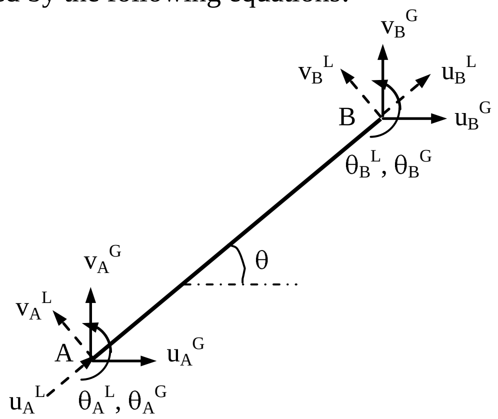
$\mathbf{u_m^L}$ = The displacement vector of the member in the local axis system, and

$\mathbf{p_m^L}$ = The force vector of the member in the local axis system

(= $\mathbf{q_m^L} - \mathbf{f_m^L}$ = Imposed load vector – Fixed end reaction vector)

Transformation of Stiffness Matrix from Local to Global Axes

The member matrices formed in the local axes system can be transformed into the global axes system by considering the angles they make with the horizontal. The local displacements/rotations and global displacements/rotations are related by the following equations.



Local and global joint displacements and rotations of a frame member

$$u_A^L = u_A^G \cos\theta + v_A^G \sin\theta \quad \text{…………………(8)}$$

$$v_A^L = -u_A^G \sin\theta + v_A^G \cos\theta \quad \text{…………………(9)}$$

$$\theta_A^L = \theta_A^G \quad \text{..………………(10)}$$

$$u_B^L = u_B^G \cos\theta + v_B^G \sin\theta \quad \text{………………..(11)}$$

$$v_B^L = -u_B^G \sin\theta + v_B^G \cos\theta \quad \text{………………..(12)}$$

$$\theta_B^L = \theta_B^G \quad \text{………………..(13)}$$

In matrix form, using C = cos θ, S = sin θ

$$\begin{Bmatrix} u_A^L \\ v_A^L \\ \theta_A^L \\ u_B^L \\ v_B^L \\ \theta_B^L \end{Bmatrix} = \begin{pmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} u_A^G \\ v_A^G \\ \theta_A^G \\ u_B^G \\ v_B^G \\ \theta_B^G \end{Bmatrix}$$

$$\Rightarrow \mathbf{u_m^L} = \mathbf{T_m}\, \mathbf{u_m^G} \quad \text{…………………(14)}$$

where $\mathbf{T_m}$ is the transformation matrix for member AB, which connects the displacement vector $\mathbf{u_m^L}$ in the local axes of AB with the displacement vector $\mathbf{u_m^G}$ in the global axes.

A similar expression can be obtained for the force vectors $\mathbf{p_m^L}$ and $\mathbf{p_m^G}$; i.e.,

$$\Rightarrow \mathbf{p_m^L} = \mathbf{T_m}\, \mathbf{p_m^G} \quad \text{…………………(15)}$$

∴ Eq. (7) can be rewritten as $\Rightarrow \mathbf{K_m^L}\, \mathbf{T_m}\, \mathbf{u_m^G} = \mathbf{T_m}\, \mathbf{p_m^G}$ ………………..……(16)

$$\Rightarrow (\mathbf{T_m^{-1}} \mathbf{K_m^L}\, \mathbf{T_m})\, \mathbf{u_m^G} = \mathbf{p_m^G}$$

$$\Rightarrow (\mathbf{T_m^T}\, \mathbf{K_m^L}\, \mathbf{T_m})\, \mathbf{u_m^G} = \mathbf{p_m^G} \quad \text{...………….……(17)}$$

where $\mathbf{T_m^T}$ is the transpose of the transformation matrix $\mathbf{T_m}$, which is also = $\mathbf{T_m^{-1}}$

If $(\mathbf{T_m^T}\, \mathbf{K_m^L}\, \mathbf{T_m})$ is written as $\mathbf{K_m^G}$, the member stiffness matrix in the global axis system, then

$$\mathbf{K_m^G}\, \mathbf{u_m^G} = \mathbf{p_m^G} \quad \text{...………….……(18)}$$

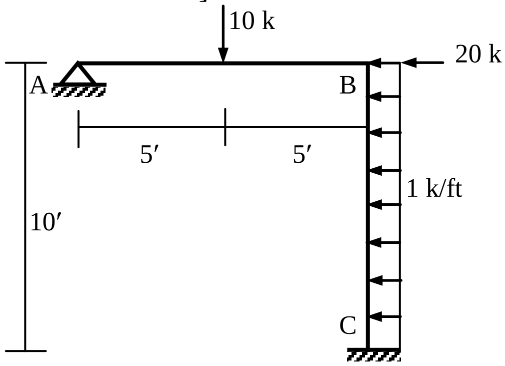
Assembly of Stiffness Matrix and Load Vector of a 2D Frame

The general form of the stiffness matrix for any member of a 2-dimensional frame is

$$\mathbf{K_m}^{\mathbf{G}} = \begin{pmatrix} S_xC^2+S_1S^2 & (S_x-S_1)CS & -S_2S & -(S_xC^2+S_1S^2) & -(S_x-S_1)CS & -S_2S \\ (S_x-S_1)CS & S_xS^2+S_1C^2 & S_2C & -(S_x-S_1)CS & -(S_xS^2+S_1C^2) & S_2C \\ -S_2S & S_2C & S_3 & S_2S & -S_2C & S_4 \\ -(S_xC^2+S_1S^2) & -(S_x-S_1)CS & S_2S & S_xC^2+S_1S^2 & (S_x-S_1)CS & S_2S \\ -(S_x-S_1)CS & -(S_xS^2+S_1C^2) & -S_2C & (S_x-S_1)CS & (S_xS^2+S_1C^2) & -S_2C \\ -S_2S & S_2C & S_4 & S_2S & -S_2C & S_3 \end{pmatrix}$$

<u>Example</u>

Assemble the global stiffness matrix and write the global load vector of the frame shown below. Also write the boundary conditions [E, A, I are constant for all the members].



Since E, A, I and L are uniform, so are S_x, S_1, S_2, S_3 and S_4

If E = 500×10^3 ksf, A = 1 ft^2, I = 0.10 ft^4, L = 10 ft
S_x = EA/L = 50,000 k/ft
$S_1 = 12EI/L^3$ = 600 k/ft, $S_2 = 6EI/L^2$ = 3,000 k/rad
S_3 = 4EI/L = 20,000 k-ft/rad, S_4 = 2EI/L =10,000 k-ft/rad

For member AB, C = 1, S = 0
For member BC, C = 0, S = –1

d.o.k.i = 3×3 = 9, which are (u_A, v_A, θ_A), (u_B, v_B, θ_B) and (u_C, v_C, θ_C), denoted by u_1~u_9.

$$\mathbf{K_{AB}}^{\mathbf{G}} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} S_x & 0 & 0 & -S_x & 0 & 0 \\ 0 & S_1 & S_2 & 0 & -S_1 & S_2 \\ 0 & S_2 & S_3 & 0 & -S_2 & S_4 \\ -S_x & 0 & 0 & S_x & 0 & 0 \\ 0 & -S_1 & -S_2 & 0 & S_1 & -S_2 \\ 0 & S_2 & S_4 & 0 & -S_2 & S_3 \end{pmatrix} \end{array} \qquad \mathbf{K_{BC}}^{\mathbf{G}} = \begin{array}{c} \begin{matrix} 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \begin{pmatrix} S_1 & 0 & S_2 & -S_1 & 0 & S_2 \\ 0 & S_x & 0 & 0 & -S_x & 0 \\ S_2 & 0 & S_3 & -S_2 & 0 & S_4 \\ -S_1 & 0 & -S_2 & S_1 & 0 & -S_2 \\ 0 & -S_x & 0 & 0 & S_x & 0 \\ S_2 & 0 & S_4 & -S_2 & 0 & S_3 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \begin{pmatrix} S_x & 0 & 0 & -S_x & 0 & 0 & & & \\ 0 & S_1 & S_2 & 0 & -S_1 & S_2 & & & \\ 0 & S_2 & S_3 & 0 & -S_2 & S_4 & & & \\ -S_x & 0 & 0 & S_x+S_1 & 0+0 & 0+S_2 & -S_1 & 0 & S_2 \\ 0 & -S_1 & -S_2 & 0+0 & S_1+S_x & -S_2+0 & 0 & -S_x & 0 \\ 0 & S_2 & S_4 & 0+S_2 & -S_2+0 & S_3+S_3 & -S_2 & 0 & S_4 \\ & & & -S_1 & 0 & -S_2 & S_1 & 0 & -S_2 \\ & & & 0 & -S_x & 0 & 0 & S_x & 0 \\ & & & S_2 & 0 & S_4 & -S_2 & 0 & S_3 \end{pmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} X_A \\ Y_A \\ 0 \\ -20 \\ 0 \\ 0 \\ X_C \\ Y_C \\ M_C \end{Bmatrix} - \begin{Bmatrix} 0 \\ 5.0 \\ 12.5 \\ 0 + 5.0 \\ 5.0 + 0 \\ -12.5+8.33 \\ 5.0 \\ 0 \\ -8.33 \end{Bmatrix}$$

Boundary Conditions: $u_1 = 0$, $u_2 = 0$, $u_7 = 0$, $u_8 = 0$, $u_9 = 0$

Therefore, the matrices and vectors can be modified accordingly (similar to the analysis of truss).
Solving the resulting (4×4) matrix, the following displacements and rotations are obtained

$u_3 = -8.12\times10^{-4}$ rad, $u_4 = -5.14\times10^{-4}$ ft, $u_5 = -1.27\times10^{-4}$ ft, $u_6 = 3.36\times10^{-4}$ rad

Stiffness Method for 2-D Frame neglecting Axial Deformations

If axial deformations are neglected in the problem shown before, the displacements u_4 and u_5 are zero and the only unknown displacements are the rotations u_3 and u_6. In that case, the modified equilibrium equations are

$S_3\,u_3 + S_4\,u_6 = -12.5 \Rightarrow 20\times10^3\,u_3 + 10\times10^3\,u_6 = -12.5$

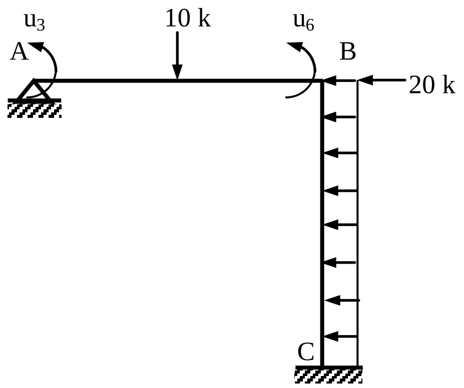
and $S_4\,u_3 + 2S_3\,u_6 = 4.17 \Rightarrow 10\times10^3\,u_3 + 40\times10^3\,u_6 = 4.17$

[Note: S_1 = 600 k/ft, S_2 = 3,000 k/rad, S_3 = 20,000 k-ft/rad, S_4 = 10,000 k-ft/rad]

Solving, $u_3 = -7.74\times10^{-4}$ rad, $u_6 = 2.98\times10^{-4}$ rad [instead of -8.12×10^{-4}, 3.36×10^{-4} found before]

$\therefore$ If the axial deformations are neglected, the calculations and formulations are much simplified without significant loss of accuracy.

Neglecting the axial deformations, the earlier problem can be formulated as shown below



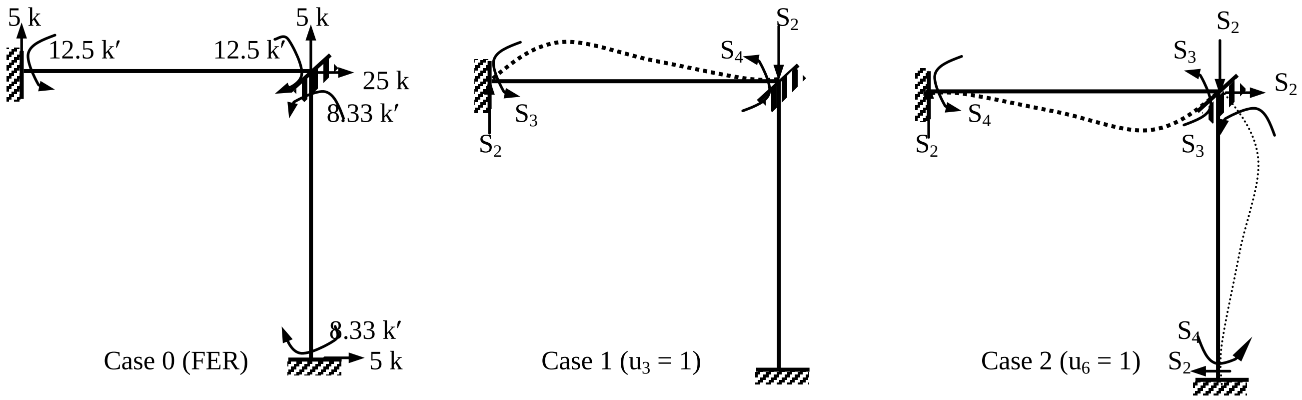
Here, d.o.k.i. = 2

There can be three cases of response

(i) Case0: The fixed-end reactions

(ii) Case1: The reactions due to u_3

(iii) Case2: The reactions due to u_6



<u>Equilibrium equations</u>:

$\sum M_{z(A)} = 0 \Rightarrow 12.5 + S_3\,u_3 + S_4\,u_6 = 0 \qquad \Rightarrow 20\times10^3\,u_3 + 10\times10^3\,u_6 = -12.5$

$\sum M_{z(B)} = 0 \Rightarrow -12.5 + 8.33 + S_4\,u_3 + (S_3+S_3)\,u_6 = 0 \qquad \Rightarrow 10\times10^3\,u_3 + 40\times10^3\,u_6 = 4.17$

Solving the two equations, $u_3 = -7.74\times10^{-4}$ rad, $u_6 = 2.98\times10^{-4}$ rad

<u>Calculation of Internal Forces (SF and BM)</u>:

$SF_{(A)} = 5 + S_2\,u_3 + S_2\,u_6 = 5 + 3{,}000\times(-7.74\times10^{-4}) + 3{,}000\times(2.98\times10^{-4}) = 3.54$ k

$SF_{(B)}$ (in AB) $= 5 - S_2\,u_3 - S_2\,u_6 = 5 - 3{,}000\times(-7.74\times10^{-4}) - 3{,}000\times(2.98\times10^{-4}) = 6.46$ k

$SF_{(B)}$ (in BC) $= 25 + 0 + S_2\,u_6 = 25 + 3{,}000\times(2.98\times10^{-4}) = 25.89$ k

$SF_{(C)}$ (in BC) $= 5 + 0 - S_2\,u_6 = 5 - 3{,}000\times(2.98\times10^{-4}) = 4.11$ k

should be zero

$BM_{(A)} = 12.5 + S_3\,u_3 + S_4\,u_6 = 12.5 + 20{,}000\times(-7.74\times10^{-4}) + 10{,}000\times(2.98\times10^{-4}) = 0$

$BM_{(B)}$ (in AB) $= -12.5 + S_4\,u_3 + S_3\,u_6 = -12.5 + 10{,}000\times(-7.74\times10^{-4}) + 20{,}000\times(2.98\times10^{-4}) = -14.28$ k′

$BM_{(B)}$ (in BC) $= 8.33 + 0 + S_3\,u_6 = 8.33 + 20{,}000\times(2.98\times10^{-4}) = 14.29$ k′

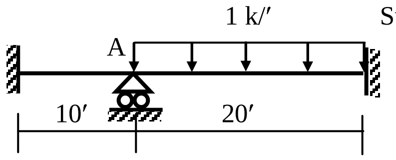
$BM_{(C)} = -8.33 + 0 + S_4\,u_6 = -8.33 + 10{,}000\times(2.98\times10^{-4}) = -5.35$ k′

should be equal

Problems on Stiffness Method for Beams/Frames

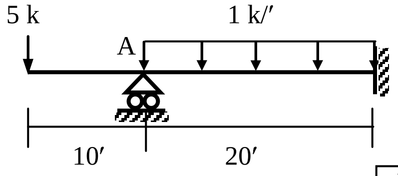
Neglect axial deformations and assume EI = 40,000 k-ft^2

1.

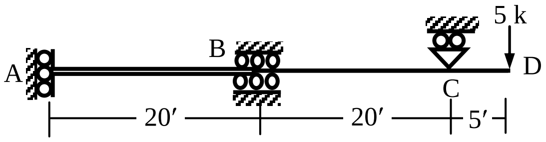


Support A settles 0.05′

2.

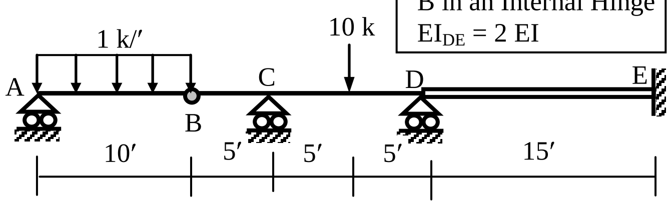


3.



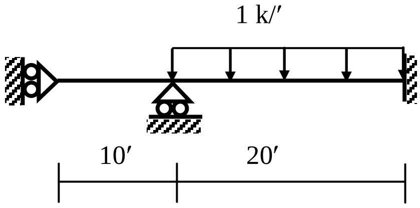
A and B are guided roller supports; EI_{AB} = 2 EI

4.

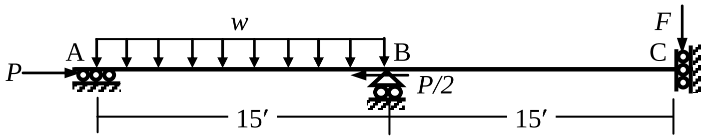


B in an Internal Hinge
EI_{DE} = 2 EI

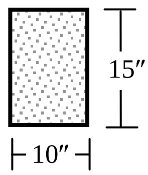
5.



6. Assemble the stiffness matrix, load vector and calculate the unknown joint deflections and rotations of the beam ABC shown below, considering flexural and axial deformations as well as boundary conditions [Given: P = 250 k, w = 1 k/ft, F = 10 k, $E = 400 \times 10^3$ k/ft^2].

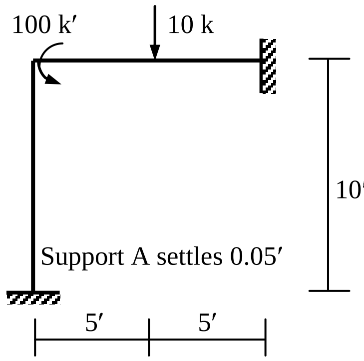


A and C are Guided Rollers



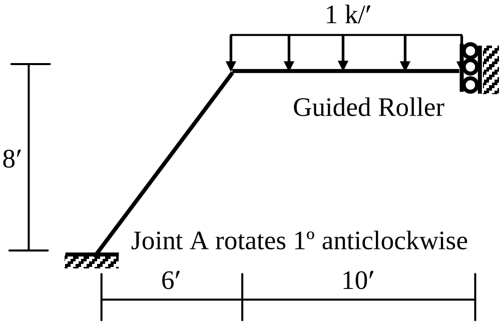
Beam Section

7.



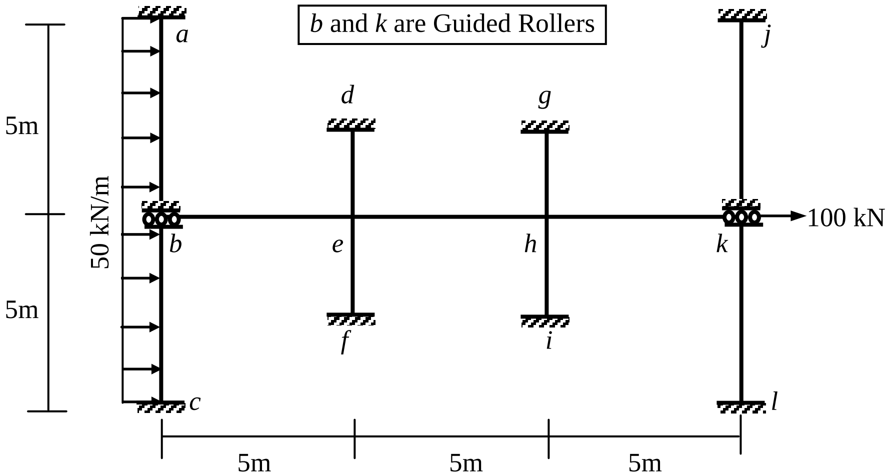
Support A settles 0.05′

8.



Joint A rotates 1° anticlockwise

9. Use the Stiffness Method (considering flexural deformations only) to calculate the unknown joint deflections and rotations of the frame loaded as shown below [Given: EI = constant = 10×10^3 kN-m^2].



b and *k* are Guided Rollers

Analysis of Three-Dimensional Trusses and Frames

1. <u>Three-Dimensional Trusses</u>

Three-dimensional trusses have 3 unknown displacements at each joint; i.e., the deflection u along x-axis, deflection v along y-axis and deflection w along z-axis. Therefore the size of the member stiffness matrix is (6×6). If $S_x = EA/L$, then the stiffness matrix in the local axes system is

$$\mathbf{K_m}^{\mathbf{L}} = S_x \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The member stiffness matrix in the global axes system is

$$\mathbf{K_m}^{\mathbf{G}} = S_x \begin{pmatrix} C_x^2 & C_xC_y & C_xC_z & -C_x^2 & -C_xC_y & -C_xC_z \\ C_yC_x & C_y^2 & C_yC_z & -C_yC_x & -C_y^2 & -C_yC_z \\ C_zC_x & C_zC_y & C_z^2 & -C_zC_x & -C_zC_y & -C_z^2 \\ -C_x^2 & -C_xC_y & -C_xC_z & C_x^2 & C_xC_y & C_xC_z \\ -C_yC_x & -C_y^2 & -C_yC_z & C_yC_x & C_y^2 & C_yC_z \\ -C_zC_x & -C_zC_y & -C_z^2 & C_zC_x & C_zC_y & C_z^2 \end{pmatrix}$$

where $C_x = \cos\alpha$, $C_y = \cos\beta$, $C_z = \cos\gamma$
[α, β and γ are the angles the member makes with the coordinate axes x, y and z respectively]

After assembling the stiffness matrix and load vector and applying known boundary conditions, the unknown displacements are calculated by any standard method of solving simultaneous equations.

Once the displacements are known, the member forces are calculated by the following equation

$$P_{AB} = S_x\,[(u_B-u_A)\,C_x + (v_B-v_A)\,C_y + (w_B-w_A)\,C_z]$$

2. <u>Three-Dimensional Frames</u>

Three-dimensional frames have 6 unknown displacements at each joint; i.e., the deflections (u, v, w) along the x, y and z-axis and rotations (θ_x, θ_y, θ_z) around the x, y and z-axis. Therefore the size of the member stiffness matrix is (12×12), which has the following form in the local axes system

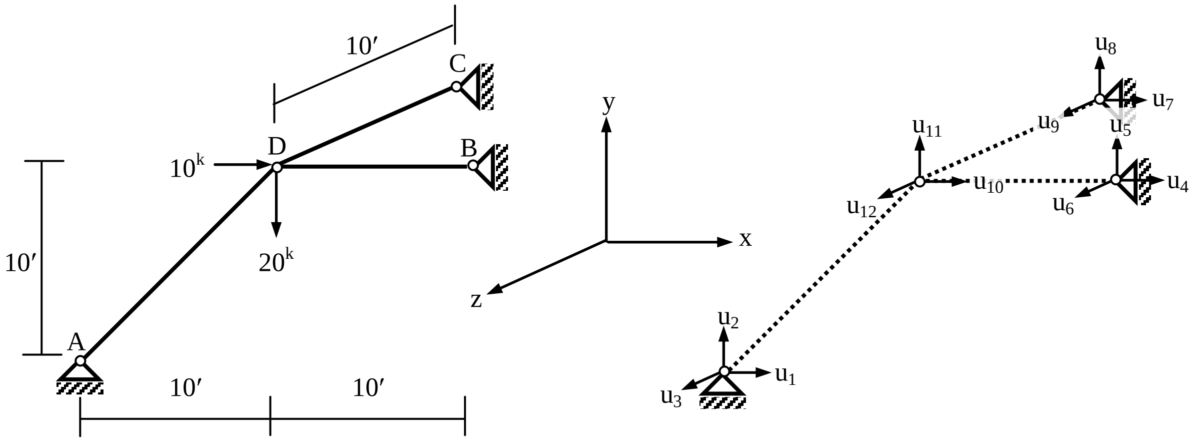
$\mathbf{K_m}^{\mathbf{L}}$ =

S_x						$-S_x$					
	S_{1z}				S_{2z}		$-S_{1z}$				S_{2z}
		S_{1y}		$-S_{2y}$				$-S_{1y}$		$-S_{2y}$	
			T_x						$-T_x$		
		$-S_{2y}$		S_{3y}				S_{2y}		S_{4y}	
	S_{2z}				S_{3z}		$-S_{2z}$				S_{4z}
$-S_x$						S_x					
	$-S_{1z}$				$-S_{2z}$		S_{1z}				$-S_{2z}$
		$-S_{1y}$		S_{2y}				S_{1y}		S_{2y}	
			$-T_x$						T_x		
		$-S_{2y}$		S_{4y}				S_{2y}		S_{3y}	
	S_{2z}				S_{4z}		$-S_{2z}$				S_{3z}

The transformation matrix $\mathbf{T_m}$ and the transformed stiffness matrix $\mathbf{K_m}^{\mathbf{G}}$ in the global axes system are complicated and not written here. However, the method of applying boundary conditions and solving for the unknown displacements are similar to the methods mentioned earlier.

Assembly of Stiffness Matrix and Load Vector of a Three-Dimensional Truss

Assemble the global stiffness matrix and write the global load vector of the three dimensional truss shown below. Also write the boundary conditions [EA/L = Constant = 500 kip/ft].



Member DB: ($C_x = 1$, $C_y = 0$, $C_z = 0$)

$$\mathbf{K_{DB}}^G = 500 \begin{array}{c} \begin{array}{cccccc} 10 & 11 & 12 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 10 \\ 11 \\ 12 \\ 4 \\ 5 \\ 6 \end{array}$$

Member DC: ($C_x = 0$, $C_y = 0$, $C_z = -1$)

$$\mathbf{K_{DC}}^G = 500 \begin{array}{c} \begin{array}{cccccc} 10 & 11 & 12 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ 10 \\ 11 \\ 12 \\ 7 \\ 8 \\ 9 \end{array}$$

Member DA: ($C_x = -0.707$, $C_y = -0.707$, $C_z = 0$)

$$\mathbf{K_{DA}}^G = 500 \begin{array}{c} \begin{array}{cccccc} 10 & 11 & 12 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccccc} 0.5 & 0.5 & 0 & -0.5 & -0.5 & 0 \\ 0.5 & 0.5 & 0 & -0.5 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 10 \\ 11 \\ 12 \\ 1 \\ 2 \\ 3 \end{array}$$

$$\mathbf{K}^G = 500 \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \left(\begin{array}{cccccccccccc} 0.5 & 0.5 & 0 & & & & & & & -0.5 & -0.5 & 0 \\ 0.5 & 0.5 & 0 & & & & & & & -0.5 & -0.5 & 0 \\ 0 & 0 & 0 & & & & & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & & & & -1 & 0 & 0 \\ & & & 0 & 0 & 0 & & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 1 & 0 & 0 & -1 \\ -0.5 & -0.5 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1+0+0.5 & 0+0+0.5 & 0+0+0 \\ -0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0+0+0.5 & 0+0+0.5 & 0+0+0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0+0+0 & 0+0+0 & 0+1+0 \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

$$\mathbf{p}^G = \left\{\begin{array}{c} X_A \\ Y_A \\ Z_A \\ X_B \\ Y_B \\ Z_B \\ X_C \\ Y_C \\ Z_C \\ 10 \\ -20 \\ 0 \end{array}\right\}$$

Boundary Conditions: $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$, $u_5 = 0$, $u_6 = 0$, $u_7 = 0$, $u_8 = 0$, $u_9 = 0$

Applying boundary conditions

$$500\begin{pmatrix}1.5 & 0.5 & 0\\ 0.5 & 0.5 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{Bmatrix}u_{10}\\ u_{11}\\ u_{12}\end{Bmatrix} = \begin{Bmatrix}10\\ -20\\ 0\end{Bmatrix}$$

Solving the three equations $\Rightarrow u_{10} = 0.06'$, $u_{11} = -0.14'$, $u_{12} = 0$

Support Reactions

$X_A = 250\ u_1 + 250\ u_2 - 250\ u_{10} - 250\ u_{11} = 20$ k

$Y_A = 250\ u_1 + 250\ u_2 - 250\ u_{10} - 250\ u_{11} = 20$ k

$Z_A = 0$

$X_B = 500\ u_4 - 500\ u_{10} = -30$ k

$Y_B = 0$

$Z_B = 0$

$X_C = 0$

$Y_C = 0$

$Z_C = 500\ u_9 - 500\ u_{12} = 0$

Member Forces

$F_{DA} = 500\ \{-0.707\ (u_1 - u_{10}) - 0.707\ (u_2 - u_{11}) + 0\ (u_3 - u_{12})\} = -28.28$ k

$F_{DB} = 500\ \{1\ (u_4 - u_{10}) + 0\ (u_5 - u_{11}) + 0\ (u_6 - u_{12})\} = -30$ k

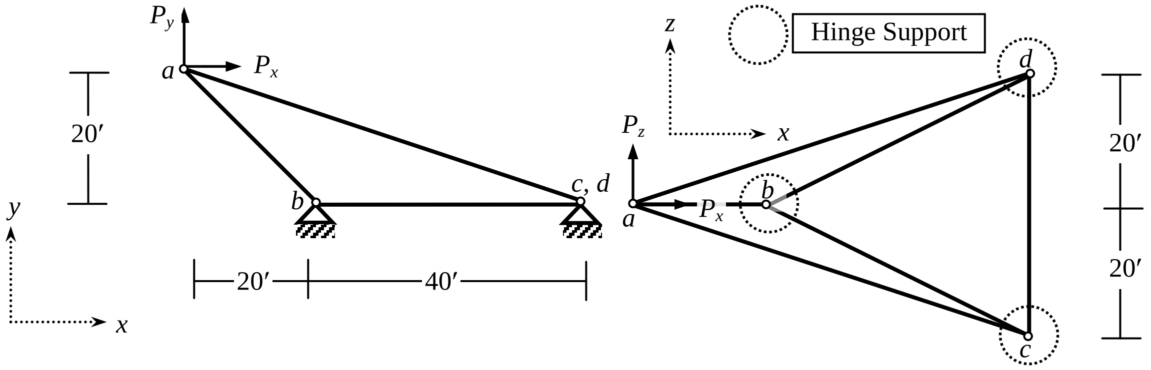
$F_{DC} = 500\ \{0\ (u_7 - u_{10}) + 0\ (u_8 - u_{11}) - 1\ (u_9 - u_{12})\} = 0$

Problems on the Analysis of Three-Dimensional Trusses

Apply boundary conditions and ignore zero-force members whenever necessary/convenient
[Assume EA/L = constant = 500 k/ft for all questions]

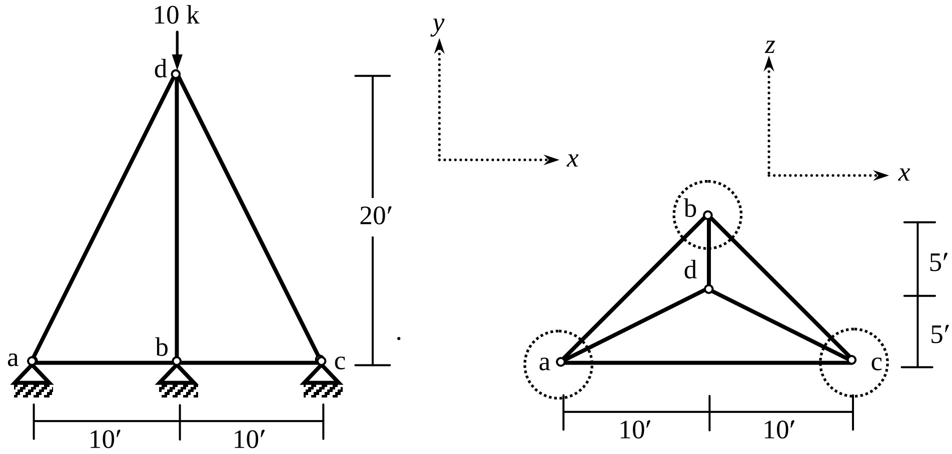
1. Calculate the joint deflections, support reactions and member forces of the space truss analyzed in class if support A settles 0.10′ vertically downwards.

2. Calculate the member forces of the space truss *abcd* loaded as shown below, if $P_x = 0$, $P_y = 10$ k, $P_z = 0$.

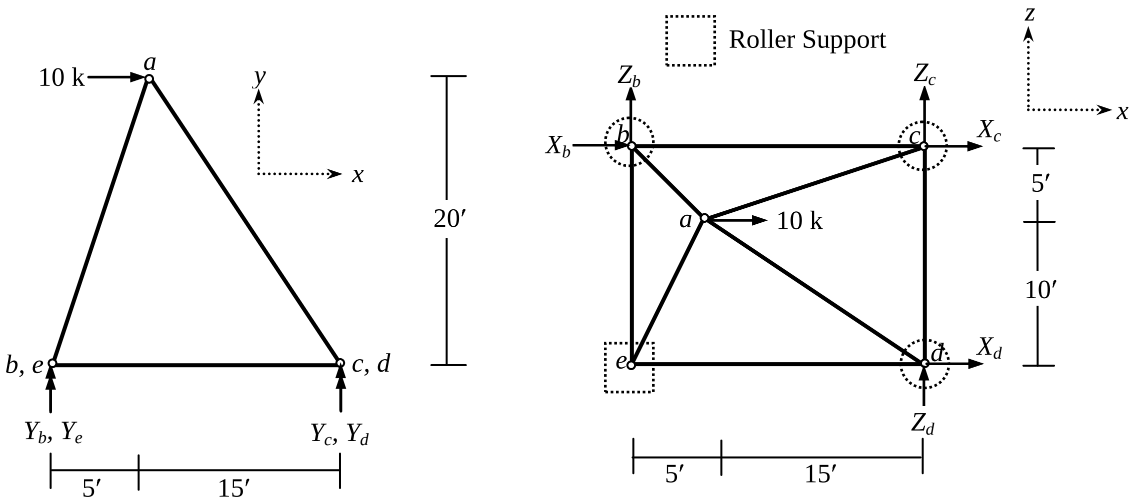


3. Calculate the member forces and applied loads P_x, P_y, P_z in the space truss *abcd* shown in Question 2, if the joint *a* moves 0.10′ right wards and 0.05′ downwards due to the applied loads (i.e., no displacement in *z*-direction).

4. Calculate the support reactions and member forces of the space truss loaded as shown below.

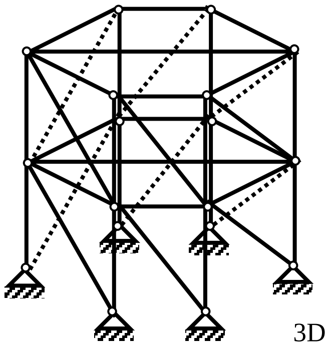


5. Assemble the stiffness matrix, load vector and write down the boundary reactions of the three-dimensional truss loaded as shown below.

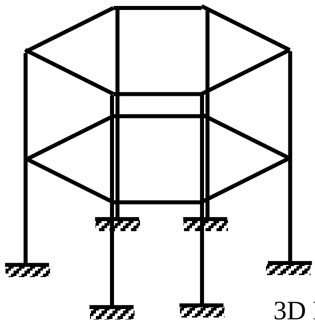


Calculation of Degree of Kinematic Indeterminacy (Doki)

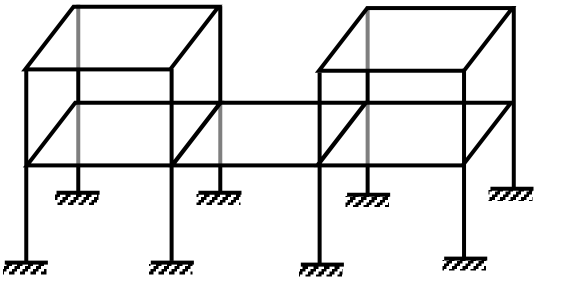
Determine the doki (i.e., size of the stiffness matrix) for the structures shown below, considering boundary conditions. For the frames, also determine the doki if axial deformations are neglected.



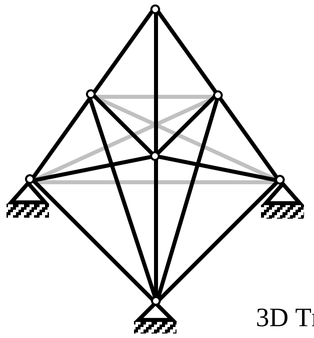
3D Truss



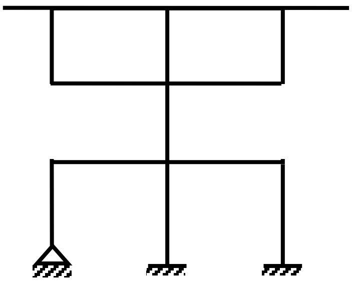
3D Frame



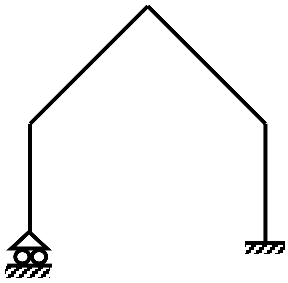
3D Frame



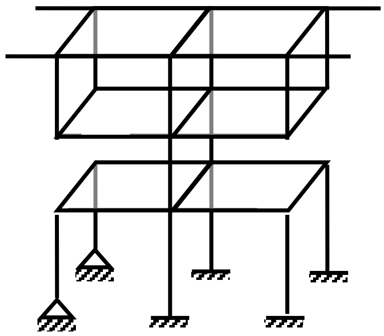
3D Truss



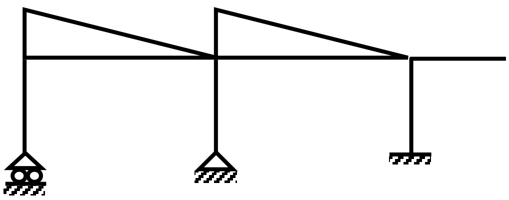
2D Frame



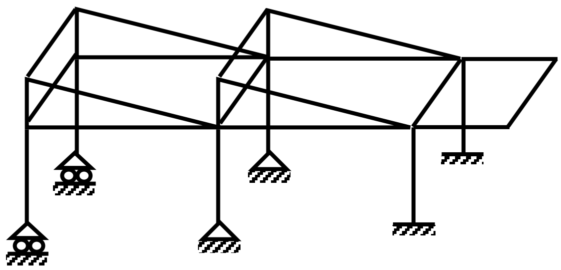
2D Frame



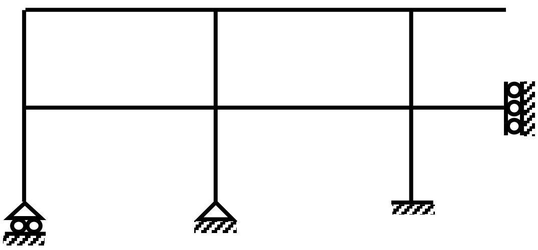
3D Frame



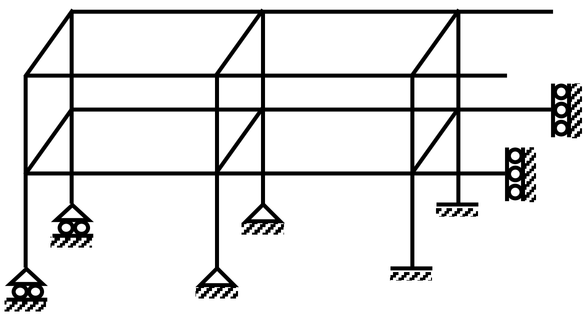
2D Frame



3D Frame



2D Frame



3D Frame

Energy Formulation of Geometric Nonlinearity

Linear structural analysis is based on the assumption of small deformations and linear elastic behavior of materials. The analysis is performed on the initial undeformed shape of the structure. As the applied loads increase, this assumption is no longer accurate, because the deformations may cause significant changes in the structural shape. Geometric nonlinearity is the change in the elastic load-deformation characteristics of the structure caused by the change in the structural shape.

Among various types of geometric nonlinearity, the structural instability or moment magnification caused by large compressive forces, stiffening of structures due to large tensile forces, change in structural parameters due to applied dynamic loads are significant. Rather than using equilibrium equations, it is often more convenient to formulate geometrically nonlinear problems by the *Method of Virtual Work*.

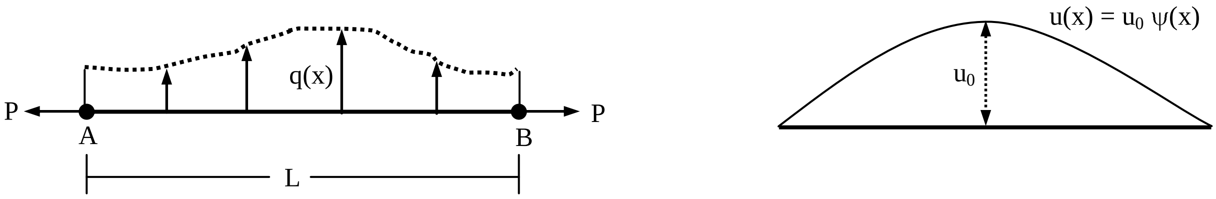
<u>Method of Virtual Work</u>

Another way of representing Newton's equation of equilibrium is by energy methods, which is based on the law of conservation of energy. According to the principle of virtual work, if a system in equilibrium is subjected to virtual displacements δu, the virtual work done by the external forces (δW_E) is equal to the virtual work done by the internal forces (δW_I)

$$\delta W_I = \delta W_E \qquad \text{......................(1)}$$

where the symbol δ is used to indicate 'virtual'. This term is used to indicate hypothetical increments of displacements and works that are assumed to happen in order to formulate the problem.

<u>Energy Formulation and Buckling of Beams-columns</u>



Transversely Loaded Member and Assumed Shape

Applying the method of virtual work to flexural members subjected to transverse load of q(x) per unit length and axial (tensile) force P $\Rightarrow \int u''EI\,\delta u''\,dx + \int u'\,P\,\delta u'\,dx = \int q(x)\,dx\,\delta u$ ....(2)

Using the energy formulation assuming $u(x) = u_0\psi(x)$ provides the following equation

$$\int u_0\,\psi''(x)\,EI\,\delta u_0\,\psi''(x)\,dx + \int u_0\,\psi'(x)\,P\,\delta u_0\,\psi'(x)\,dx = \int q(x)\,dx\,\delta u_0\,\psi(x)$$

$$\Rightarrow \{\int EI\,[\psi''(x)]^2\,dx + \int P\,[\psi'(x)]^2\,dx\}\,u_0 = \int q(x)\,\psi(x)\,dx \qquad \text{.....................(3)}$$

$\therefore$ Carrying out the integrations after knowing (or assuming) $\psi(x)$, Eq. (3) can be rewritten as,

$$k_{Total}^*\,u_0 = f^* \qquad \text{.......................(4)}$$

where k*, f*are the 'effective' stiffness and force of the system, with

$$k_{Total}^* = \int EI\,[\psi''(x)]^2\,dx + \int P\,[\psi'(x)]^2\,dx \qquad \text{.....................(5.1)}$$

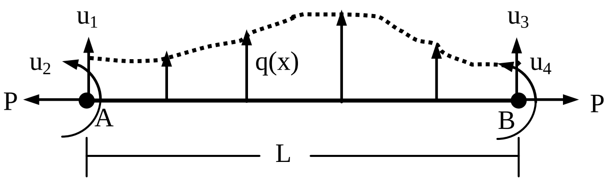
$$f^* = \int q(x)\,\psi(x)\,dx \qquad \text{.....................(5.2)}$$

Therefore a tensile force P (i.e., positive P) will further stiffen the beam-column (i.e., increase its stiffness) and a compressive force (i.e., negative P) will make it more flexible and increase the resulting deflection and internal forces compared to linear analysis. In the extreme case, the beam-column will buckle if the effective stiffness k* becomes zero, which is possible only for a compressive force

$$P_{cr} = -\{\int EI\,[\psi''(x)]^2\,dx\}/\int[\psi'(x)]^2\,dx \qquad \text{.......................(6)}$$

It is obvious that the accuracy of the formulation depends on the accuracy of the assumed shape function $\psi(x)$, which must at least satisfy the natural boundary conditions. However, other than assuming a more appropriate shape function, its accuracy cannot be improved by any other means.

Stiffness Matrix and Geometric Stiffness Matrix of Beams-columns



Transversely Loaded Beam-Column

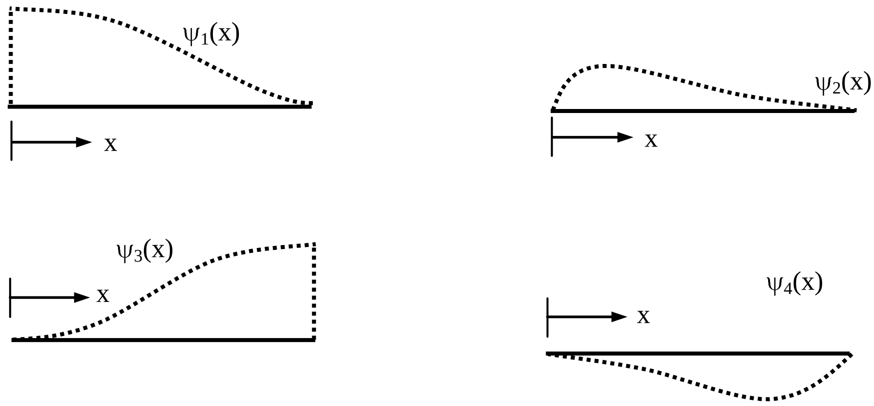
Two-noded elements with cubic interpolation functions for u_1, u_2, u_3 and u_4 are typically chosen in such cases, so that $u(x) = u_1\,\psi_1 + u_2\,\psi_2 + u_3\,\psi_3 + u_4\,\psi_4$ ……………(7)

where $\psi_1(x) = 1-3(x/L)^2+2(x/L)^3$, $\psi_2(x) = x\{1-(x/L)\}^2$

$\psi_3(x) = 3(x/L)^2-2(x/L)^3$, $\psi_4(x) = (x-L)(x/L)^2$ ……………(8)

$\therefore u' = u_1\,\psi_1' + u_2\,\psi_2' + u_3\,\psi_3' + u_4\,\psi_4'$; $\delta u' = \delta u_1\,\psi_1' + \delta u_2\,\psi_2' + \delta u_3\,\psi_3' + \delta u_4\,\psi_4'$ ……………..(9)

$\therefore u'' = u_1\,\psi_1'' + u_2\,\psi_2'' + u_3\,\psi_3'' + u_4\,\psi_4''$; $\delta u'' = \delta u_1\,\psi_1'' + \delta u_2\,\psi_2'' + \delta u_3\,\psi_3'' + \delta u_4\,\psi_4''$ ……..…..…(10)



Shape functions $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$ and $\psi_4(x)$

Inserting the values of u', $\delta u'$, u'' and $\delta u''$ in Eq. (1), and equating the coefficients of $\psi_1 \Rightarrow$

$(\int E\,I\,\psi_1''\,\psi_1''\,dx + \int P\,\psi_1'\,\psi_1'\,dx)\,u_1 + (\int E\,I\,\psi_1''\,\psi_2''\,dx + \int P\,\psi_1'\,\psi_2'\,dx)\,u_2$
$+ (\int E\,I\,\psi_1''\,\psi_3''\,dx + \int P\,\psi_1'\,\psi_3'\,dx)\,u_3 + (\int E\,I\,\psi_1''\,\psi_4''\,dx + \int P\,\psi_1'\,\psi_4'\,dx)\,u_4 = \int q(x)\,\psi_1\,dx$ …..…....(11)

Similarly, equating the coefficients of ψ_2, ψ_3 and ψ_4 will produce two (4×4) matrices $\mathbf{K_m}$ and $\mathbf{G_m}$, along with a (4×1) load vector $\mathbf{p_m}$ here, and their elements are given by

$K_{mij} = \int EI\,\psi_i''\,\psi_j''\,dx$ $G_{mij} = \int P\,\psi_i'\psi_j'\,dx$ $p_{mi} = \int q(x)\,\psi_i\,dx$ ……..…(12)

The equations of the stiffness matrix and geometric stiffness matrix for flexural members guarantee that for 'linear' problems,

(i) The stiffness and geometric stiffness matrices are symmetric [i.e., element (i,j) = element (j,i)],

(ii) The diagonal elements of the matrices are positive [as the element (i,i) involves squares].

As mentioned, for structural analysis the effect of axial load on flexural behavior can be approximated by simplified formulations of the geometric nonlinearity problem. For this purpose, a new matrix called the geometric stiffness matrix (**G**) has been added to the original stiffness matrix **K** obtained from linear analysis of the undeformed deflected shape of the structure. Therefore, the total stiffness matrix of a flexural member is the sum of these two matrices; i.e.,

$\mathbf{K_{total}} = \mathbf{K} + \mathbf{G}$ ……………………..(13)

Using the same shape functions ψ_i (i = 1~4) as done for the linear analyses of beams and frames, the following geometric stiffness matrix is formed in the local axes system of a member of length L.

$$G_m^L = (P/30L)\begin{pmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{pmatrix} \qquad \text{..........(14)}$$

This geometric stiffness matrix can be added to the linear stiffness matrix shown before, and the total stiffness matrix is transformed and assembled using the equations and formulations mentioned in earlier lectures. Once the total stiffness matrix $\mathbf{K_{total}}$ is obtained in the global axes after applying appropriate boundary conditions, the structural analyses can be carried out using the procedures mentioned before.

The governing equations of motion can be written in matrix form as

$$\mathbf{K_{total}\, u = f} \qquad \text{.....................(15)}$$

However it should be noted that the presence of axial force P in the geometric stiffness matrix makes the problem nonlinear because P is obtained from member deformations, which cannot be found before performing the structural analysis. Therefore the system properties and output are interdependent, which calls for iterative methods of structural analysis. However, P is known for special cases (e.g., single column subjected to a known axial load) so that the problem is not nonlinear any more.

Buckling occurs when the structure loses its stiffness, i.e., when the total stiffness matrix $\mathbf{K_{total}}$ becomes singular. Therefore, the buckling load can be obtained by solving the eigenvalue problem

$$|\mathbf{K_{total}}| = 0 \Rightarrow |\mathbf{K}+\mathbf{G}| = 0 \qquad \text{..........................(16)}$$

Since the stiffness and geometric stiffness matrix are derived from approximate shape functions, the critical buckling load obtained from Eq. (16) is also approximate and can be improved if the beam-column is divided into more segments throughout its length.

Just as an axial compressive load can reduce the effective stiffness of a structural member, a tensile load may increase it. This will cause stiffening of the member and a corresponding decrease in deformations.

Example

For $EI = 40\times10^3$ k-ft^2, $L = 10$ ft, calculate the approximate first buckling load for

(i) a simply supported beam, (ii) a cantilever beam

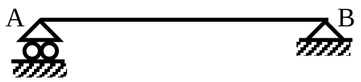
(iii) Suggest how to improve the results.

(iv) Also calculate the tip deflection and rotation of the cantilever beam when subjected to a uniformly distributed transverse load of 1 k/ft along with a compressive load of 400 kips.

Solution

(i) For the simply supported beam, the two d.o.f. are θ_A and θ_B, so the **K** and **G** matrices are

$$\mathbf{K} = 10^3\begin{pmatrix}16 & 8\\ 8 & 16\end{pmatrix} \quad \text{and} \quad \mathbf{G} = (P/300)\begin{pmatrix}400 & -100\\ -100 & 400\end{pmatrix}$$



$\therefore$ For critical buckling load P, the determinant of (**K** + **G**) is = 0

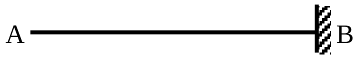
$\Rightarrow (16000 + 4P/3)^2 - (8000 - P/3)^2 = 0 \Rightarrow 16000 + 4P/3 = \pm\,(8000 - P/3)$

$\Rightarrow 5P/3 = -8000$; i.e., $P = -4800$ k, or $P + 24000 = 0$; i.e., $P = -24000$ k (negative $\Rightarrow$ compression)

Compared to the first two 'exact' buckling loads, π^2EI/L^2 and $4\pi^2EI/L^2$; i.e., -3948 k and -15791 k

(ii) For the cantilever beam, the two d.o.f. are v_A and θ_A, the **K** and **G** matrices being

$$\mathbf{K} = 10^3\begin{pmatrix}0.48 & 2.4\\ 2.4 & 16\end{pmatrix} \quad \text{and} \quad \mathbf{G} = (P/300)\begin{pmatrix}36 & 30\\ 30 & 400\end{pmatrix}$$



$\therefore$ For critical buckling load P, the determinant of (**K** + **G**) is = 0

$\Rightarrow (480 + 0.12\,P)\,(16000 + 4\,P/3) - (2400 + 0.10\,P)^2 = 0 \Rightarrow 0.15\,P^2 + 2080\,P + 192\times10^4 = 0$

$\Rightarrow P = [-2080 \pm \sqrt{\{(-2080)^2 - 4\times0.15\times192\times10^4\}}]/0.30 = -994$ k and -12872 k

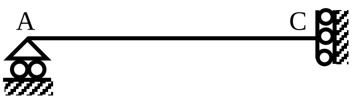
Compared to the first two 'exact' buckling loads, $\pi^2EI/(2L)^2$ and $9\pi^2EI/(2L)^2$; i.e. -987 k, -8883 k

(iii) The predictions can be improved by dividing the beams into more segments or using more appropriate shape functions. For the simply supported beam

Dividing into two segments, half of the symmetric beam takes the form

For the simply supported beam, the two d.o.f. are θ_A and v_C, so the **K** and **G** matrices are

$$\mathbf{K} = 10^3\begin{pmatrix}32 & -9.6\\ -9.6 & 3.84\end{pmatrix} \quad \text{and} \quad \mathbf{G} = (P/150)\begin{pmatrix}100 & -15\\ -15 & 36\end{pmatrix}$$



$\therefore$ For critical buckling load P, the determinant of (**K** + **G**) is = 0

$\Rightarrow (32000 + 2P/3)\,(3840 + 0.24P) - (-9600 - P/10)^2 = 0 \Rightarrow 0.15\,P^2 + 8320\,P + 30.72\times10^6 = 0$

$\Rightarrow P = -3978$ k, or -51489 k (negative $\Rightarrow$ compression)

They only represent the first two 'odd' buckling loads, π^2EI/L^2 and $9\pi^2EI/L^2$; i.e., -3948 k, -35531 k

Assuming $\psi(x) = \sin(\pi x/L)$ [with $\psi(0) = \psi(L) = 0$, $\psi'(x) = (\pi/L)\cos(\pi x/L)$, $\psi''(x) = -(\pi/L)^2\sin(\pi x/L)$]

Effective stiffness $k^* = \int EI\,[\psi''(x)]^2\,dx = (\pi/L)^4\,EI\,L/2$

Effective geometric stiffness $g^* = \int P\,[\psi'(x)]^2\,dx = (\pi/L)^2\,P\,L/2$

$\therefore k_{Total}^* = k^* + g^* = 0 \Rightarrow$ Buckling load $P_{cr} = -\{(\pi/L)^4\,EI\,L/2\}/\{(\pi/L)^2\,L/2\} = -\pi^2\,EI//L^2 = -3948$ k, which is the exact first buckling load.

(iv) If $P = -400$ kips for the cantilever beam, the total stiffness matrix $\mathbf{K_{total}}$ and load vector **f** are

$$\mathbf{K_{total}} = \begin{pmatrix}480 - 48 & 2400 - 40\\ 2400 - 40 & 16000 - 533.33\end{pmatrix} = \begin{pmatrix}432 & 2360\\ 2360 & 15466.67\end{pmatrix} \qquad \mathbf{f} = \begin{Bmatrix}-5.00\\ -8.33\end{Bmatrix}$$

Solving the two equations, $v_A = -51.86\times10^{-3}$ ft, and $\theta_A = 7.374\times10^{-3}$ rad

(compared to the results when $P = 0$, i.e., $v_A = -31.25\times10^{-3}$ ft, and $\theta_A = 4.167\times10^{-3}$ rad)

Assuming $\psi(x) = 1 - \sin(\pi x/2L)$, $\psi'(x) = -(\pi/2L)\cos(\pi x/2L)$

Effective stiffness $k^* = \int EI\,[\psi''(x)]^2\,dx = (\pi/2L)^4\,EI\,L/2 = 121.76$ k/ft

Effective geometric stiffness $g^* = \int P\,[\psi'(x)]^2\,dx = (\pi/2L)^2\,P\,L/2 = -49.35$ k/ft

Effective force $f^* = \int q(x)\,\psi(x)\,dx = -qL(1 - 2/\pi) = -3.63$ kips

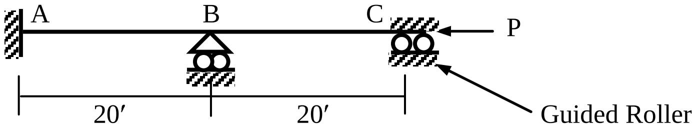
$\therefore (121.76 - 49.35)\,u_2 = -3.63 \Rightarrow u_2 = -50.18\times10^{-3}$ ft

$\Rightarrow v_A = u_2\,\psi(0) = u_2 = -50.18\times10^{-3}$ ft, and $\theta_A = u_2\,\psi'(0) = -u_2\,(\pi/2L) = 7.882\times10^{-3}$ rad

Assembling Stiffness Matrix and Geometric Stiffness Matrix

Assume EI = 40×10^3 k-ft^2 for the following problems

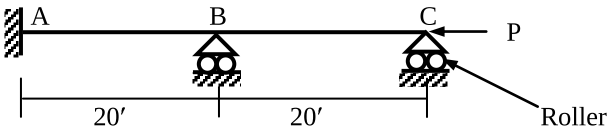
1.



Stiffness Matrix **K** = $\begin{pmatrix} 8000 + 8000 \end{pmatrix} = \begin{pmatrix} 16000 \end{pmatrix}$

Geometric Stiffness Matrix **G** = $-\,(P/600)\begin{pmatrix} 1600 + 1600 \end{pmatrix} = -\,(P/600)\begin{pmatrix} 3200 \end{pmatrix}$

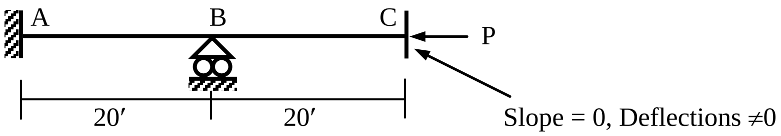
2.



Stiffness Matrix **K** = $\begin{pmatrix} 8000 + 8000 & 4000 \\ 4000 & 8000 \end{pmatrix} = \begin{pmatrix} 16000 & 4000 \\ 4000 & 8000 \end{pmatrix}$

Geometric Stiffness Matrix **G** = $-\,(P/600)\begin{pmatrix} 1600 + 1600 & -400 \\ -400 & 1600 \end{pmatrix} = -\,(P/0.6)\begin{pmatrix} 3.2 & -0.4 \\ -0.4 & 1.6 \end{pmatrix}$

3.

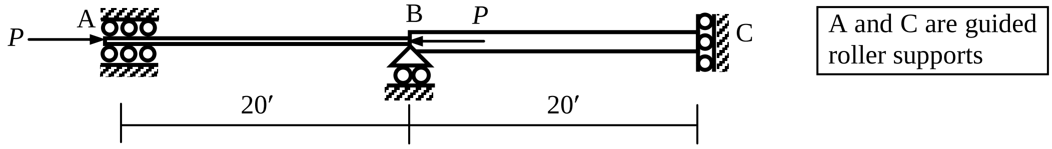


Stiffness Matrix **K** = $\begin{pmatrix} 8000 + 8000 & -600 \\ -600 & 60 \end{pmatrix} = \begin{pmatrix} 16000 & -600 \\ -600 & 60 \end{pmatrix}$

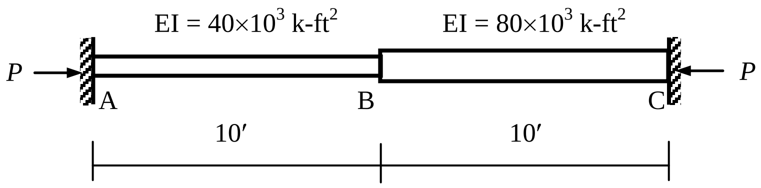
Geometric Stiffness Matrix **G** = $-\,(P/600)\begin{pmatrix} 1600 + 1600 & -60 \\ -60 & 36 \end{pmatrix} = -\,(P/600)\begin{pmatrix} 3200 & -60 \\ -60 & 36 \end{pmatrix}$

Practice Problems on Geometrically Nonlinear Structures

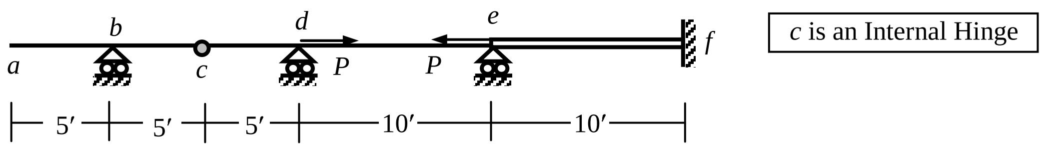
1. Calculate the force P needed to cause buckling of the beam ABC shown below [Given: $EI_{AB} = 20 \times 10^3$ k-ft^2, $EI_{BC} = 40 \times 10^3$ k-ft^2].



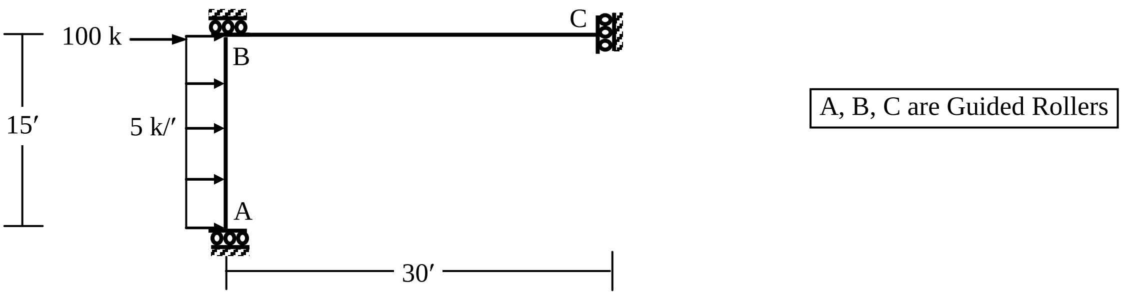
2. Approximately calculate the critical buckling load of the beam ABC shown below.



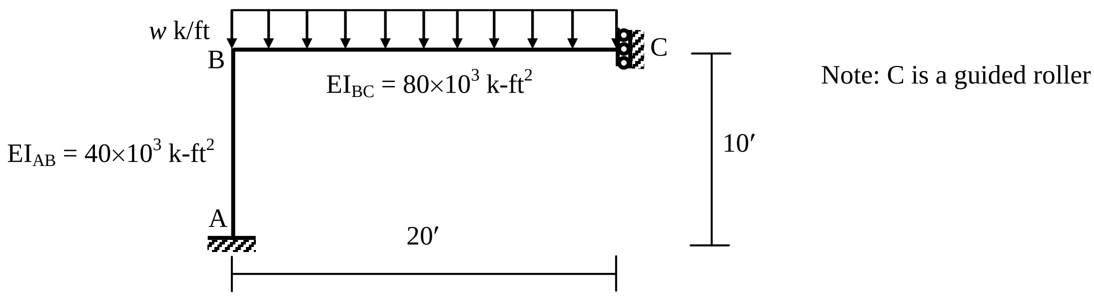
3. Calculate the value of force P needed to cause buckling of the beam *abcdef* shown below [Given: $EI_{ae} = 20 \times 10^3$ k-ft^2, $EI_{ef} = 2EI_{ae}$].



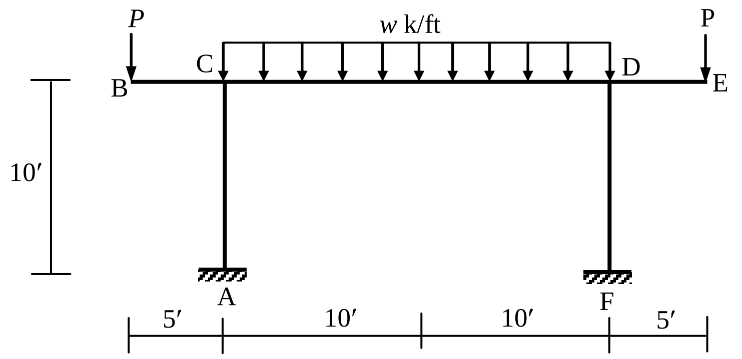
4. Use the Stiffness Method (considering geometric nonlinearity) to calculate the horizontal deflection at A and vertical deflection at C of the frame loaded as shown below [Given: EI = constant = 15×10^3 k-ft^2].



5. Calculate the load w to cause buckling of the frame ABC shown below.



6. Calculate the force P to cause buckling of the frame shown below, using $w = 0.15P$ [$EI = 40\times10^3$ k-ft^2].



Material Nonlinearity and Plastic Moment

As mentioned in the previous section, structural properties cannot be assumed to remain constant in many practical situations. In addition to the geometric nonlinearity that may lead to instability of structures with linearly materials properties, the variation in material properties itself can make the structural analysis nonlinear. For example, yielding of the structural materials, a likely situation in a severe loading conditions or ground vibrations, may alter the stiffness properties, which needs to be updated with structural deformations.

Material Nonlinearity in Concrete, Steel and Reinforced Concrete

Concrete and steel are the most common among the construction materials used for Civil Engineering constructions. Among them, concrete is much stronger in compression than in tension (tensile strength is of the order of one-tenth of compressive strength). While its tensile stress-strain relationship is almost linear, the stress-strain relationship in compression is nonlinear from the beginning (Fig. 1).

Steel on the other hand, has similar stress-strain properties in tension and compression. After an initial linearly-elastic portion, the stress remains almost constant while the strain increases significantly (a phenomenon called yielding). This is typically followed by some increase in stress (strain hardening) at a reducing elasticity, and finally a decrease in stress leading to breaking of the specimen (Fig. 2).

Reinforced Concrete or RC is a unique combination of these two materials where the complexities of their constitutive behavior come into effect. The behavior of RC cannot be modeled properly by linear elastic behavior. Recognizing this, the design of RC structures has gradually shifted over the years from the 'elastic' Working Stress Design (WSD) to the more rational Ultimate Strength Design (USD). The design of steel structures has also undergone similar transition from the *Allowable Stress Design* (ASD) method to the *Load and Resistant Factor Design* (LRFD) method.

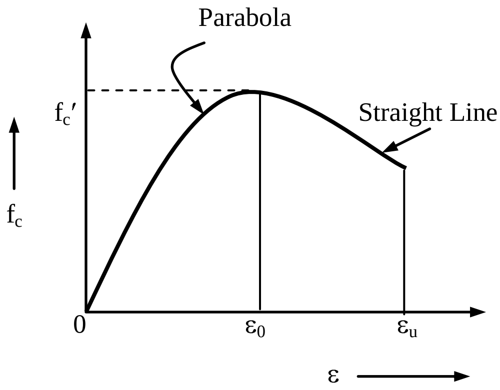


Fig. 1: Stress-Strain Model for Concrete (Compression)

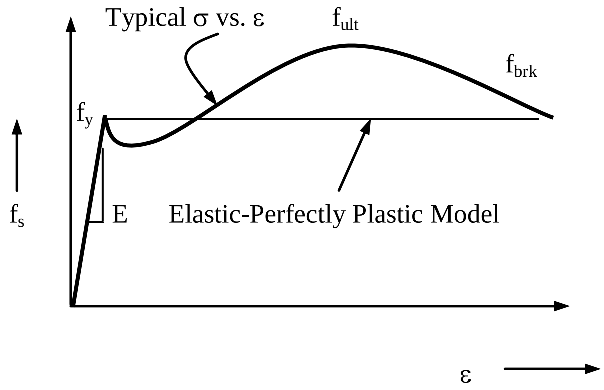


Fig. 2: Typical Stress vs. Strain for Steel

Analysis of Linearly Elastic and Inelastic Systems

For a linearly elastic system the relationship between the applied force f_s and the resulting deformation u is linear, i.e.,

$$f_s = k\,u \qquad \text{……………………(1)}$$

where k is the linear stiffness of the system; its units are force/length. Implicit in Eq. (1) is the assumption that the linear f_s-u relationship determined for small deformations of structure is also valid for large deformations. Because the resisting force is a single valued function of u, the system is elastic; hence the term k can be used in linearly elastic system. This is however not valid when the load-deformation relationship is nonlinear, i.e., when the stiffness itself is not constant but is a function of u. Thus the resisting force can be expressed as

$$f_s = f_s(u) \qquad \text{……………………....(2)}$$

and the system is called inelastic dynamic system. The structural analysis of such systems can only be performed by iterative methods.

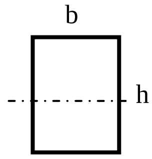
Plastic Moment of Typical Sections

The iterative method required to analyze nonlinear systems is quite laborious, time consuming and its convergence to the exact solution is not always guaranteed, it is usually not followed in typical structural analyses other than for very important projects. However, the calculation of the ultimate moment capacity of a cross-section or the ultimate load carrying capacity of a structure is usually much simpler, and is of more interest to a structural designer.

The following examples show the calculation of yielding and ultimate moment capacities of typical steel and RC sections.

Example 1

Calculate the Yield Moment and Plastic Moment capacity of the sections shown below if they are made of elastic-fully plastic material (e.g., steel model shown in Fig. 2).



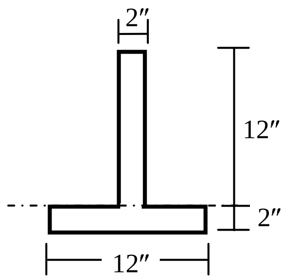
For the rectangular section, the neutral axis divides the area into two segments of (b × h/2)

$\therefore$ Compressive force = Tensile force = σ_{yp} (bh/2)

$\therefore$ Plastic moment M_p = Tensile (or compressive) force × Moment arm = σ_{yp} (bh/2) × h/2

$\therefore M_p = \sigma_{yp}\,(bh^2/4)$

The yield moment is $M_y = \sigma_{yp}\,(S) = \sigma_{yp}\,(bh^3/12)/(h/2) = \sigma_{yp}\,(bh^2/6)$



For the T-section, the equal-area axis divides the area along the flange line.

$\therefore$ Compressive force = Tensile force = $\sigma_{yp}\,(12 \times 2) = 24\sigma_{yp}$

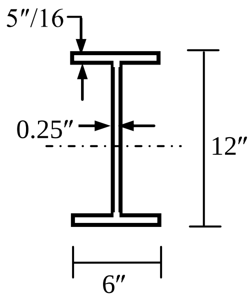
$\therefore$ Plastic moment M_p = Tensile (or compressive) force × Moment arm = $24\sigma_{yp} \times (1 + 6)$

$\therefore M_p = \sigma_{yp}\,(168)$ = 6048 k-in = 504 k-ft [assuming σ_{yp} = 36 ksi]

Also, $\bar{y} = (24 \times 1 + 24 \times 8)/48 = 4.5''$; $c = 14 - 4.5 = 9.5''$

$\bar{I} = 12 \times 2^3/12 + 24\,(1 - 4.5)^2 + 2 \times 12^3/12 + 24\,(8 - 4.5)^2 = 884 \text{ in}^4$

$S = 884/9.5 = 93.05 \text{ in}^3 \Rightarrow M_y = \sigma_{yp}\,(93.05) = 279.15$ k-ft



For the I-section, the equal-area axis divides the area symmetrically.

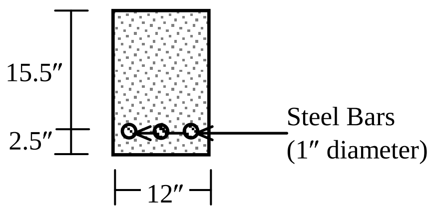
$\therefore$ Compressive force = Tensile force = $\sigma_{yp}\,\{6 \times 5/16 + (6 - 5/16) \times 0.25\} = 3.297\sigma_{yp}$

$\therefore$ Plastic moment $M_p = \sigma_{yp}\,\{1.875 \times (6 - 5/32) + 1.422 \times (6 - 5/16)/2\} \times 2 = 30\sigma_{yp}$

$\therefore M_p = \sigma_{yp}\,(30)$ = 1800 k-in = 150 k-ft [assuming σ_{yp} = 60 ksi]

Example 2

Calculate the Ultimate Moment capacity of the rectangular RC beam section shown below
[Given: f_c' = 4 ksi, f_y = 60 ksi].



For b = 12″, d = 15.5″, $A_s = 3 \times \pi(1)^2/4 = 2.36 \text{ in}^2$

$\Rightarrow a = A_s f_y/(0.85 f_c'\, b) = 2.36 \times 60/(0.85 \times 4 \times 12) = 3.46''$

$\therefore M_{ult} = A_s f_y\,(d - a/2) = 2.36 \times 60\,(15.5 - 3.46/2) = 1946$ k-in = 162.2 k-ft

Ultimate Load of Simple Beams

Plastic Hinge and Ultimate Load

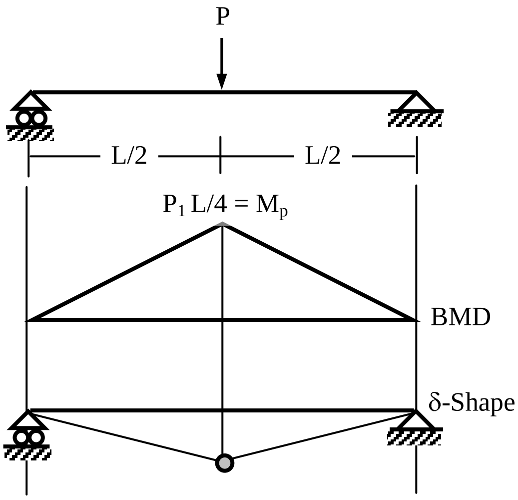
Since Plastic Moment of a section is its ultimate moment capacity, it cannot take any more moment beyond this. As such, the section behaves almost like an internal hinge within a structure. Such a hypothetical internal hinge is called *Plastic Hinge*; and by adding a new equation of statics, it reduces by one the degree of statical indeterminacy of the structure. Therefore, formation of such hinges can make the structure statically determinate, and eventually lead to its instability, which can cause the ultimate collapse of the structure, at the formation of *Collapse Mechanism*.

By calculating the external loads necessary to form such hinges, it is possible to calculate the loads needed to form *Collapse Mechanism* of the structure. This load is called the Ultimate Load of the structure and is important to a designer because it provides information about the load that the structure can possibly sustain, as demonstrated by the following examples.

Example 3

Calculate the ultimate load capacity of the simply supported beams loaded as shown below

[Given: Plastic Moment (M_p) of the section = 150 k-ft, as calculated for the I-section in Example 1].

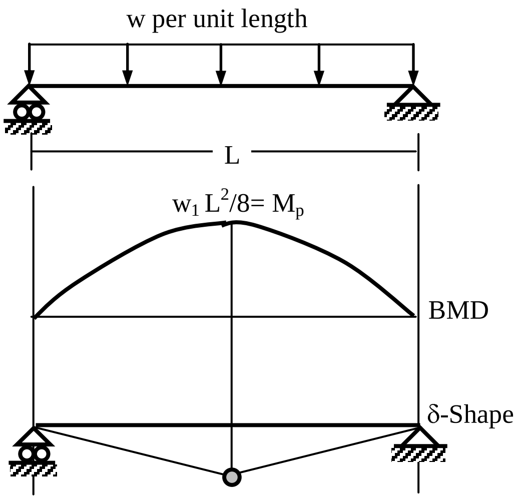


When P = P_1, Plastic Hinge forms at the midspan of the beam at a moment $P_1L/4$.

$\therefore P_1L/4 = M_p \Rightarrow P_1 = 4M_p/L$

$\therefore L = 25'$ and $M_p = 150$ k′

$\Rightarrow P_{ult} = P_1 = 4 \times 150/25 = 24$ k



When w = w_1, Plastic Hinge forms again at the midspan at a moment of $w_1L^2/8$.

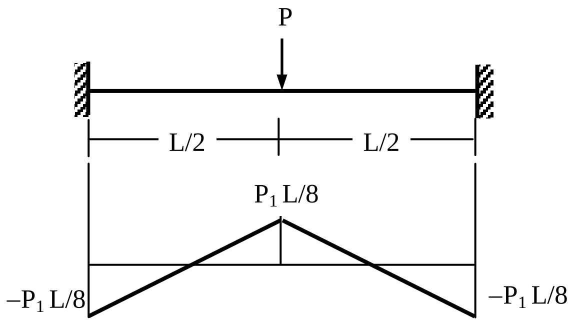
$\therefore w_1L^2/8 = M_p \Rightarrow w_1 = 8M_p/L^2$

$\therefore L = 25'$ and $M_p = 150$ k′

$\Rightarrow w_{ult} = w_1 = 8 \times 150/25^2 = 1.92$ k/ft

Example 4

Calculate the ultimate load capacity of the fixed-ended beams loaded as shown below.

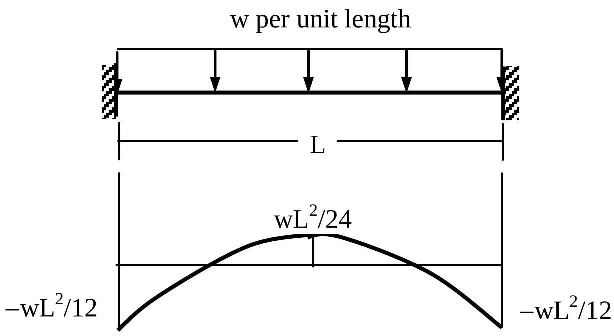


When P = P_1, Plastic Hinges form at both ends and midspan of the beam at moments of $P_1L/8$.

$\therefore P_1L/8 = M_p \Rightarrow P_1 = 8M_p/L$, when a Collapse Mechanism is formed

$\therefore L = 25'$, $M_p = 150$ k′

$\Rightarrow P_{ult} = P_1 = 8 \times 150/25 = 48$ k



When w = w_1, the first Plastic Hinges form at both ends at moments of $w_1L^2/12$.

$\therefore w_1L^2/12 = M_p \Rightarrow w_1 = 12M_p/L^2$

But a Collapse Mechanism is not formed until another hinge forms at midspan at a load w = w_2; i.e., when $w_2L^2/8 - M_p = M_p \Rightarrow w_2 = 16M_p/L^2$

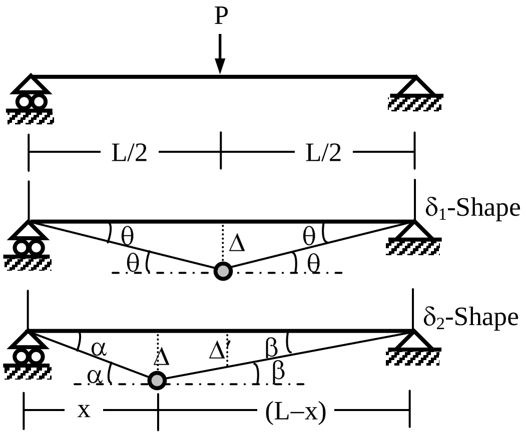
$\therefore L = 25'$, $M_p = 150$ k′ $\Rightarrow w_{ult} = w_2 = 3.84$ k/ft

Energy Formulation of Collapse Mechanism

The calculation of ultimate load capacity based on bending moment diagrams demonstrates the actual sequence of plastic hinge formulation in a structure leading to its ultimate failure. However, it requires the bending moment diagram after each hinge formation, which may not always be convenient to form. A more direct (though not as detailed) calculation of the ultimate load capacity is possible by using the virtual work method on assumed collapse mechanisms of structures. As mentioned in previous formulations, if a system in equilibrium is subjected to virtual displacements δu, the virtual work done by the external forces (δW_E) is equal to the virtual work done by the internal forces (δW_I); i.e., $\delta W_E = \delta W_I$

Example 5

Use Energy Formulation to calculate the ultimate load capacity of the simply supported beams shown below.



For the deflected shape δ_1

External work done = $P\,\Delta$

Internal work done = $M_p(\theta + \theta) = 2M_p\theta$

$\therefore P\,\Delta = 2M_p\,\theta = 2M_p\{\Delta/(L/2)\}$

$\Rightarrow P = 4M_p/L$

For the deflected shape δ_2

External work done = $P\,\Delta' = P\,\beta L/2$

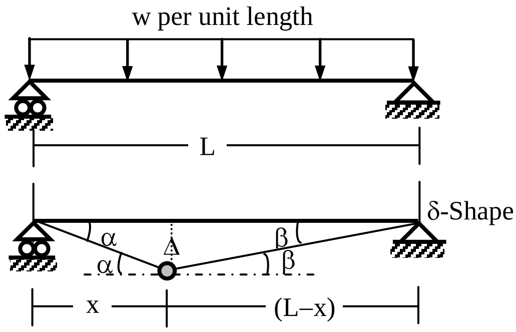
Internal work done = $M_p(\alpha+\beta)$

$\therefore P\,\beta L/2 = M_p(\alpha+\beta)$

$\Rightarrow P\{\Delta/(L-x)\}L/2 = M_p\{\Delta/x + \Delta/(L-x)\}$

$\Rightarrow P = 2M_p/L\{(L/x - 1) + 1\} = 2M_p/x$

$P_{min} = 2M_p/(L/2) = 4M_p/L$



External work done = $wL\,\Delta/2$

Internal work done = $M_p(\alpha+\beta)$

$\therefore wL\,\Delta/2 = M_p(\alpha + \beta) = M_p\{\Delta/x + \Delta/(L-x)\}$

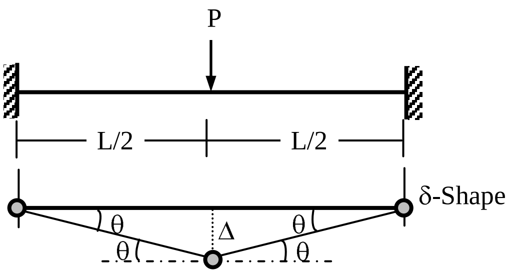
$\Rightarrow w = (2M_p/L)\{1/x + 1/(L-x)\}$

$\partial w/\partial x = 0 \Rightarrow -1/x^2 + 1/(L-x)^2 = 0 \Rightarrow x = L/2$

$\Rightarrow w_{ult} = w_{min} = (2M_p/L^2)\,(2 + 2) = 8M_p/L^2$

Example 6

Use Energy Formulation to calculate the ultimate load capacity of the beams shown below.



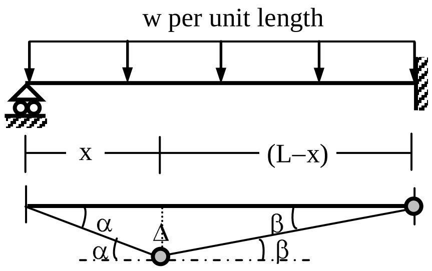
External work done = $P\,\Delta$

Internal work done = $M_p(2\theta) + M_p\theta + M_p\theta$

$= 4M_p\theta$

$\therefore P\,\Delta = 4M_p\,\theta = 2M_p\{\Delta/(L/2)\}$

$\Rightarrow P_{ult} = 8M_p/L$



External work done = $wL\,\Delta/2$

Internal work done = $M_p(\alpha+\beta) + M_p\beta = M_p(\alpha+2\beta)$

$\therefore wL\,\Delta/2 = M_p(\alpha+2\beta) = M_p\{\Delta/x + 2\Delta/(L-x)\}$

$\Rightarrow w = (2M_p/L)\{1/x + 2/(L-x)\}$

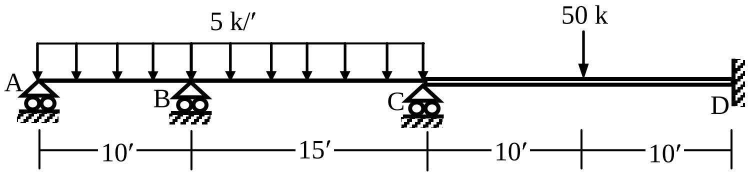
$\partial w/\partial x = 0 \Rightarrow -1/x^2 + 2/(L-x)^2 = 0 \Rightarrow x = L/(\sqrt{2}+1)$

$\Rightarrow w_{ult} = w_{min} = (2M_p/L^2)\{\sqrt{2}+1+2+\sqrt{2}\} = 11.66M_p/L^2$

Ultimate Load of Continuous Beams, Frames

Example 7

Use the Energy Method to calculate the plastic moment M_p needed to prevent the development of plastic hinge mechanism in the beam ABCD loaded as shown below [Given: $M_{p(AB)} = M_{p\,(BC)} = M_p$, $M_{p(CD)} = 2M_p$].



For span AB: $w = 11.66M_{p(AB)}/L^2 \Rightarrow 5 = 11.66\ M_p/10^2 \Rightarrow M_p = 42.88$ k-ft

For span BC: $w = 16M_{p(BC)}/L^2 \Rightarrow 5 = 16M_p/15^2 \Rightarrow M_p = 70.31$ k-ft

For span CD: $P = 8M_{p(CD)}/L \Rightarrow 50 = 8(2M_p)/20 \Rightarrow M_p = 62.50$ k-ft

$\therefore\ M_{p(req)} = 70.31$ k-ft

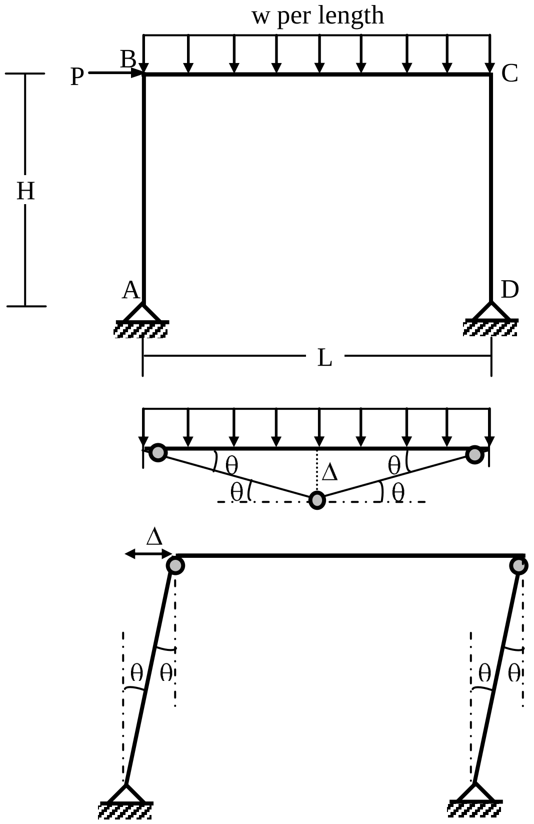
If $M_{p(req)} = 70.31$ k-ft, $w_{(all)AB} = 11.66 \times 70.31/10^2 = 8.20$ k/ft

$w_{(all)BC} = 16 \times 70.31/15^2 = 5.00$ k/ft

$P_{(all)CD} = 8 \times 2 \times 70.31/20 = 56.25$ k

Example 8

Use the Energy Method to calculate the load (i) w needed to form beam mechanism, (ii) P needed to form the sidesway mechanism in the frames ABCD loaded as shown below [Given: $M_{pb} \neq M_{pc}$].

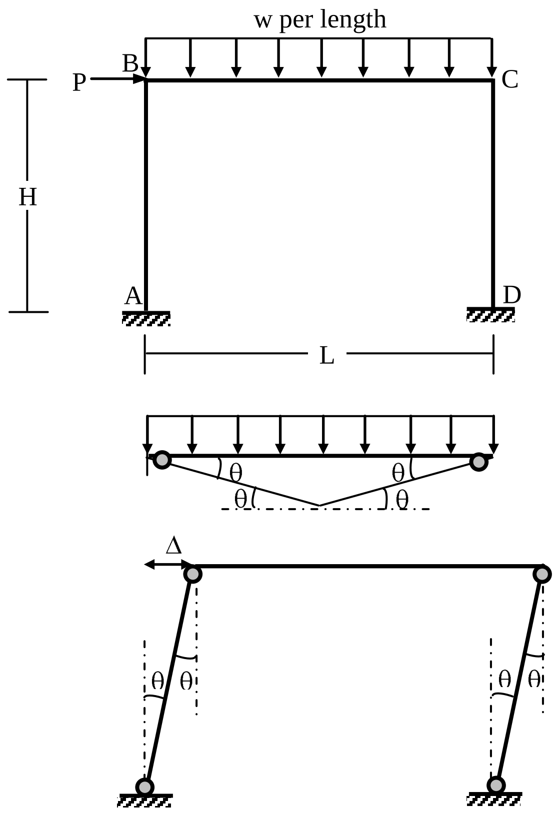


For beam mechanism, $w_{ult} = 16M_{pb}/L^2$

For sidesway mechanism,

$P\Delta = M_{pc}\theta + M_{pc}\theta = 2M_{pc}\theta = 2M_{pc}\Delta/H$

$\therefore P_{ult} = 2M_{pc}/H$



For beam mechanism, $w_{ult} = 16M_{pb}/L^2$

For sidesway mechanism,

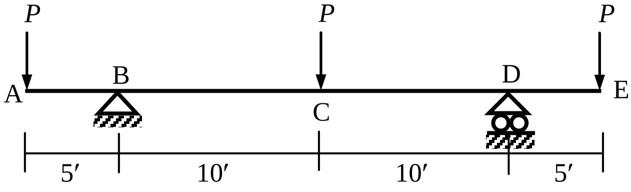
$P\Delta = M_{pc}\theta + M_{pc}\theta + M_{pc}\theta + M_{pc}\theta$

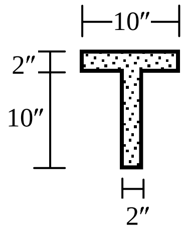
$= 4M_{pc}\theta = 4M_{pc}\Delta/H$

$\therefore P_{ult} = 4M_{pc}/H$

Practice Problems on Material Nonlinearity and Plastic Moment

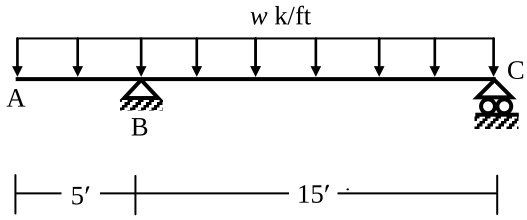
1. Use bending moment diagram of the beam ABCDE loaded as shown below to calculate the force P needed to develop plastic hinge mechanism [Given: σ_{yp} = 40 ksi].

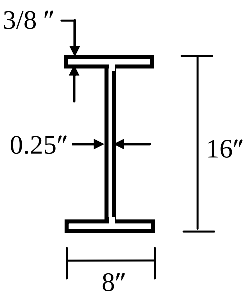




Cross-section of the beam

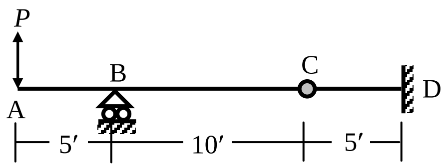
2. Calculate the distributed load w k/ft needed to develop plastic hinge mechanism of the beam ABC loaded as shown below (by using the bending moment diagram) [Given: σ_{yp} = 40 ksi].

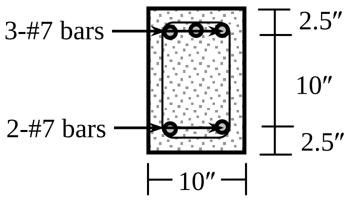




Cross-section of the beam

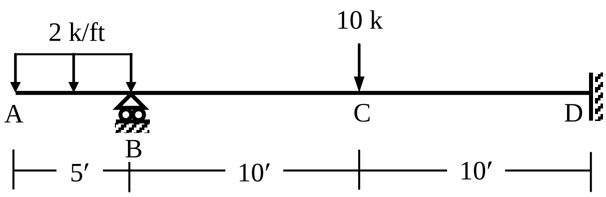
3. Use the bending moment diagram of the reinforced concrete beam ABCD loaded as shown below to calculate the concentrated load P needed to develop plastic hinge mechanism, assuming P to act (i) upward, (ii) downward [Given: f_c' = 3 ksi, f_y = 50 ksi].



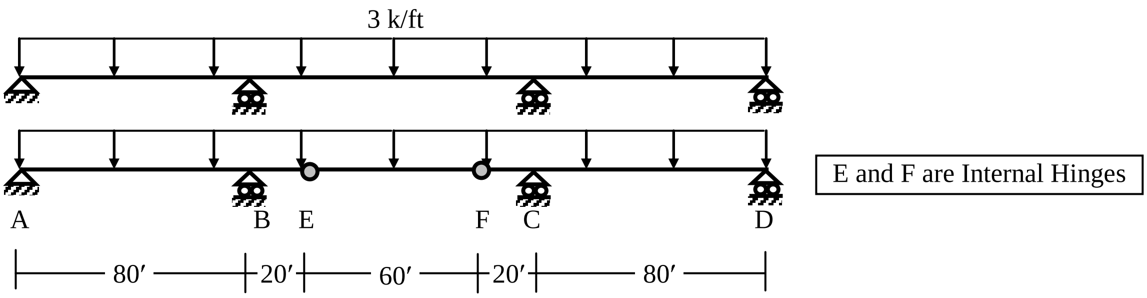


Beam Cross-section

4. Answer Question 1, 2 and 3 using the Energy Method of Collapse Mechanism.
5. Calculate the plastic moment M_p needed to prevent the development of plastic hinge mechanism in the beam ABCD loaded as shown below (by using the Energy Method) [Given: $M_{p(AB)} = M_{p\,(BCD)} = M_p$].



6. Use the Energy Method to calculate the plastic moment M_p of the cross-sections necessary to prevent the development of collapse mechanism in the (i) continuous bridge ABCD, and (ii) balanced cantilever bridge ABEFCD loaded as shown below.



Dynamic Equations of Motion for Lumped Mass Systems

Formulation of the Single-Degree-of-Freedom (SDOF) Equation

A dynamic system resists external forces by a combination of forces due to its stiffness (spring force), damping (viscous force) and mass (inertia force). For the system shown in Fig. 1.1, k is the stiffness, c the viscous damping, m the mass and u(t) is the dynamic displacement due to the time-varying excitation force f(t). Such systems are called Single-Degree-of-Freedom (SDOF) systems because they have only one dynamic displacement [u(t) here].

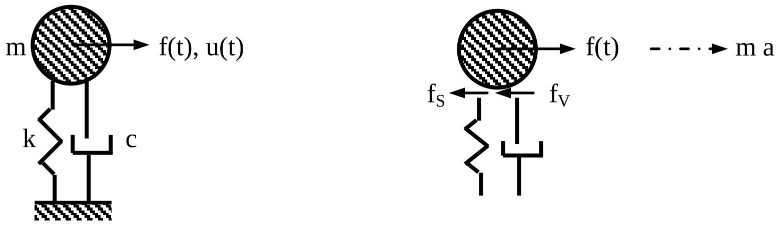


Fig. 1.1: Dynamic SDOF system subjected to dynamic force f(t)

Considering the free body diagram of the system, $f(t) - f_S - f_V = ma$ …………..(1.1)

where f_S = Spring force = Stiffness times the displacement = k u …..………(1.2)

f_V = Viscous force = Viscous damping times the velocity = c du/dt …..………(1.3)

f_I = Inertia force = Mass times the acceleration = $m\, d^2u/dt^2$ ..…………(1.4)

Combining the equations (1.2)-(1.4) with (1.1), the equation of motion for a SDOF system is derived as,

$$m\, d^2u/dt^2 + c\, du/dt + ku = f(t) \qquad (1.5)$$

This is a 2nd order ordinary differential equation (ODE), which needs to be solved in order to obtain the dynamic displacement u(t). As will be shown subsequently, this can be done analytically or numerically.

Eq. (1.5) has several limitations; e.g., it is assumed on linear input-output relationship [constant spring (k) and dashpot (c)]. It is only a special case of the more general equation (1.1), which is an equilibrium equation and is valid for linear or nonlinear systems. Despite these, Eq. (1.5) has wide applications in Structural Dynamics. Several important derivations and conclusions in this field have been based on it.

Governing Equation of Motion for Systems under Seismic Vibration

The loads induced by earthquake are not body-forces; rather it is a ground vibration that induces certain forces in the structure. For the SDOF system subjected to ground displacement $u_g(t)$

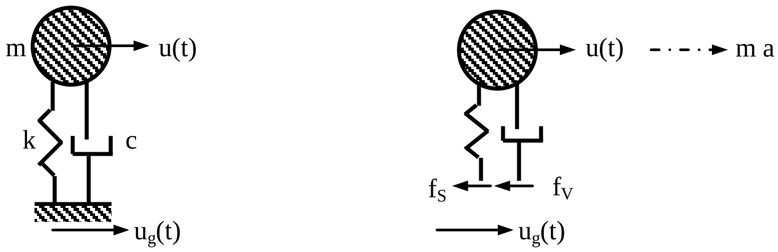


Fig. 1.2: Dynamic SDOF system subjected to ground displacement $u_g(t)$

f_S = Spring force = $k\,(u-u_g)$, f_V = Viscous force = $c\,(du/dt-du_g/dt)$, f_I = Inertia force = $m\, d^2u/dt^2$

Combining the equations, the equation of motion for a SDOF system is derived as,

$$m\, d^2u/dt^2 + c\,(du/dt-du_g/dt) + k\,(u-u_g) = 0 \Rightarrow m\, d^2u/dt^2 + c\, du/dt + k\, u = c\, du_g/dt + k\, u_g \qquad (1.6)$$

$$\Rightarrow m\, d^2u_r/dt^2 + c\, du_r/dt + k\, u_r = -m\, d^2u_g/dt^2 \qquad (1.7)$$

where $u_r = u-u_g$ is the relative displacement of the SDOF system with respect to the ground displacement. Eqs. (1.6) and (1.7) show that the ground motion appears on the right side of the equation of motion just like a time-dependent load. Therefore, although there is no body-force on the system, it is still subjected to dynamic excitation by the ground displacement.

Formulation of the Two-Degrees-of-Freedom (2-DOF) Equation

The simplest extension of the SDOF system is a two-degrees-of-freedom (2-DOF) system, i.e., a system with two unknown displacements for two masses. The two masses may be connected to each other by several spring-dashpot systems, which will lead to two differential equations of motion, the solution of which gives the displacements and internal forces in the system.

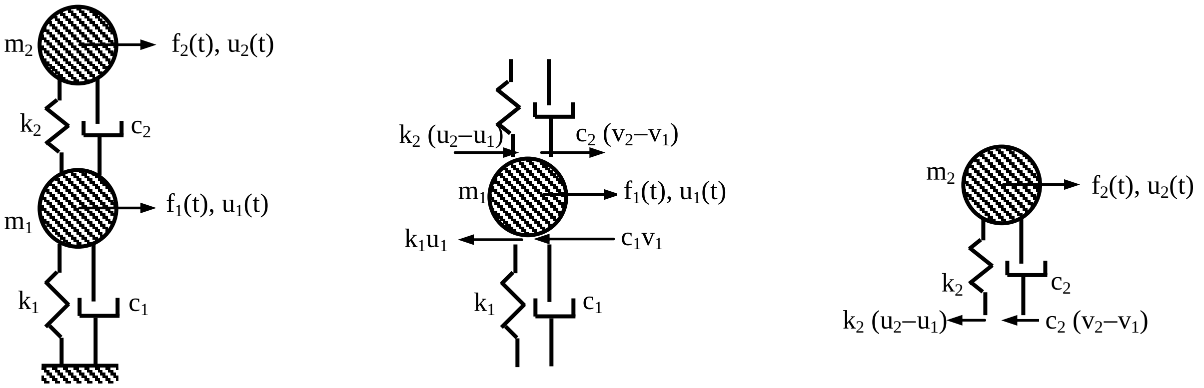


Fig. 2.1: Dynamic 2-DOF system and free body diagrams of m_1 and m_2

Fig. 2.1 shows a 2-DOF dynamic system and the free body diagrams of the two masses m_1 and m_2. In the figure, 'u' stands for displacement (i.e., u_1 and u_2) while 'v' stands for velocity (v_1 and v_2). Denoting accelerations by a_1 and a_2, the differential equations of motion are formed by applying Newton's 2nd law of motion to m_1 and m_2; i.e.,

$$m_1a_1 = f_1(t) + k_2(u_2–u_1) + c_2(v_2–v_1) – k_1u_1 – c_1v_1$$

$$\Rightarrow m_1a_1 + (c_1+c_2)\, v_1 + (k_1+k_2)u_1 – c_2v_2 – k_2u_2 = f_1(t) \qquad \text{……..…(2.1)}$$

and $m_2a_2 = f_2(t) – k_2\, (u_2–u_1) – c_2(v_2–v_1) \Rightarrow m_2a_2 – c_2v_1 + c_2v_2 – k_2u_1 + k_2u_2 = f_2(t)$ ……..…(2.2)

Putting $v = du/dt$ (i.e., $v_1 = du_1/dt$, $v_2 = du_2/dt$) and $a = d^2u/dt^2$ (i.e., $a_1 = d^2u_1/dt^2$, $a_2 = d^2u_2/dt^2$) in Eqs. (2.1) and (2.2), the following equations are obtained

$$m_1\, d^2u_1/dt^2 + (c_1+c_2)\, du_1/dt – c_2\, du_2/dt + (k_1+k_2)\, u_1 – k_2\, u_2 = f_1(t) \qquad \text{………....(2.3)}$$

$$m_2\, d^2u_2/dt^2 – c_2\, du_1/dt + c_2\, du_2/dt – k_2\, u_1 + k_2\, u_2 = f_2\,(t) \qquad \text{…………(2.4)}$$

Eqs. (2.3) and (2.4) can be arranged in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} d^2u_1/dt^2 \\ d^2u_2/dt^2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & –c_2 \\ –c_2 & c_2 \end{bmatrix} \begin{Bmatrix} du_1/dt \\ du_2/dt \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & –k_2 \\ –k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \qquad \text{………....(2.5)}$$

Eqs. (2.5) represent in matrix form the set of equations [i.e. (2.3) and (2.4)] to evaluate the displacements $u_1(t)$ and $u_2(t)$. In this set, the matrix consisting of the masses (m_1 and m_2) is called the *mass matrix*, the one consisting of the dampings (c_1 and c_2) is called the *damping matrix* and the one consisting of the stiffnesses (k_1 and k_2) is called the *stiffness matrix* of this particular system. These matrices are different for various 2-DOF systems, so that Eq. (2.5) cannot be taken as a general form for any 2-DOF system.

For a lumped 2-DOF system subjected to ground displacement $u_g(t)$, velocity $v_g(t)$ and acceleration $a_g(t)$, the following equations are obtained in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} d^2u_1/dt^2 \\ d^2u_2/dt^2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & –c_2 \\ –c_2 & c_2 \end{bmatrix} \begin{Bmatrix} du_1/dt \\ du_2/dt \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & –k_2 \\ –k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} c_1v_g + k_1u_g \\ 0 \end{Bmatrix} \qquad \text{…………(2.6)}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} d^2u_{1r}/dt^2 \\ d^2u_{2r}/dt^2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & –c_2 \\ –c_2 & c_2 \end{bmatrix} \begin{Bmatrix} du_{1r}/dt \\ du_{2r}/dt \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & –k_2 \\ –k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_{1r} \\ u_{2r} \end{Bmatrix} = – \begin{Bmatrix} m_1\, a_g \\ m_2\, a_g \end{Bmatrix} \qquad \text{…………(2.7)}$$

For a MDOF system, Eq. (2.5) can be written in the general form of the dynamic equations of motion,

$$\mathbf{M}\, d^2\mathbf{u}/dt^2 + \mathbf{C}\, d\mathbf{u}/dt + \mathbf{K}\, \mathbf{u} = \mathbf{f}(t) \qquad \text{……….….(2.8)}$$

Numerical Solution of SDOF Equation

The equation of motion for a SDOF system can be solved analytically for different loading functions. Even if the assumptions of linear structural properties are satisfied; the practical loading situations can be more complicated and not convenient to solve analytically. Numerical methods must be used in such situations.

The most widely used numerical approach for solving dynamic problems is the *Newmark-β method*. Actually, it is a set of solution methods with different physical interpretations for different values of β. The total simulation time is divided into a number of intervals (usually of equal duration Δt) and the unknown displacement (as well as velocity and acceleration) is solved at each instant of time. The method solves the dynamic equation of motion in the $(i + 1)^{th}$ time step based on the results of the i^{th} step.

The equation of motion for the $(i +1)^{th}$ time step is

$$m\,(d^2u/dt^2)_{i+1} + c\,(du/dt)_{i+1} + k\,(u)_{i+1} = f_{i+1} \Rightarrow m\,a_{i+1} + c\,v_{i+1} + k\,u_{i+1} = f_{i+1} \qquad \ldots\ldots\ldots\ldots(3.1)$$

where ‘a’ stands for the acceleration, ‘v’ for velocity and ‘u’ for displacement.

To solve for the displacement or acceleration at the $(i + 1)^{th}$ time step, the following equations are assumed for the velocity and displacement at the $(i + 1)^{th}$ step in terms of the values at the i^{th} step.

$$v_{i+1} = v_i + \{(1-\alpha)\,a_i + \alpha\,a_{i+1}\}\Delta t \qquad \ldots\ldots\ldots\ldots(3.2)$$

$$u_{i+1} = u_i + v_i\,\Delta t + \{(0.5-\beta)\,a_i + \beta\,a_{i+1}\}\Delta t^2 \qquad \ldots\ldots\ldots\ldots(3.3)$$

By putting the value of v_{i+1} from Eq. (3.2) and u_{i+1} from Eq. (3.3) in Eq. (3.1), the only unknown variable a_{i+1} can be solved from Eq. (3.1).

In the solution set suggested by the Newmark-β method, the *Constant Average Acceleration (CAA) method* is the most popular because of the stability of its solutions and the simple physical interpretations it provides. This method assumes the acceleration to remain constant during each small time interval Δt, and this constant is assumed to be the average of the accelerations at the two instants of time t_i and t_{i+1}. The CAA is a special case of Newmark-β method where $\alpha = 0.50$ and $\beta = 0.25$. Thus in the CAA method, the equations for velocity and displacement [Eqs. (3.2) and (3.3)] become

$$v_{i+1} = v_i + (a_i + a_{i+1})\Delta t/2 \qquad \ldots\ldots\ldots\ldots(3.4)$$

$$u_{i+1} = u_i + v_i\,\Delta t + (a_i + a_{i+1})\Delta t^2/4 \qquad \ldots\ldots\ldots\ldots(3.5)$$

Inserting these values in Eq. (3.1) and rearranging the coefficients, the following equation is obtained,

$$(m + c\Delta t/2 + k\Delta t^2/4)a_{i+1} = f_{i+1} - ku_i - (c + k\Delta t)v_i - (c\Delta t/2 + k\Delta t^2/4)a_i \qquad \ldots\ldots\ldots\ldots(3.6)$$

$$(m_{eff})\,a_{i+1} = f_{i+1} - ku_i - (c_{eff})\,v_i - (m_{eff1})\,a_i \qquad \ldots\ldots\ldots\ldots(3.6)'$$

To obtain the acceleration a_{i+1} at an instant of time t_{i+1} using Eq. (3.6), the values of u_i, v_i and a_i at the previous instant t_i have to be known (or calculated) before. Once a_{i+1} is obtained, Eqs. (3.4) and (3.5) can be used to calculate the velocity v_{i+1} and displacement u_{i+1} at time t_{i+1}. All these values can be used to obtain the results at time t_{i+2}. The method can be used for subsequent time-steps also.

The simulation should start with two initial conditions, like the displacement u_0 and velocity v_0 at time $t_0 = 0$. The initial acceleration can be obtained from the equation of motion at time $t_0 = 0$ as

$$a_0 = (f_0 - cv_0 - ku_0)/m \qquad \ldots\ldots\ldots\ldots(3.7)$$

Example 3.1
For the undamped SDOF system described before (m = 1 k-sec^2/ft, k = 25 k/ft, c = 0 k-sec/ft), calculate the dynamic response for a *Ramped Step Loading* with p0 = 25 kips and t0 = 0.5 sec [i.e., p(t) = 50 t ≤ 25 kips]

Results using the CAA Method (for time interval Δt = 0.05 sec) as well as the exact analytical equation are shown below in tabular form.

Table 3.1: Acceleration, Velocity and Displacement for Δt = 0.05 sec

m (k-sec^2/ft)	c (k-sec/ft)	k (k/ft)	t_0 (sec)	dt (sec)	m_{eff} (k-sec^2/ft)	c_{eff} (k-sec/ft)	m_{eff1} (k-sec^2/ft)
1.00	0.00	25.00	0.50	0.05	1.0156	1.2500	0.0156

i	t (sec)	f_i (kips)	a_i (ft/sec^2)	v_i (ft/sec)	u_i (ft)	u_{ex} (ft)
0	0.00	0.0	0.0000	0.0000	0.0000	0.0000
1	0.05	2.5	2.4615	0.0615	0.0015	0.0010
2	0.10	5.0	4.7716	0.2424	0.0091	0.0082
3	0.15	7.5	6.7880	0.5314	0.0285	0.0273
4	0.20	10.0	8.3867	0.9107	0.0645	0.0634
5	0.25	12.5	9.4693	1.3571	0.1212	0.1204
6	0.30	15.0	9.9692	1.8431	0.2012	0.2010
7	0.35	17.5	9.8556	2.3387	0.3058	0.3064
8	0.40	20.0	9.1354	2.8135	0.4346	0.4363
9	0.45	22.5	7.8531	3.2382	0.5859	0.5888
10	0.50	25.0	6.0876	3.5867	0.7565	0.7606
11	0.55	25.0	1.4858	3.7760	0.9406	0.9463
12	0.60	25.0	-3.2073	3.7330	1.1283	1.1353
13	0.65	25.0	-7.7031	3.4603	1.3081	1.3159
14	0.70	25.0	-11.7249	2.9746	1.4690	1.4769
15	0.75	25.0	-15.0251	2.3058	1.6010	1.6082
16	0.80	25.0	-17.4007	1.4952	1.6960	1.7017
17	0.85	25.0	-18.7055	0.5925	1.7482	1.7516
18	0.90	25.0	-18.8592	-0.3466	1.7544	1.7547
19	0.95	25.0	-17.8523	-1.2644	1.7141	1.7109
20	1.00	25.0	-15.7468	-2.1044	1.6299	1.6230
21	1.05	25.0	-12.6723	-2.8149	1.5069	1.4962
22	1.10	25.0	-8.8179	-3.3521	1.3527	1.3387
23	1.15	25.0	-4.4209	-3.6831	1.1768	1.1600
24	1.20	25.0	0.2481	-3.7874	0.9901	0.9715
25	1.25	25.0	4.9019	-3.6586	0.8039	0.7846
26	1.30	25.0	9.2540	-3.3048	0.6298	0.6112
27	1.35	25.0	13.0367	-2.7475	0.4785	0.4620
28	1.40	25.0	16.0171	-2.0211	0.3593	0.3462
29	1.45	25.0	18.0118	-1.1704	0.2795	0.2711
30	1.50	25.0	18.8981	-0.2477	0.2441	0.2412
31	1.55	25.0	18.6214	0.6903	0.2551	0.2586
32	1.60	25.0	17.1989	1.5858	0.3120	0.3220
33	1.65	25.0	14.7179	2.3837	0.4113	0.4276
34	1.70	25.0	11.3312	3.0350	0.5468	0.5688
35	1.75	25.0	7.2472	3.4994	0.7101	0.7368
36	1.80	25.0	2.7172	3.7485	0.8913	0.9212
37	1.85	25.0	-1.9800	3.7670	1.0792	1.1105
38	1.90	25.0	-6.5553	3.5536	1.2622	1.2929
39	1.95	25.0	-10.7273	3.1215	1.4291	1.4570
40	2.00	25.0	-14.2391	2.4974	1.5696	1.5928

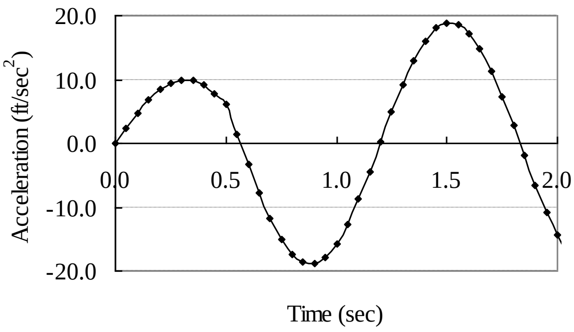


Fig. 3.1: Acceleration vs. Time

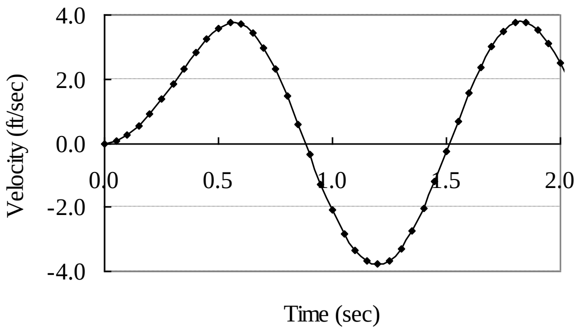


Fig. 3.2: Velocity vs. Time

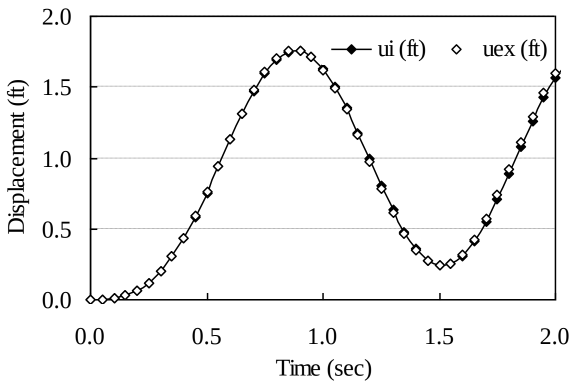


Fig. 3.3: Displacement vs. Time

Stiffness and Mass Matrices of Continuous Systems

Axial Members

Applying the method of virtual work to undamped members subjected to axial load of p(x,t) per unit length,

$\delta W_I = \delta W_E \Rightarrow \int m\, dx\, d^2u/dt^2\, \delta u + \int u'\, E\, A\, \delta u'\, dx = \int p(x,t)\, dx\, \delta u$ ……………….….(4.1)

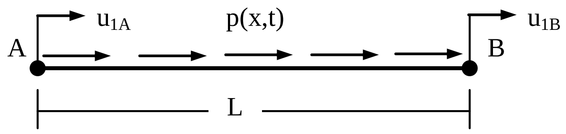


Fig. 4.1: Axially Loaded Member

If the displacements of a member AB (Fig. 4.1) are assumed to be interpolating functions [$\phi_1(x)$ and $\phi_2(x)$] of two nodal displacements u_{1A} and u_{1B},

$u = u_{1A}\, \phi_1 + u_{1B}\, \phi_2 \Rightarrow u' = u_{1A}\, \phi_1' + u_{1B}\, \phi_2'$ ….…….…...………(4.2), (4.3)

$d^2u/dt^2 = d^2u_{1A}/dt^2\, \phi_1 + d^2u_{1B}/dt^2\, \phi_2$ ……………………..……(4.4)

$\delta u = \delta u_{1A}\, \phi_1 + \delta u_{1B}\, \phi_2 \Rightarrow \delta u' = \delta u_{1A}\, \phi_1' + \delta u_{1B}\, \phi_2'$ …………...……….(4.5), (4.6)

$\therefore$ Eq. (4.1) can be written in matrix form as,

$$\begin{pmatrix} \int m\, \phi_1\, \phi_1 dx & \int m\, \phi_1\, \phi_2 dx \\ \int m\, \phi_2\, \phi_1 dx & \int m\, \phi_2\, \phi_2 dx \end{pmatrix} \begin{Bmatrix} d^2u_{1A}/dt^2 \\ d^2u_{1B}/dt^2 \end{Bmatrix} + \begin{pmatrix} \int EA\, \phi_1'\phi_1' dx & \int EA\, \phi_1'\phi_2' dx \\ \int EA\, \phi_2'\phi_1' dx & \int EA\, \phi_2'\phi_2' dx \end{pmatrix} \begin{Bmatrix} u_{1A} \\ u_{1B} \end{Bmatrix} = \begin{Bmatrix} \int p(x,t)\phi_1 dx \\ \int p(x,t)\phi_2\, dx \end{Bmatrix}$$

...…...…..(4.7)

For concentrated loads p(x,t) is a delta function of x, as mentioned before. If loads X_A and X_B are applied at joints A and B, they can be added to the right side of Eq. (4.7).

Eq. (4.7) can be rewritten as, $\mathbf{M_m}\, d^2\mathbf{u_m}/dt^2 + \mathbf{K_m}\, \mathbf{u_m} = \mathbf{f_m}$ ………………(4.8)

where $\mathbf{M_m}$ and $\mathbf{K_m}$ are the mass and stiffness matrices of the member respectively, while $d^2\mathbf{u_m}/dt^2$, $\mathbf{u_m}$ and $\mathbf{f_m}$ are the member acceleration, displacement and load vectors. They can be formed once the shape functions ϕ_1 and ϕ_2 are known or assumed.

$M_{mij} = \int m\, \phi_i\, \phi_j\, dx$, and $K_{mij} = \int EA\, \phi_i'\phi_j'\, dx$ ………………………(4.9)

Flexural Members

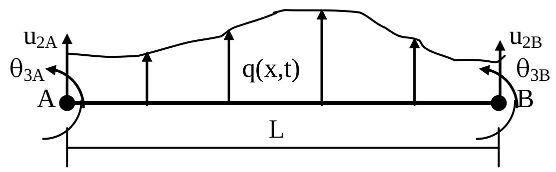


Fig. 4.2: Transversely Loaded Member

Applying the method of virtual work to undamped members subjected to flexural load of q(x,t) per unit length $\Rightarrow \int m\, dx\, d^2u/dt^2\, \delta u + \int u''E\, I\, \delta u''\, dx = \int q(x,t)\, dx\, \delta u$ …….…………….(4.10)

Following the same type of formulation as for axial members, the member equations for undamped flexural members subjected to transverse load of q(x,t) per unit length (Fig. 4.2) can be written in matrix form like Eq. (4.8), but the member matrices are different here.

Interpolation functions for u_{2A}, θ_{3A}, u_{2B} and θ_{3B} are typically chosen in such cases, so that

$u(x) = u_{2A}\, \psi_1 + \theta_{3A}\, \psi_2 + u_{2B}\, \psi_3 + \theta_{3B}\, \psi_4$ ………..………………(4.11)

The size of the matrices is (4×4) here, due to transverse joint displacements (u_{2A}, u_{2B}) joint rotations (θ_{3A}, θ_{3B}) and their elements are given by

$M_{mij} = \int m\, \psi_i\, \psi_j\, dx$, and $K_{mij} = \int EI\, \psi_i''\, \psi_j''\, dx$ ………………………(4.12)

Example 4.1

For modulus of elasticity E = 450000 ksf, cross-sectional area A = 1 ft^2, length L = 10 ft, mass per length m = 0.0045 k-sec^2/ft^2, calculate the natural frequencies of a cantilever beam in axial direction, analyzing with (i) one lumped-mass element, (ii) one consistent-mass element, (iii) two lumped-mass elements.

Solution

(i) For lumped-mass elements

$$\mathbf{M_m} = \begin{pmatrix} mL/2 & 0 \\ 0 & mL/2 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} EA/L & -EA/L \\ -EA/L & EA/L \end{pmatrix}$$

Assuming one linear element with properties mentioned, mL/2 = 0.0225 k-sec^2/ft, EA/L = 45000 k/ft

$$\therefore \mathbf{M_m} = \begin{pmatrix} 0.0225 & 0 \\ 0 & 0.0225 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} 45000 & -45000 \\ -45000 & 45000 \end{pmatrix}$$

Applying the boundary conditions that the only non-zero DOF is the axial deformation at B (u_{1B}), the mass and stiffness matrices are reduced to (1×1) matrices $\mathbf{M} = 0.0225$, $\mathbf{K} = 45000$

$$\therefore |\,\mathbf{K} - \omega_n^2\,\mathbf{M}\,| = 0 \Rightarrow 45000 - \omega_n^2\, 0.0225 = 0 \Rightarrow \omega_n^2 = 2 \times 10^6 \Rightarrow \omega_n = 1414 \text{ rad/sec}$$

(ii) For linear functions $\phi_1(x) = 1 - x/L$, $\phi_2(x) = x/L$, the mass and stiffness matrices obtained from Eq. (4.7)

$$\mathbf{M_m} = \begin{pmatrix} mL/3 & mL/6 \\ mL/6 & mL/3 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} EA/L & -EA/L \\ -EA/L & EA/L \end{pmatrix}$$

Assuming one linear element with properties mentioned, mL/3 = 0.015 k-sec^2/ft, EA/L = 45000 k/ft

$$\therefore \mathbf{M_m} = \begin{pmatrix} 0.0150 & 0.0075 \\ 0.0075 & 0.0150 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} 45000 & -45000 \\ -45000 & 45000 \end{pmatrix}$$

Applying the boundary conditions that the only non-zero DOF is the axial deformation at B (u_{1B}), the mass and stiffness matrices are reduced to (1×1) matrices $\mathbf{M} = 0.015$, $\mathbf{K} = 45000$

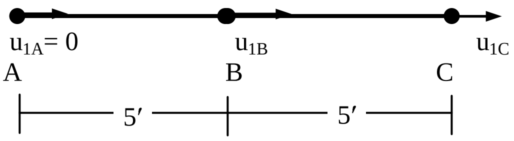
$$\therefore |\,\mathbf{K} - \omega_n^2\,\mathbf{M}\,| = 0 \Rightarrow 45000 - \omega_n^2\, 0.015 = 0 \Rightarrow \omega_n^2 = 3 \times 10^6 \Rightarrow \omega_n = 1732 \text{ rad/sec}$$

(iii) For two lumped-mass elements of length 5′ each, mL/2 = 0.01125 k-sec^2/ft, EA/L = 90000 k/ft

The following mass and stiffness matrices are obtained for each element

$$\mathbf{M_m} = \begin{pmatrix} 0.01125 & 0 \\ 0 & 0.01125 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} 90000 & -90000 \\ -90000 & 90000 \end{pmatrix}$$

Applying the boundary conditions that axial deformation at A (u_{1A}) is zero, only the axial deformations at B (u_{1B}) and C (u_{1C}) are non-zero, the mass and stiffness matrices are reduced to (2×2) matrices.



$$\mathbf{M} = \begin{pmatrix} 0.01125 & 0 \\ 0 & 0.0225 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 90000 & -90000 \\ -90000 & 180000 \end{pmatrix}$$

$$\therefore |\,\mathbf{K} - \omega_n^2\,\mathbf{M}\,| = 0 \Rightarrow (90000 - \omega_n^2\, 0.01125)(180000 - \omega_n^2\, 0.0225) - (-90000)^2 = 0$$

$$\Rightarrow \omega_n = 1531 \text{ rad/sec}, 3696 \text{ rad/sec}$$

Analytical solutions for the first two natural frequencies are 1571 rad/sec, 4712 rad/sec respectively.

Example 4.2

For the member properties E = 450000 ksf, I = 0.08 ft^4, L = 10 ft, m = 0.0045 k-sec^2/ft^2, calculate the approximate first natural frequency of the cantilever beam in transverse direction, analyzing with
(i) one lumped-mass element, (ii) one consistent-mass element.

Solution

(i) For cubic polynomial functions

$\psi_1(x) = 1-3(x/L)^2+2(x/L)^3$, $\psi_2(x) = x\{1-(x/L)\}^2$, $\psi_3(x) = 3(x/L)^2-2(x/L)^3$, $\psi_4(x) = (x-L)(x/L)^2$

with lumped mass mL/2 at both ends and constant EI, the following matrices are obtained from Eq. (4.12)

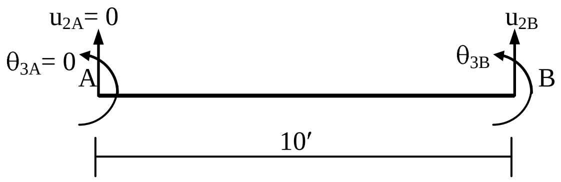
$$\mathbf{M_m} = (mL/2)\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} \qquad \mathbf{K_m} = (EI/L^3)\begin{pmatrix}12&6L&-12&6L\\6L&4L^2&-6L&2L^2\\-12&-6L&12&-6L\\6L&2L^2&-6L&4L^2\end{pmatrix}$$

$\therefore$ In this case, mL = 0.045 k-sec^2/ft, EI/L^3 = 36 k/ft

$$\mathbf{M_m} = 0.0225\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} \qquad \mathbf{K_m} = 36\begin{pmatrix}12&60&-12&60\\60&400&-60&200\\-12&-60&12&-60\\60&200&-60&400\end{pmatrix}$$

Applying the boundary conditions that the only non-zero degrees of freedom are the vertical deflection and rotation at B (u_{2B} and θ_{3B}), the mass and stiffness matrices are reduced to (2 × 2) matrices

$$\mathbf{M} = 0.0225\begin{pmatrix}1&0\\0&0\end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix}432&-2160\\-2160&14400\end{pmatrix}$$



$\therefore |\mathbf{K} - \omega_n^2\,\mathbf{M}| = 0 \Rightarrow (432 - \omega_n^2\, 0.0225)\, 14400 - (-2160)^2 = 0 \Rightarrow \omega_n = 69.28$ rad/sec

(ii) For cubic polynomial functions with uniform m

$$\mathbf{M_m} = (mL/420)\begin{pmatrix}156&22L&54&-13L\\22L&4L^2&13L&-3L^2\\54&13L&156&-22L\\-13L&-3L^2&-22L&4L^2\end{pmatrix} = 1.071\times10^{-4}\begin{pmatrix}156&220&54&-130\\220&400&130&-300\\54&130&156&-220\\-130&-300&-220&400\end{pmatrix}$$

Applying the boundary conditions, the mass and stiffness matrices are reduced to

$$\mathbf{M} = 1.071\times10^{-4}\begin{pmatrix}156&-220\\-220&400\end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix}432&-2160\\-2160&14400\end{pmatrix}$$

$\therefore |\mathbf{K} - \omega_n^2\,\mathbf{M}| = 0 \Rightarrow (432 - \omega_n^2\, 0.01671)(14400 - \omega_n^2\, 0.04286) - (-2160 + \omega_n^2\, 0.02357)^2 = 0$

$\Rightarrow \omega_n = 99.92$ rad/sec, 984.49 rad/sec

The exact results for the first two natural frequencies are 99.45 rad/sec and 623.10 rad/sec respectively. Therefore, as was the case for axial vibrations, the natural frequencies are under-estimated for lumped-mass element and over-estimated for consistent-mass element.

Dynamic Analysis of Trusses and Frames

Two-dimensional Trusses

The mass and stiffness matrices derived for axially loaded members can be used for the dynamic analysis of two-dimensional trusses. One difference is that here the transverse displacements (u_{2A}, u_{2B}) are also considered in forming the matrices, so that the size of the matrices is (4 × 4) instead of (2 × 2).

$$\underset{\text{(Lumped)}}{\mathbf{M_m}^{\mathbf{L}}} = \begin{pmatrix} mL/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & mL/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \underset{\text{(Consistent)}}{\mathbf{M_m}^{\mathbf{L}}} = \begin{pmatrix} mL/3 & 0 & mL/6 & 0 \\ 0 & 0 & 0 & 0 \\ mL/6 & 0 & mL/3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \ldots\ldots\ldots(5.1)$$

The member matrices formed in the local axes system by Eq. (5.1) can be transformed into the global axes system by considering the angles they make with the horizontal.

$$\mathbf{M_m}^{\mathbf{G}} = (mL/2) \begin{pmatrix} \vartheta & \mathbf{0} \\ \mathbf{0} & \vartheta \end{pmatrix} \quad \text{or} \quad \mathbf{M_m}^{\mathbf{G}} = (mL/3) \begin{pmatrix} \vartheta & \vartheta/2 \\ \vartheta/2 & \vartheta \end{pmatrix} \quad \ldots\ldots\ldots(5.2)$$

where ϑ is a (2 × 2) matrix of coefficients given by

$$\vartheta = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5.3)$$

The mass and stiffness matrices (from previous formulations) and load vector of the whole structure can be assembled from the member matrices and vector ($\mathbf{M_m}^{\mathbf{G}}$, $\mathbf{K_m}^{\mathbf{G}}$ and $\mathbf{f_m}^{\mathbf{G}}$). They are obtained in their final forms only after applying appropriate boundary conditions.

Two-dimensional Frames

The matrices formed for flexural members and already used for a cantilever beam can be used for the dynamic analysis of two-dimensional frames. The elements of the i^{th} row and j^{th} column of the mass and stiffness matrices are given by Eq. (4.12) in integral form and can be evaluated once the shape functions ψ_i and ψ_j are known or assumed [as shown in Example 4.2]. However, the axial displacements of joints (u_{1A}, u_{1B}) are also considered for frames in addition to the transverse displacements (u_{2A}, u_{2B}) and rotations (θ_{3A}, θ_{3B}) about the out-of-plane axis considered in forming the matrices for beams, so that the size of the matrices is (6 × 6) instead of the (4 × 4) matrices shown for beams.

If shape functions of Example 4.2 are assumed for frame members of uniform cross-section, the member mass and stiffness matrices take the following forms in the local axes system

$$\underset{\text{(Lumped)}}{\mathbf{M_m}^{\mathbf{L}}} = (mL/2) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \underset{\text{(Consistent)}}{\mathbf{M_m}^{\mathbf{L}}} = (mL/420) \begin{pmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{pmatrix} \quad \ldots\ldots\ldots(5.4)$$

Denoting the global structural matrices by **M** and **K** respectively and assuming appropriate damping ratios, the damping matrix **C** can be obtained as,

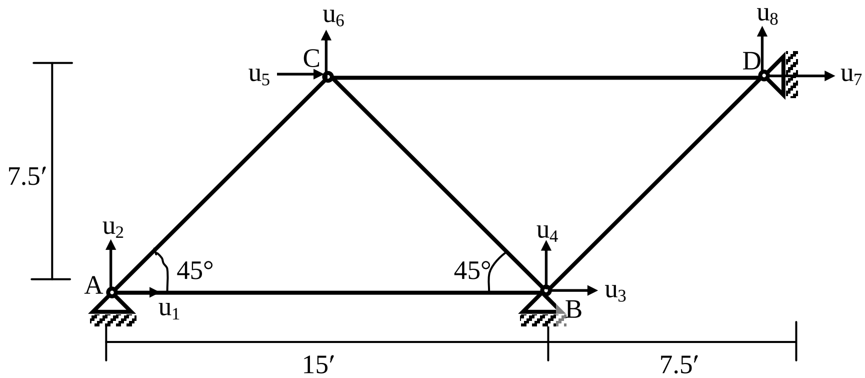
$$\mathbf{C} = a_0\,\mathbf{M} + a_1\,\mathbf{K} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5.5)$$

The dynamic analysis can be carried out once these matrices and vector are formed.

Example 5.1

For the plane truss shown below, modulus of elasticity E = 30000 ksi, cross-sectional area A = 2 in^2, mass per length m = 1.5×10^{-6} k-sec^2/in^2. Calculate its natural frequencies using consistent mass matrices.

Solution



The truss has 8 DOF. The displacements u_1~u_4 and u_7, u_8 are restrained, so that only two DOF (u_5, u_6) are non-zero. There are five members in the truss (including two zero-force members), all with the same cross-sectional properties, but different lengths. The member mass and stiffness matrices can be obtained from

$$\mathbf{M_m}^G = (mL/3)\begin{pmatrix} C^2 & CS & C^2/2 & CS/2 \\ CS & S^2 & CS/2 & S^2/2 \\ C^2/2 & CS/2 & C^2 & CS \\ CS/2 & S^2/2 & CS & S^2 \end{pmatrix} \qquad \mathbf{K_m}^G = (EA/L)\begin{pmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{pmatrix}$$

For member AB, C = 1, S = 0, L = 15′ = 180″, $\therefore$ mL/3 = 9.0×10^{-5} k-sec^2/in, EA/L = 333.33 k/in

$$\mathbf{M_{AB}}^G = 9.0\times10^{-5}\begin{pmatrix} 1.0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K_{AB}}^G = 333.33\begin{pmatrix} 1.0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

DOF [1 2 3 4]

The matrices for AB and CD are the same, but the latter connects displacements 5, 6, 7 and 8

For member AC, C = 0.707, S = 0.707, L = 10.607′ = 127.28″

$\therefore$ mL/3 = 6.37×10^{-5} k-sec^2/in, EA/L = 471.41 k/in

$$\mathbf{M_{AC}}^G = 6.37\times10^{-5}\begin{pmatrix} 0.5 & 0.5 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 & 0.5 \end{pmatrix} \qquad \mathbf{K_{AC}}^G = 471.41\begin{pmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{pmatrix}$$

[1 2 5 6]

The matrices for AC and BD are the same, but the latter connects displacements 3, 4, 7 and 8

For member BC, C = -0.707, S = 0.707, L = 10.607′ = 127.28″

$\therefore$ mL/3 = 6.37×10^{-5} k-sec^2/in, EA/L = 471.41 k/in

$$\mathbf{M_{BC}}^G = 6.37\times10^{-5}\begin{pmatrix} 0.5 & -0.5 & 0.25 & -0.25 \\ -0.5 & 0.5 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.5 & -0.5 \\ 0.25 & -0.25 & -0.5 & 0.5 \end{pmatrix} \qquad \mathbf{K_{BC}}^G = 471.41\begin{pmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix}$$

[3 4 5 6]

Applying boundary conditions, the mass and stiffness matrices for the whole truss can be assembled as

$$\mathbf{M} = 10^{-5}\begin{pmatrix} 15.37 & 0 \\ 0 & 6.37 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 804.74 & 0 \\ 0 & 471.41 \end{pmatrix}$$

$\therefore |\,\mathbf{K} - \omega_n^2\,\mathbf{M}\,| = 0 \Rightarrow (804.74 - \omega_n^2\,0.0001537)\,(471.41 - \omega_n^2\,0.0000637) = 0$

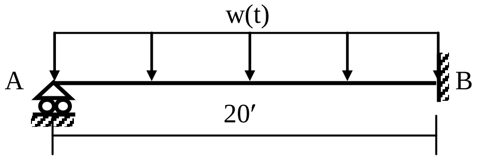
$\Rightarrow \omega_n$ = 2288 rad/sec, 2720 rad/sec

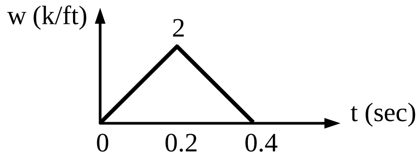
Practice Problems on Structural Dynamics

1. A SDOF system with k = 10 k/ft, m = 1 k-sec^2/ft, c = 0.5 k-sec/ft is subjected to a force (in kip) given by (i) p(t) = 50, (ii) p(t) = 100 t, (iii) p(t) = 50 cos(3t). In each case, use the CAA method to calculate the displacement of the system at time t = 0.10 seconds, if the initial displacement and velocity are both zero.

2. For a (20′ × 20′ × 20′) overhead water tank supported by a (25″ × 25″) square column, use the CAA method (c = 0) to calculate the displacement at time t = 0.20 seconds, when subjected to
(i) a sustained wind pressure of 40 psf, (ii) a harmonic wind pressure of 40 cos(2t) psf [use k = 3EI/L^3]. Assume the total weight of the system to be concentrated in the tank, and initial displacement and velocity are both zero [Given: E of concrete = 400 × 10^3 k/ft^2, Unit weight of water = 62.5 lb/ft^3].

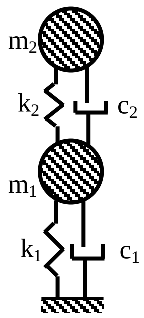
3. For beam AB loaded as shown below, use the CAA method to calculate rotation at A at time t = 0.10 sec (starting with zero initial displacement and velocity) [Given: EI = 40 × 10^3 k-ft^2, m = 0.0045 k-sec^2/ft^2].





4. Calculate the natural frequencies and periods of the structures shown below (in axial/transverse vibration)

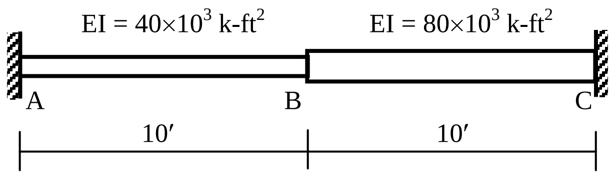
(i)



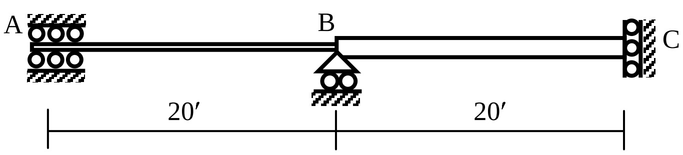
$m_1 = m_2 = 1$ k-sec^2/ft
$k_1 = k_2 = 10$ k/ft

(ii) A 10-ft long simply supported beam with EI = 40 × 10^3 k-ft^2, m = 0.005 k-sec^2/ft^2.

(iii) Given: EA_{AB} = 400 × 10^3 k, EA_{BC} = 800 × 10^3 k; EI_{AB} = 40 × 10^3 k-ft^2, EI_{BC} = 80 × 10^3 k-ft^2;
m_{AB} = 0.005 k-sec^2/ft^2, m_{BC} = 0.010 k-sec^2/ft^2.



(iv) Given: EA_{AB} = 200 × 10^3 k, EA_{BC} = 400 × 10^3 k; EI_{AB} = 20 × 10^3 k-ft^2, EI_{BC} = 40 × 10^3 k-ft^2;
m_{AB} = 0.005 k-sec^2/ft^2, m_{BC} = 0.010 k-sec^2/ft^2.



A and C are guided roller supports

Structures on Flexible Foundations

Rather than idealized support conditions (i.e., roller, hinged or fixed), it is more rational to assume structures to be supported on flexible supports. In addition to real springs, foundations on flexible supports (e.g., columns) or soils can also be modeled by springs for horizontal, vertical displacements, as well as bending and torsional rotations.

For example, if $EI = 80 \times 10^3$ k-ft^2, the Stiffness Matrix for the beam in Fig. 1.1 is $S_3 = 16 \times 10^3$ k-ft/rad

$\therefore$ The stiffness formulation is, $S_3\,\theta_A = -\,P_0L/8 \Rightarrow 16 \times 10^3\,\theta_A = -25 \Rightarrow \theta_A = -1.56 \times 10^{-3}$ rad

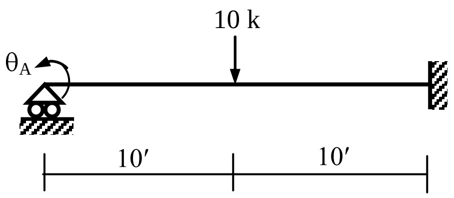


Fig. 1.1

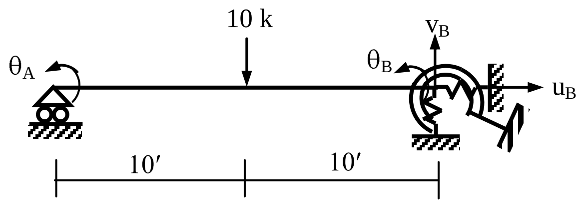


Fig. 1.2

If springs of stiffness K_h, K_v and K_θ replace the 'fixed' support (Fig. 1.2), the stiffness formulation becomes

$$\begin{pmatrix} S_3 & 0 & -S_2 & S_4 \\ 0 & S_x + K_h & 0 & 0 \\ -S_2 & 0 & S_1 + K_v & -S_2 \\ S_4 & 0 & -S_2 & S_3 + K_\theta \end{pmatrix} \begin{Bmatrix} \theta_A \\ u_B \\ v_B \\ \theta_B \end{Bmatrix} = -\begin{Bmatrix} P_0L/8 \\ 0 \\ P_0/2 \\ -P_0L/8 \end{Bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(1.1)$$

Stiffness of Circular Foundations and Long Pile Foundations

Motion	$K_{Halfspace}$	K_{Embed}	K_{Pile}
h	$8GR/(2\text{-}\nu)$	$4\,G'E$	$4\,G^{0.75}\,(E_pI_p)^{0.25}$
v	$4GR/(1\text{-}\nu)$	$2.75\,G'E$	$1.5\,G^{0.5}\,(E_pA_p)^{0.5}$
θ	$8GR^3/(3\text{-}3\nu)$	$[8+4(E/R)^2]\,G'ER^2/3$	$2\,G^{0.25}\,(E_pI_p)^{0.75}$
t	$16GR^3/3$	$12\,G'ER^2$	$3\,R\,G^{0.5}\,(E_pJ_p)^{0.5}$

[h for horizontal, v for vertical , θ for bending and t for torsional motion]

If $EA = 800 \times 10^3$ k, $S_x = 40 \times 10^3$ k/ft, $S_1 = 120$ k/ft, $S_2 = 1200$ k-ft/ft, $S_4 = 8 \times 10^3$ k-ft/rad

G = Shear modulus of sub-soil, R = Radius of circular foundation, ν = Poisson's ratio

$K_h = 8GR/(2-\nu)$, $K_v = 4GR/(1-\nu)$, $K_\theta = 8GR^3/(3-3\nu)$

Assuming shear-wave velocity $v_s = 1000$ ft/s, $G = \rho_s v_s^2 = (0.12/32.2) \times (1000)^2 = 3.73 \times 10^3$ k/ft^2

$\therefore R = 2$ ft, $\nu = 0.30 \Rightarrow K_h = 8GR/(2\text{-}\nu) = 35.08 \times 10^3$ k/ft, $K_v = 4GR/(1\text{-}\nu) = 42.59 \times 10^3$ k/ft,

$K_\theta = 8GR^3/(3\text{-}3\nu) = 113.58 \times 10^3$ k-ft/rad

$$10^3 \begin{pmatrix} 16.0 & 0 & -1.2 & 8.0 \\ 0 & 75.07 & 0 & 0 \\ -1.2 & 0 & 42.71 & -1.2 \\ 8.0 & 0 & -1.2 & 129.58 \end{pmatrix} \begin{Bmatrix} \theta_A \\ u_B \\ v_B \\ \theta_B \end{Bmatrix} = -\begin{Bmatrix} 25 \\ 0 \\ 5 \\ -25 \end{Bmatrix} \Rightarrow \boxed{\begin{matrix} \theta_A = -1.721 \times 10^{-3}\text{ rad} \\ u_B = 0 \\ v_B = -0.133 \times 10^{-3}\text{ ft} \\ \theta_B = 0.298 \times 10^{-3}\text{ rad} \end{matrix}}$$

These values may vary significantly with the stiffness(es) of foundation, which are directly proportional to the value of G (shear modulus of sub-soil), which in turn depends on the shear-wave velocity v_s of sub-soil. Table below shows the variation of θ_A, v_B and θ_B with v_s.

v_s (ft/s)	θ_A (10^{-3} rad)	v_B (10^{-3} ft)	θ_B (10^{-3} rad)
∝	1.563	0	0
1000	1.721	–0.133	0.298
300	2.476	–1.138	1.657
100	3.372	–7.325	2.520

Effect on Dynamic Properties

The effect of foundation flexibility on the dynamic properties of a structural system can be illustrated by a simple analysis of a 2-DOF system with the equations of motion in matrix form

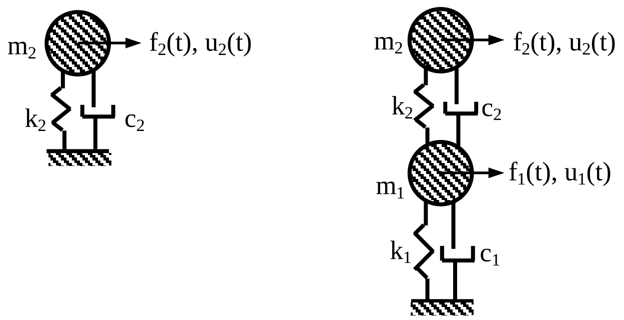


Fig. 1.3: 'Fixed-based' and 'Flexible-based' foundation-structure systems

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{Bmatrix} d^2u_1/dt^2 \\ d^2u_2/dt^2 \end{Bmatrix} + \begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix} \begin{Bmatrix} du_1/dt \\ du_2/dt \end{Bmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

……………..(1.2)

If u_1 and u_2 are the horizontal displacements at the foundation and 1st floor level and the foundation is assumed massless (i.e., $m_1 = 0$) but to consist of a spring k_1 and dashpot c_1 [Fig. 1.3], Eq. (1.2) reduces to

$$\begin{pmatrix} 0 & 0 \\ 0 & m_2 \end{pmatrix} \begin{Bmatrix} d^2u_1/dt^2 \\ d^2u_2/dt^2 \end{Bmatrix} + \begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix} \begin{Bmatrix} du_1/dt \\ du_2/dt \end{Bmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

………..……(1.3)

Therefore, the natural frequencies of the system can be calculated from $(k_1 + k_2 - 0)(k_2 - \omega_n^2 m_2) - (-k_2)^2 = 0$

$\Rightarrow \omega_n = \sqrt{\{k_1 k_2/(k_1+k_2)/m_2\}} = \sqrt{\{k_2/m_2/(1+k_2/k_1)\}}$ ……………….(1.4)

Since there is only one mass in the system, the foundation-structure system reduces to a SDOF system. The natural frequency given by Eq. (1.4) is less than the 'fixed-based' frequency [$\omega_n = \sqrt{(k_2/m_2)}$] of the system.

Moreover, instead of the 'fixed-based' damping ratio $\xi = c_2/2\sqrt{(k_2 m_2)}$, the damping ratio now

$\xi = c_2/2\sqrt{\{k_2 m_2 (1+k_2/k_1)^3\}} + c_1/2\sqrt{\{k_1 m_2 (1+k_1/k_2)^3\}}$ ………………(1.5)

This simple illustration shows some important features of foundation flexibility

(i) Natural frequency of the structure is reduced.
(ii) The damping ratio of the structure may increase or decrease.
(iii) Whether it is beneficial or harmful to the structure depends on the frequency of applied loads.

For example, if $k_1 = k_2 = 10$ k/ft, $m_1 = 0$, $m_2 = 1$ k-sec^2/ft, $c_1 = c_2 = 0.316$ k-sec/ft

ω_n for 'fixed-based' system = $\sqrt{\{k_2/m_2\}}$ = 3.16 rad/sec

ω_n for 'flexible-based' system = $\sqrt{\{k_1k_2/(k_1+k_2)/m_2\}}$ = 2.24 rad/sec

ξ for 'fixed-based' system = $c_2/2\sqrt{(k_2 m_2)}$ = 0.05

ξ for 'flexible-based' system = $0.316/2\sqrt{(80)} + 0.316/2\sqrt{(80)} = 0.035$

Short Questions and Explanations

Stiffness Method for 3D Trusses vs. 3D Frames

1. Unknowns: Deflections only vs. Deflections + Rotations
2. No. of Unknowns: *doki* = 3j vs. *doki* = 6j
3. Member Stiffness Matrix: (6×6) vs. (12×12)
4. Member Properties: E, A vs. G, E, A, J, I_y, I_z
5. Forces Calculated: Member Axial Forces vs. Member Axial, Shear Forces, Torsions, BM's

Stiffness Method for

- 2D Trusses vs. 2D Frames
- 2D Trusses vs. 3D Trusses
- 2D Frames vs. 3D Frames
- Linear vs. Nonlinear Analysis
- Analysis for Geometric vs. Material Nonlinearity

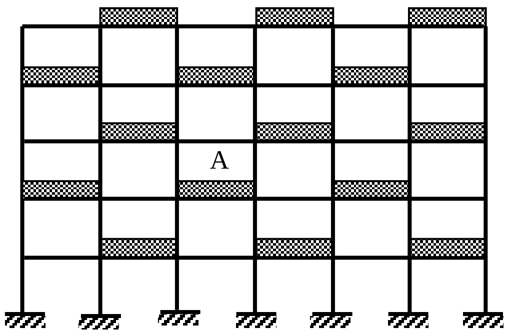
Briefly explain

- axial deformations are sometimes neglected for the structural analysis of frames but not trusses
- joint rotations are considered in calculating the *doki* of frames, but not trusses
- stiffness matrix of a 3D truss member is (6×6) while that of a 3D frame member is (12×12)
- the matrices **K** and **G** used for the nonlinear analysis of frames are only approximate
- the formulation of the geometric stiffness matrix **G** is a nonlinear problem
- a structure becomes unstable at buckling load (explain in terms of stiffness matrix)
- the terms material nonlinearity, plastic moment and collapse mechanism
- frames can be approximately modeled by lumped-mass systems
- the effect of foundation flexibility can be beneficial or harmful to the structure

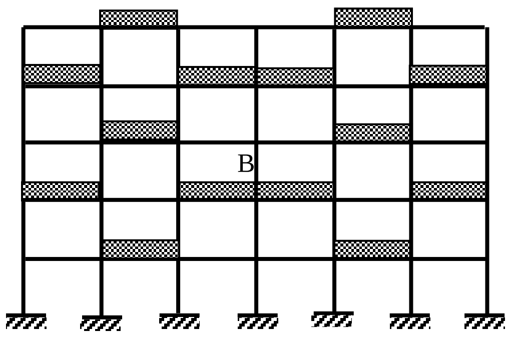
Influence Lines for Frames and Trusses

1. Pattern Loading for Multi-storied Frames

(a) Beam Moments

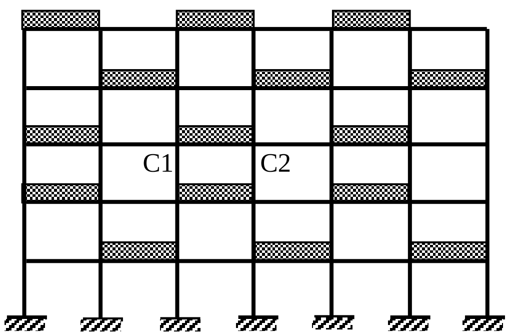


Live Loading Pattern for Maximum M_A (+ ve)

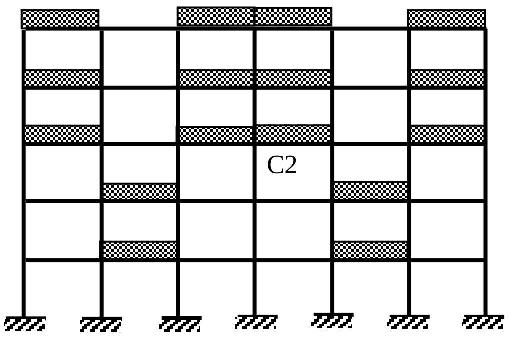


Live Loading Pattern for Maximum M_B (– ve)

(b) Column Moment and Axial Force

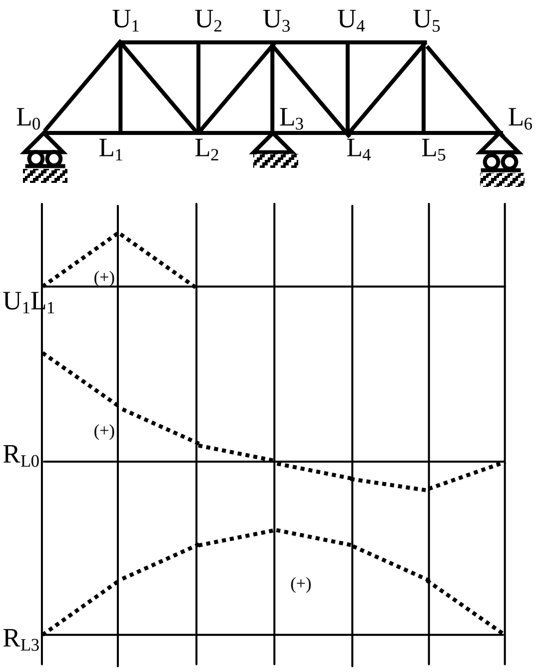


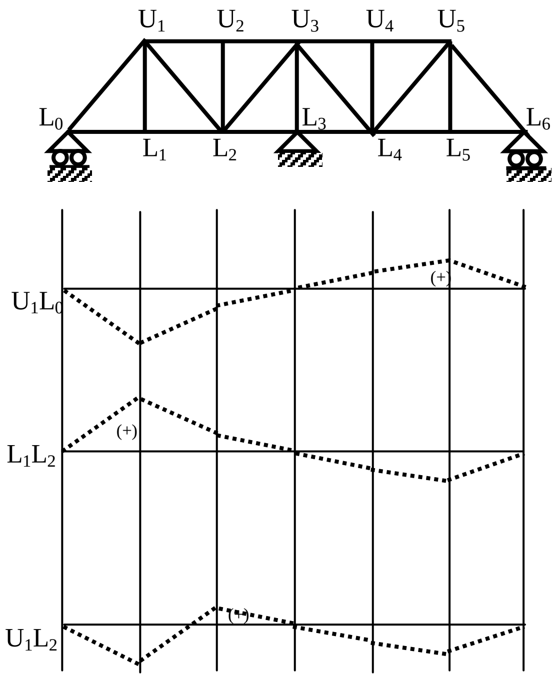
Live Loading Pattern for Maximum M_{C1} and M_{C2}



Live Loading Pattern for Maximum P_{C2}

2. Qualitative Influence Lines for Truss Reactions and Member Forces

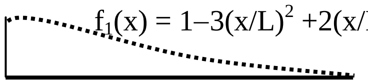




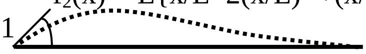
Quantitative Influence Lines using Shape Functions

Quantitative Influence Lines can also be drawn using the shape functions

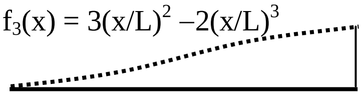
(1) $f_1(x) = 1 - 3(x/L)^2 + 2(x/L)^3$



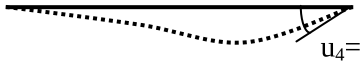
(2) $f_2(x) = L\{x/L - 2(x/L)^2 + (x/L)^3\}$



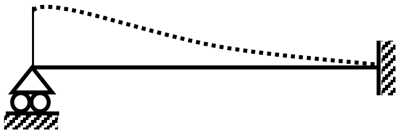
(3) $f_3(x) = 3(x/L)^2 - 2(x/L)^3$



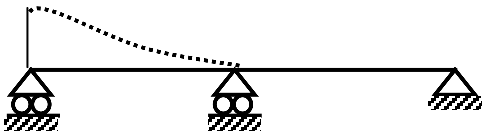
(4) $f_4(x) = L\{-(x/L)^2 + (x/L)^3\}$



The deflected shape is $u(x) = u_1 f_1(x) + u_2 f_2(x) + u_3 f_3(x) + u_4 f_4(x)$, once u_1, u_2, u_3, u_4 are known.



$[u_1 = 1]$ $6/L^2$ $6/L^2$
$(u_2 = -1.5/L)$
$-6/L^2$ ⟶ $-3/L^2$
$\therefore u(x) = (1)\, f_1(x) - (1.5/L)\, f_2(x)$
$= 1 - 1.5(x/L) + 0.5(x/L)^3$

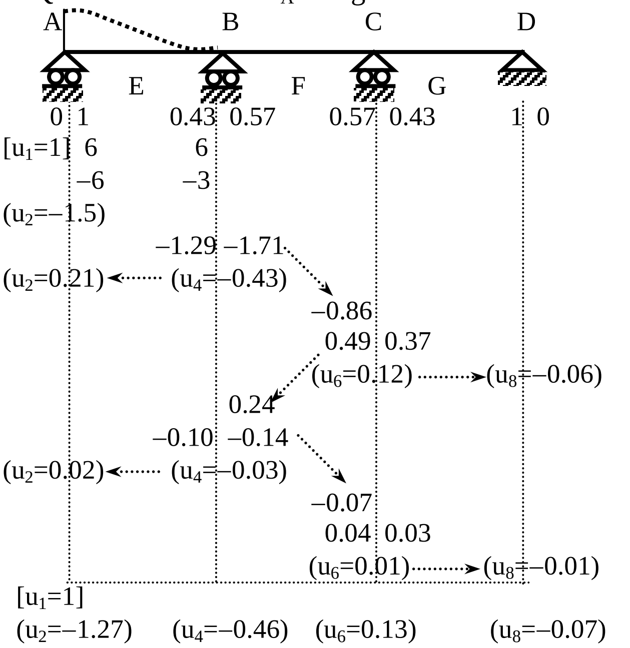


$[u_1 = 1]$ $6/L^2$ $6/L^2$
$(u_2 = -1.5/L) - 6/L^2$ ⟶ $-3/L^2$
$-1.5/L^2$ $-1.5/L^2$
$(u_2 = 0.25/L)$ ⟵ $(u_4 = -0.5/L)$ ⟶ $(u_6 = 0.25/L)$
$\therefore u(x) = (1) f_1(x) - (1.25/L) f_2(x) - (0.5/L)\, f_4(x)$; left span
$u(x) = -(0.5/L)\, f_2(x) + (0.25/L)\, f_4(x)$; right span

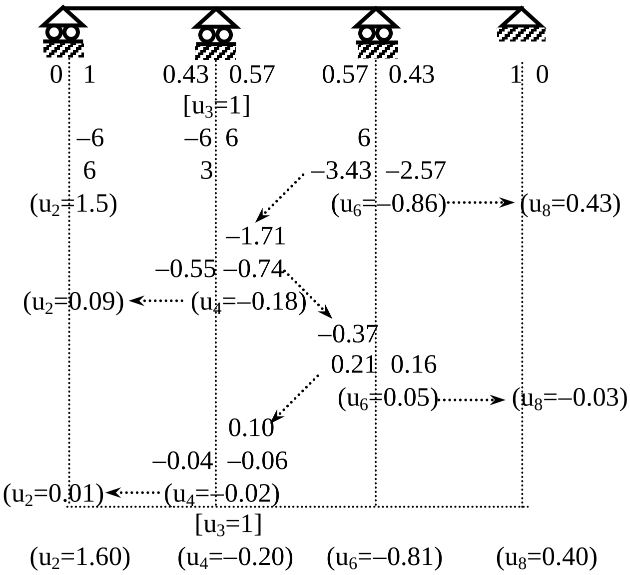
Quantitative Influence Lines for Three-Span Continuous Beam

Assume EI =1, L = 1 for each span

1. Quantitative IL for R_A using Moment Distribution



2. Quantitative IL for R_B using Moment Distribution



3. Once the reactions R_A and R_B [Fig. 1] are known
(i) Shear Forces V_E, $V_B^{(-)}$, $V_B^{(+)}$ [Fig. 2] and
(ii) Bending Moments M_E, M_B, M_F [Fig. 3] can be calculated from Statics.
[e.g., $M_F = R_A \times 1.5 + R_B \times 0.5 - 1 \times <1.5 - x/L>$]

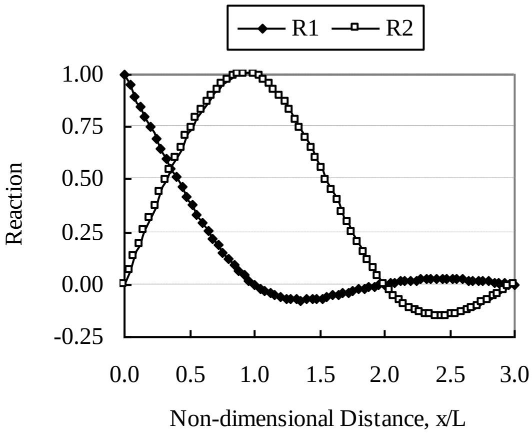


Fig. 1: IL for Reactions

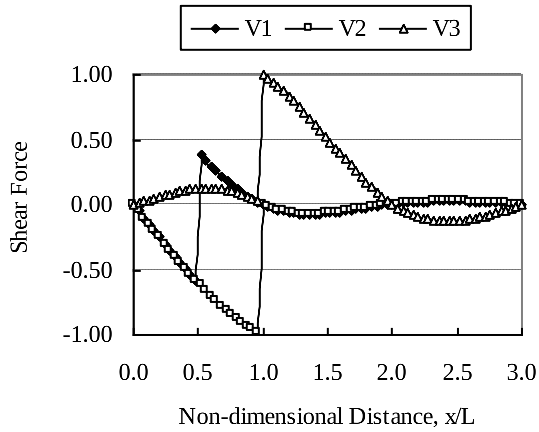


Fig. 2: IL for Shear Forces

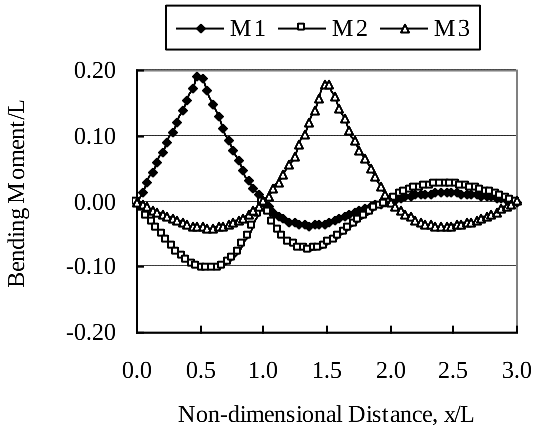


Fig. 3: IL for Bending Moments

Computer Algorithm for the Linear Static Analysis of 2D Trusses

1. Declare parameters and arrays (e.g., x, y, E, A)

2. Read
   (i) number of nodes (Nnod)
   (ii) number of members (Nmem)

3. Read the nodal coordinates (x, y) (for Nnod nodes)

4. Read for (Nmem members)
   (i) member properties (E, A)
   (ii) member nodal numbers (ni, nj)

5. Size of the stiffness matrix, Ndf = 2 Nnod

6. Assemble structural stiffness matrix SK
   i.e., formulate member stiffness matrix and assign them to appropriate locations of SK
   $L = \sqrt{\{(\Delta x)^2+(\Delta y)^2\}}$
   $S_x = EA/L$
   $C = (\Delta x)/L$, $S = (\Delta y)/L$
   i2 = 2(ni-1)
   j2 = 2(nj-1)
   $SK(i2+1, i2+1) = SK(i2+1, i2+1) + S_x C^2$
   $SK(i2+1, i2+2) = SK(i2+1, i2+2) + S_x CS$
   $SK(i2+2, i2+1) = SK(i2+2, i2+1) + S_x CS$
   $SK(i2+2, i2+2) = SK(i2+2, i2+2) + S_x S^2$, etc.

7. Read the number of loads (Nload) and formulate the load vector p
   [Actually p is denoted by u in the program]

8. Apply boundary conditions
   (i) Read the number and known values of the known displacements
   (ii) Modify the corresponding rows and load vector elements

9. Solve the matrix equations SKu = p (using Gauss Elimination), to obtain displacement vector u
   [Actually the equations are solved in this program so that the new u replaces the old u]

10. Calculate the member forces using
    $P_{AB} = S_x\{C(u_B - u_A) + S(v_B - v_A)\}$

Computer Program (in Fortran 90) for the Linear Static Analysis of 2D Trusses

```
      PROGRAM TRUSS2
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(800),Y(800)
      DIMENSION ELAS(600),AREA(600),NI(600),NJ(600)
      DIMENSION DX(600),DY(600),P(600)
      DIMENSION IKNOW(50),DKNOW(50)
      COMMON/SOLVER/STK(990,990),U(990),NDF
      CHARACTER*16 FILOUT

      PRINT*,' ENTER OUTPUT FILENAME'
      READ*,FILOUT

      OPEN(1,FILE='TRUSS2.IN',STATUS='OLD')
      OPEN(2,FILE=FILOUT,STATUS='OLD')

C*******NUMBER OF NODES, NUMBER OF MEMBERS************************
      READ(1,*)NNOD,NMEM

C*******NODAL PROPERTIES******************************************
      READ(1,*)(X(I),Y(I),I=1,NNOD)

C*******MEMBER PROPERTIES*****************************************
      DO 11 I=1,NMEM
       READ(1,*)ELAS(I),AREA(I),NI(I),NJ(I)
       NII=NI(I)
       NJI=NJ(I)
       DX(I)=X(NJI)-X(NII)
       DY(I)=Y(NJI)-Y(NII)
   11 CONTINUE

C*******ASSEMBLING STIFFNESS MATRIX*******************************
      NDF=2*NNOD

      DO 30 I=1,NDF
       U(I)=0.
       DO 30 J=1,NDF
        STK(I,J)=0.
   30 CONTINUE

      DO 12 I=1,NMEM
       TLEN=SQRT(DX(I)*DX(I)+DY(I)*DY(I))
       STX=ELAS(I)*AREA(I)/TLEN
       I2=(NI(I)-1)*2
       J2=(NJ(I)-1)*2
       C=DX(I)/TLEN
       S=DY(I)/TLEN
       CC=C*C
       CS=C*S
       SS=S*S

       STK(I2+1,I2+1)=STK(I2+1,I2+1)+STX*CC
       STK(I2+1,I2+2)=STK(I2+1,I2+2)+STX*CS
       STK(I2+1,J2+1)=STK(I2+1,J2+1)-STX*CC
       STK(I2+1,J2+2)=STK(I2+1,J2+2)-STX*CS

       STK(I2+2,I2+1)=STK(I2+2,I2+1)+STX*CS
       STK(I2+2,I2+2)=STK(I2+2,I2+2)+STX*SS
       STK(I2+2,J2+1)=STK(I2+2,J2+1)-STX*CS
       STK(I2+2,J2+2)=STK(I2+2,J2+2)-STX*SS

       STK(J2+1,I2+1)=STK(J2+1,I2+1)-STX*CC
       STK(J2+1,I2+2)=STK(J2+1,I2+2)-STX*CS
       STK(J2+1,J2+1)=STK(J2+1,J2+1)+STX*CC
       STK(J2+1,J2+2)=STK(J2+1,J2+2)+STX*CS
```

```
        STK(J2+2,I2+1)=STK(J2+2,I2+1)-STX*CS
        STK(J2+2,I2+2)=STK(J2+2,I2+2)-STX*SS
        STK(J2+2,J2+1)=STK(J2+2,J2+1)+STX*CS
        STK(J2+2,J2+2)=STK(J2+2,J2+2)+STX*SS
   12 CONTINUE

C     WRITE(2,8)(STK(I,I),I=1,NDF)

C*******LOADS CORRESPONDING TO DEGREES OF FREEDOM*********************
      READ(1,*)NLOAD
      IF(NLOAD.GT.0)READ(1,*)(J,U(J),I=1,NLOAD)

C*******RESTRAINTS****************************************************
      READ(1,*)NKND
      READ(1,*)(IKNOW(I),DKNOW(I),I=1,NKND)
      DO 15 I=1,NKND
       IKND=IKNOW(I)
       DKN=DKNOW(I)

       DO 16 J=1,NDF
        U(J)=U(J)-STK(J,IKND)*DKN
   16  CONTINUE

       DO 15 J=1,NDF
        IF(J.NE.IKND)THEN
         STK(J,IKND)=0.
         STK(IKND,J)=0.
        ENDIF
        U(IKND)=DKN*STK(IKND,IKND)
   15 CONTINUE

      CALL GAUSS

C*******DISPLACEMENTS*************************************************
    6 FORMAT(10(1X,F8.2))
    7 FORMAT(1X,I4,10(1X,F10.6))
    8 FORMAT(1X,I4,10(1X,F10.4))
      WRITE(2,*)'DISPLACEMENTS ARE'

      DO 17 I=1,NDF
       WRITE(2,7)I,U(I)
   17 CONTINUE

C*******MEMBER FORCES*************************************************
      WRITE(2,*)
      WRITE(2,*)'MEMBER FORCES ARE'
      DO 18 I=1,NMEM
       TLEN=SQRT(DX(I)*DX(I)+DY(I)*DY(I))
       STX=ELAS(I)*AREA(I)/TLEN
       I2=(NI(I)-1)*2
       J2=(NJ(I)-1)*2
       C=DX(I)/TLEN
       S=DY(I)/TLEN
       AXDIS=(U(J2+1)-U(I2+1))*C+(U(J2+2)-U(I2+2))*S
       P(I)=STX*AXDIS
       WRITE(2,8)I,P(I)
   18 CONTINUE

      END
```

```
C*********************************************************************
C*******GAUSS ELIMINATION*********************************************
      SUBROUTINE GAUSS
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SOLVER/A(990,990),B(990),N

      NHBW=N
      N1=N-1
      DO 10 K=1,N1
       K1=K+1
       KH=K+NHBW
       C=1./A(K,K)
       DO 11 I=K1,KH
        IF(I.LE.N)D=A(I,K)*C
        DO 12 J=K1,KH
   12   IF(J.LE.N)A(I,J)=A(I,J)-D*A(K,J)
   11  IF(I.LE.N)B(I)=B(I)-D*B(K)
   10 CONTINUE

      B(N)=B(N)/A(N,N)

      DO 13 I=N1,1,-1
       I1=I+1
       SUM=0.
       DO 14 K=I1,N
   14  SUM=SUM+A(I,K)*B(K)
   13 B(I)=(B(I)-SUM)/A(I,I)

      END
```

Structural Analysis using Energy Formulation

Method of Virtual Work

Another way of representing Newton's equation of equilibrium is by energy methods, which is based on the law of conservation of energy. According to the principle of virtual work, if a system in equilibrium is subjected to virtual displacements δu, the virtual work done by the external forces (δW_E) is equal to the virtual work done by the internal forces (δW_I)

$$\delta W_I = \delta W_E \quad \text{……………………(1)}$$

where the symbol δ is used to indicate 'virtual'. This term is used to indicate hypothetical increments of displacements and works that are assumed to happen in order to formulate the problem.

Axially Loaded Bar on Elastic Foundation

For a structural member loaded axially by p(x) per unit length, the external virtual work due to virtual deformation δu is $\delta W_E = \int p(x)\, dx\, \delta u$ ………………………(2)

while the internal virtual work due to virtual axial strain $d(\delta u)/dx = \delta u'$ and virtual deformation δu of the elastic foundation is $\delta W_I = \int u'\, EA\, \delta u'\, dx + \int u\, k_f\, \delta u\, dx$ ………………………(3)

where $\delta u'$ stands for differentiation of δu with respect to x (in general the symbol $'$ stands for differentiation with respect to x), E = modulus of elasticity and A = cross-sectional area of the axial member, k_f = stiffness of elastic foundation. E, A and k_f can vary with x.

$$\therefore \delta W_I = \delta W_E \Rightarrow \int u'\, EA\, \delta u'\, dx + \int u\, k_f\, \delta u\, dx = \int p(x)\, dx\, \delta u \quad \text{……………....(4)}$$

If the displacements are assumed to be function of a single displacement u_1, so that

$$u(x) = u_1\, \phi(x) \Rightarrow u' = u_1\, \phi'(x) \quad \text{……………(5), (6)}$$

$$\delta u = \delta u_1\, \phi(x) \Rightarrow \delta u' = \delta u_1\, \phi'(x) \quad \text{……………(7), (8)}$$

$$\therefore \text{Eq. (4)} \Rightarrow \int u_1\, \phi'(x)\, EA\, \delta u_1\, \phi'(x)\, dx + \int u_1\, \phi(x)\, k_f\, \delta u_1\, \phi(x)\, dx = \int p(x)\, dx\, \delta u_1\, \phi(x)$$

$$\Rightarrow \{\int EA\, [\phi'(x)]^2\, dx + \int k_f\, [\phi(x)]^2\, dx\}\, u_1 = \int p(x)\, \phi(x)\, dx \quad \text{…………………..(9)}$$

$\therefore$ If the integrations are carried out after knowing $\phi(x)$, Eq. (9) can be rewritten as,

$$k^*\, u_1 = f^* \quad \text{…………………..(10)}$$

where k*, f* are the 'effective' stiffness and force of the system.

Transversely Loaded Beam on Elastic Foundation

For a structural member loaded transversely by q(x) per unit length, the external virtual work due to virtual deformation δv is $\delta W_E = \int q(x)\, dx\, \delta v$ ……………………(11)

while the internal virtual work due to virtual curvature $d(\delta v')/dx = \delta v''$ and virtual deformation δv of the elastic foundation is $\delta W_I = \int v''\, E\, I\, \delta v''\, dx + \int v\, k_f\, \delta v\, dx$ ……………………(12)

where $\delta v''$ stands for double differentiation of δv with respect to x, E = modulus of elasticity and I = moment of inertia of the cross-sectional area of the flexural member. E, I and k_f can vary with x.

$$\therefore \delta W_I = \delta W_E \Rightarrow \int v'' E\, I\, \delta v''\, dx + \int v\, k_f\, \delta v\, dx = \int q(x)\, dx\, \delta v \quad \text{…………………….(13)}$$

If the displacements are assumed to be function of a single displacement u_2, so that

$$v(x) = u_2 \psi(x) \Rightarrow v'' = u_2\, \psi''(x) \quad \text{………………(14), (15)}$$

$$\delta v = \delta u_2\, \psi(x) \Rightarrow \delta v = \delta u_2\, \psi(x) \quad \text{……………….(16), (17)}$$

$\therefore$ Inserting these values in Eq. (13) $\Rightarrow$

$$\int u_2\, \psi''(x)\, EI\, \delta u_2\, \psi''(x)\, dx + \int u_2\, \psi(x)\, k_f\, \delta u_2\, \psi(x)\, dx = \int q(x)\, dx\, \delta u_2\, \psi(x)$$

$$\Rightarrow \{\int EI\, [\psi''(x)]^2\, dx + \int k_f\, [\psi(x)]^2\, dx\}\, u_2 = \int q(x)\, \psi(x)\, dx \quad \text{………………(18)}$$

$\therefore$ If the integrations are carried out after knowing (or assuming) $\psi(x)$, Eq. (18) can be rewritten as,

$$k^*\, u_2 = f^* \quad \text{…………………….....(19)}$$

where k*, f*are the 'effective' stiffness and force of the system.

Once k* and f* are calculated, Eq. (10) or (19) can be solved to obtain the deflection u_1 or u_2, from which the deflection u(x) or v(x) at any point can be calculated using Eq. (5) or (14). The accuracy of Eq. (10) or (19) depends on the accuracy of the shape functions $\phi(x)$ or $\psi(x)$. If the shape functions are not defined exactly, the solutions can only be approximate. These functions must be defined satisfying the natural boundary conditions; i.e., those involving displacements for axial deformation and displacements as well as rotations for flexural deformations. This method of analysis using energy principles is called the *Rayleigh-Ritz method*.

Example 1

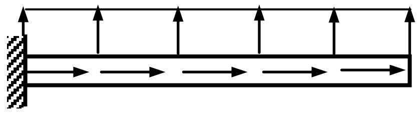
For a cantilever rod, modulus of elasticity $E = 45 \times 10^4$ ksf, cross-sectional area $A = 1$ ft^2, moment of inertia $I = 0.08$ ft^4, length $L = 10$ ft. Calculate the approximate axial and flexural deflections of the system for axial and transverse loads of 1 k/ft respectively.

Solution

Assuming shape functions (satisfying natural boundary conditions)

$\phi(x) = x/L$, $\psi(x) = (x/L)^2$

[Note that: $\phi(0) = 0$, $\psi(0) = 0$, $\psi'(0) = 0$]



For axial deformations,

Effective stiffness $k^* = \int EA\,[\phi'(x)]^2\,dx = EA/L = 45000$ k/ft

Effective force $f^* = \int p(x)\,\phi(x)\,dx = pL/2 = 5$ kips

$\therefore$ Equation for axial deformation is, $45000\,u_1 = 5$

$\Rightarrow u_1 = 1.11 \times 10^{-4}$ ft, which is the exact result

$\Rightarrow u(x) = 1.11 \times 10^{-4}\,(x/L)$, which is also the exact deformed shape of the bar

For flexural deformations,

Effective stiffness $k^* = \int EI\,[\psi''(x)]^2\,dx = 4EI/L^3 = 144$ k/ft

Effective force $f^* = \int q(x)\,\psi(x)\,dx = qL/3 = 3.33$ kips

$\therefore$ Equation for flexural deformation is, $144\,u_2 = 3.33$

$\Rightarrow u_2 = 0.02315$ ft, the exact result being $[= qL^4/(8EI)] = 0.03472$ ft

$\Rightarrow u(x) = 0.02315\,(x/L)^2$

Example 2

For the member properties mentioned in Example 1, calculate the approximate flexural deflections of

(i) a cantilever beam, assuming $\psi(x) = 1-\cos(\pi x/2L)$,

(ii) a simply supported beam, assuming $\psi(x) = \sin(\pi x/L)$, for transverse loads of 1 k/ft.

Solution

Both these shape functions satisfy the natural boundary conditions for the problems mentioned.

[i.e., (i) $\psi(0) = 0$, $\psi'(0) = 0$, (ii) $\psi(0) = 0$, $\psi(L) = 0$]

(i) For the cantilever beam,

Effective stiffness $k^* = \int EI\,[\psi''(x)]^2\,dx = 3.044\,EI/L^3 = 109.59$ k/ft

Effective force $f^* = \int q(x)\,\psi(x)\,dx = qL(1-2/\pi) = 3.63$ kips

$\therefore$ Equation for flexural deformation is, $109.59\,u_2 = 3.63$

$\Rightarrow u_2 = 0.03316$ ft, which is much better estimate of the exact result

$\Rightarrow u(x) = 0.03316\,[1-\cos(\pi x/2L)]$

(ii) For the simply supported beam,

Effective stiffness $k^* = \int EI\,[\psi''(x)]^2\,dx = (\pi/L)^4\,EI\,L/2 = 1753.36$ k/ft

Effective force $f^* = \int q(x)\,\psi(x)\,dx = 2qL/\pi = 6.367$ kips

$\therefore$ Equation for flexural deformation is, $1753.36\,u_2 = 6.367$

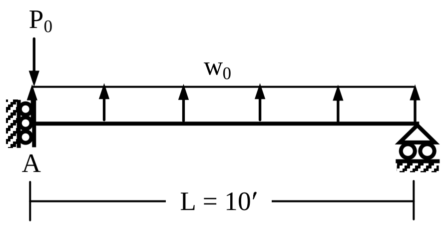
$\Rightarrow u_2 = 36.31 \times 10^{-4}$ ft, which is very close to exact result $[= 5qL^4/(384\,EI)] = 36.17 \times 10^{-4}$ ft

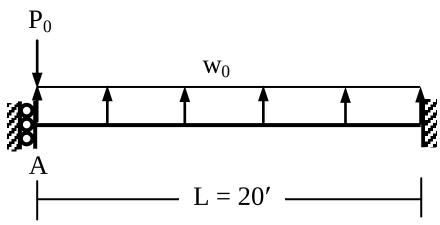
$\Rightarrow u(x) = 36.31 \times 10^{-4}\,\sin(\pi x/L)$

These results show that the accuracy of the Rayleigh-Ritz method depends on the accuracy of the assumed shape function. Based on the shape function, this method can model the structure to be too stiff (i.e., over-estimate the 'effective' stiffness and 'effective' force) or can reproduce the exact solution.

Problems on Structural Analysis using Energy Formulation

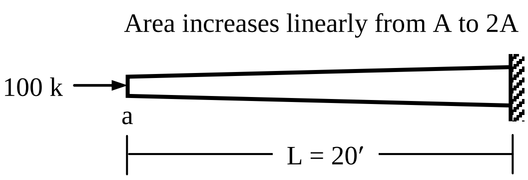
1. For the beams loaded as shown below [Given: $EI = 40 \times 10^6$ lb-ft^2]
   (a) choose an appropriate shape function (satisfying the essential boundary conditions) among (i) $\psi(x) = \cos(\pi x/2L)$, (ii) $\psi(x) = [1 + \cos(\pi x/L)]/2$ and (iii) $\psi(x) = \sin(\pi x/L)$
   (b) use the chosen shape function to calculate the deflections at A if (i) $P_0 = 10$ kips, $w_0 = 0$, (ii) $P_0 = 0$, $w_0 = 1$ kip/ft
   (c) compare the results found in (b) with the exact results.





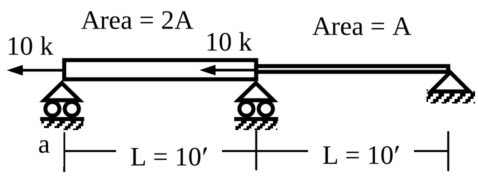
2. For the structures loaded as shown below [Given: $EI = 36 \times 10^3$ k-ft^2, $EA = 450 \times 10^3$ k]
   (a) justify the choice of shape functions

Bar1: $\phi(x) = 1-(x/L)$



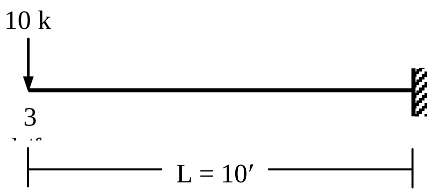
Bar1

Bar2: $\phi(x) = 1-(x/2L)$



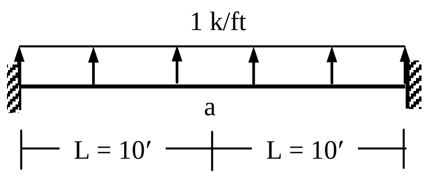
Bar2

Beam1: $\psi(x) = 2 - 3(x/L) + (x/L)^3$



Beam1

Beam2: $\psi(x) = \sin(\pi x/2L)$
$\psi(x) = \sin^2(\pi x/2L)$



Beam2

(b) calculate the corresponding elongations/deflections at 'a'

Problems on Structural Analysis using Energy Formulation

1. (a) For $\psi(x) = \cos(\pi x/2L)$, $\psi(0) = \cos(0) = 1$ $\qquad$ $\psi'(0) = -(\pi/2L)\sin(0) = 0$

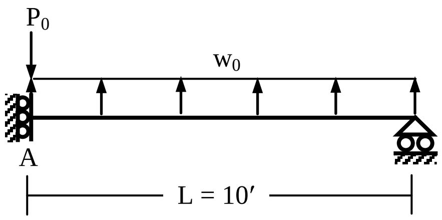
$\psi(L) = \cos(\pi/2) = 0$ $\qquad$ $\psi'(L) = -(\pi/2L)\sin(\pi/2) = -(\pi/2L)$

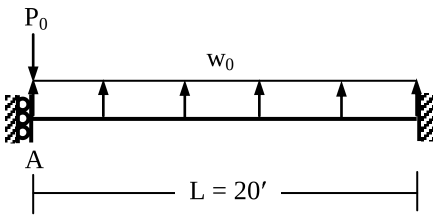
For $\psi(x) = [1 + \cos(\pi x/L)]/2$, $\psi(0) = [1 + 1]/2 = 1$ $\qquad$ $\psi'(0) = -(\pi/L)\sin(0) = 0$

$\psi(L) = [1 - 1]/2 = 0$ $\qquad$ $\psi'(L) = -(\pi/L)\sin(\pi) = 0$

For $\psi(x) = \sin(\pi x/L)$, $\psi(0) = \sin(0) = 0$ $\qquad$ $\psi'(0) = (\pi/L)\cos(0) = \pi/L$

$\psi(L) = \sin(\pi) = 0$ $\qquad$ $\psi'(L) = (\pi/L)\cos(\pi) = -\pi/L$





For the first beam

$\psi(0) \neq 0$, $\psi'(0) = 0$, $\psi(L) = 0$, $\psi'(L) \neq 0$; $\therefore$ Choose $\psi(x) = \cos(\pi x/2L)$

For the second beam

$\psi(0) \neq 0$, $\psi'(0) = 0$, $\psi(L) = 0$, $\psi'(L) = 0$; $\therefore$ Choose $\psi(x) = [1 + \cos(\pi x/L)]/2$

Since $\psi(0) = 1$ for both these functions, the deflection u_2 indicates u_A here

(b) For the first beam, $\psi(x) = \cos(\pi x/2L) \Rightarrow \psi''(x) = -(\pi/2L)^2 \cos(\pi x/2L)$

Effective stiffness $k^* = \int EI\,[\psi''(x)]^2\,dx = (\pi/2L)^4\,EI\,L/2 = 121.76$ k/ft

For $P_0 = 10$ kips, Effective force $f^* = \int q(x)\,\psi(x)\,dx = (-10)\,\psi(0) = -10$ kips

$\therefore u_2 = -10/121.76 = -0.0821$ ft $= -0.986$ in

For $w_0 = 1$ kip/ft, Effective force $f^* = \int q(x)\,\psi(x)\,dx = (1)\,(2 \times 10/\pi)\,(1) = 6.366$ kips

$\therefore u_2 = 6.366/121.76 = 0.0523$ ft $= 0.627$ in

For the second beam, $\psi(x) = [1 + \cos(\pi x/L)]/2 \Rightarrow \psi''(x) = -(\pi/L)^2 \cos(\pi x/L)/2$

Effective stiffness $k^* = \int EI\,[\psi''(x)]^2\,dx = [(\pi/L)^4/4]\,EI\,L/2 = 60.88$ k/ft

For $P_0 = 10$ kips, Effective force $f^* = \int q(x)\,\psi(x)\,dx = (-10)\,\psi(0) = -10$ kips

$\therefore u_2 = -10/60.88 = -0.164$ ft $= -1.971$ in

For $w_0 = 1$ kip/ft, Effective force $f^* = \int q(x)\,\psi(x)\,dx = (1)\,(20)/2 = 10$ kips

$\therefore u_2 = 10/60.88 = 0.164$ ft $= 1.971$ in

(c) The exact results are (using Stiffness Method)

For the first beam, $u_2 = -1.0$ in and 0.625 in

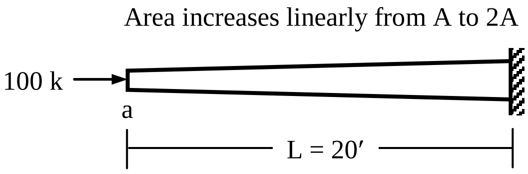
For the second beam, $u_2 = -2.0$ in and 2.0 in

2. (a) For Bar1, $\phi(0) \neq 0$, $\phi(L) = 0$
$\phi(x) = 1-(x/L) \Rightarrow \phi(0) = 1 \neq 0$, $\phi(L) = 0 \Rightarrow$ OK

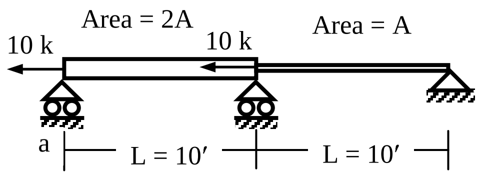
For Bar2, $\phi(0) \neq 0$, $\phi(L) \neq 0$, $\phi(2L) = 0$
$\phi(x) = 1-(x/2L) \Rightarrow \phi(0) = 1 \neq 0$, $\phi(L) = 0.5 \neq 0$, $\phi(2L) = 0 \Rightarrow$ OK

For Beam1, $\psi(0) \neq 0$, $\psi'(0) \neq 0$, $\psi(L) = 0$, $\psi'(L) = 0$
$\psi(x) = 2 - 3(x/L) + (x/L)^3$, $\psi'(x) = -3/L + 3x^2/L^3$
$\Rightarrow \psi(0) = 2$, $\psi'(0) = -3/L \neq 0$, $\psi(L) = 2-3 + 1 = 0$, $\psi'(L) = -3/L + 3L^2/L^3 = 0 \Rightarrow$ OK

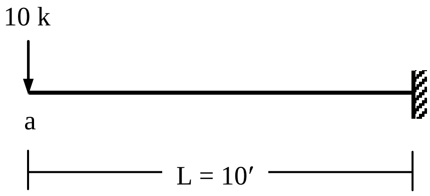
For Beam2, $\psi(0) = 0$, $\psi'(0) = 0$, $\psi(2L) = 0$, $\psi'(2L) = 0$
$\psi(x) = \sin(\pi x/2L)$, $\psi'(x) = (\pi/2L) \cos(\pi x/2L)$
$\Rightarrow \psi(0) = 0$, $\psi'(0) = \pi/2L$, $\psi(2L) = \sin(\pi) = 0$, $\psi'(2L) = (\pi/2L) \cos(\pi) = -(\pi/2L) \Rightarrow$ Not OK
$\psi(x) = \sin^2(\pi x/2L) = [1-\cos(\pi x/L)]/2$, $\psi'(x) = (\pi/2L) \sin(\pi x/L)$
$\Rightarrow \psi(0) = 0$, $\psi'(0) = \pi/2L \sin(0) = 0$, $\psi(2L) = \sin^2(\pi) = 0$, $\psi'(2L) = (\pi/2L) \sin(2\pi) = 0 \Rightarrow$ OK



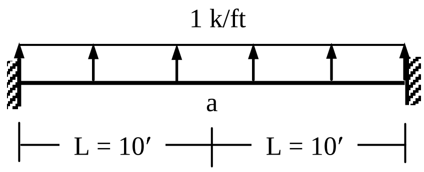
Bar1



Bar2



Beam1



Beam2

(b) For Bar1, $A(x) = A(1 + x/L)$, $\phi'(x) = -1/L$
Effective stiffness $k^* = \int EA[\phi'(x)]^2 \, dx = EA(L + L^2/2L)/L^2 = 33.75 \times 10^3$ k/ft
Effective force $f^* = \int p(x) \phi(x) \, dx = (100) \phi(0) = 100$ kips
$\therefore u_1 = 100/(33.75 \times 10^3) = 2.96 \times 10^{-3}$ ft $= 0.0356$ in $\Rightarrow u_a = u_1 \psi(0) = 0.0356$ in

For Bar2, $A_1 = 2A$, $A_2 = A$, $\phi'(x) = -1/2L$
Effective stiffness $k^* = \int EA[\phi'(x)]^2 \, dx = (E/4L^2)(2A\,L + A\,L) = 3\,EA/4L = 33.75 \times 10^3$ k/ft
Effective force $f^* = \int p(x) \phi(x) dx = (-10) \phi(0) + (-10) \phi(L) = (-10)(1) + (-10)(0.5) = -15$ kips
$\therefore u_1 = -15/(33.75 \times 10^3) = -4.44 \times 10^{-4}$ ft $= -5.33 \times 10^{-3}$ in $\Rightarrow u_a = u_1 \psi(0) = -5.33 \times 10^{-3}$ in

For Beam1, $\psi(x) = 2 - 3(x/L) + (x/L)^3$, $\psi'(x) = -3/L + 3x^2/L^3$, $\psi''(x) = 6x/L^3$
Effective stiffness $k^* = \int EI[\psi''(x)]^2 \, dx = EI \int 36 x^2/L^6 \, dx = 12\,EI/L^3 = 432$ k/ft
Effective force $f^* = \int q(x) \psi(x) \, dx = (-10) \psi(0) = (-10)(2) = -20$ kips
$\therefore u_2 = -20/432 = -0.0463$ ft $= -0.556$ in $\Rightarrow u_a = u_2 \psi(0) = -0.556 \times 2 = -1.111$ in

For Beam2, $\psi(x) = [1-\cos(\pi x/L)]/2$, $\psi'(x) = (\pi/2L) \sin(\pi x/L)$, $\psi''(x) = [(\pi/L)^2 \cos(\pi x/L)]/2$
Effective stiffness $k^* = \int EI[\psi''(x)]^2 \, dx = EI(\pi/L)^4 \int [\cos^2(\pi x/L)]/4 \, dx = \pi^4 EI/(4L^3) = 876.68$ k/ft
Effective force $f^* = \int q(x) \psi(x) \, dx = \int (1)[1-\cos(\pi x/L)]/2 \, dx = 10$ kips
$\therefore u_2 = 10/876.68 = 0.0114$ ft $= 0.137$ in $\Rightarrow u_a = u_2 \psi(L) = 0.137 \times 1 = 0.137$ in

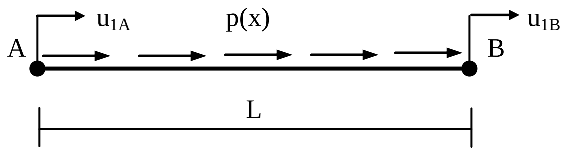
Stiffness Matrices of Axial and Flexural Members

The Rayleigh-Ritz method can handle variations in member properties and loads over the members. But its accuracy depends on the shape function chosen for the analysis, for which there is no automatic way of choosing. Moreover the choice of the function depends on the boundary conditions, thus needing a different formulation even if the structure remains the same otherwise. The Stiffnes Method, on the other hand, has the advantage of a methodical formulation and versatility in applying the boundary conditions for a large variety of linear and nonlinear problems. Like the Rayleigh-Ritz method, the formulation of Stiffness Method can also be based on energy principles, which makes its formulation more versatile. But rather than defining the displacement of the entire structure/structural member by a single function, it divides the member into a number of small elements and defines the displacements at any point in the member by interpolating between the displacements/rotations of the nodes at the ends of the member.

Axially Loaded Bar

Applying the method of virtual work to members subjected to axial load of p(x) per unit length,

$$\delta W_I = \delta W_E \Rightarrow \int u' \, E \, A \, \delta u' \, dx = \int p(x) \, dx \, \delta u \qquad \text{…………………(4)}$$



Axially Loaded Member

If the displacements of a member AB (shown above) are assumed to be interpolating functions [$\phi_1(x)$ and $\phi_2(x)$] of two nodal displacements u_{1A} and u_{1B},

$$u = u_{1A}\,\phi_1 + u_{1B}\,\phi_2 \Rightarrow u' = u_{1A}\,\phi_1' + u_{1B}\,\phi_2' \qquad \text{………….………(20), (21)}$$

$$\delta u = \delta u_{1A}\,\phi_1 + \delta u_{1B}\,\phi_2 \Rightarrow \delta u' = \delta u_{1A}\,\phi_1' + \delta u_{1B}\,\phi_2' \qquad \text{…...……..……….(22), (23)}$$

$\therefore$ Eq. (4) can be written in matrix form as,

$$\begin{pmatrix} \int EA\,\phi_1'\phi_1' dx & \int EA\,\phi_1'\phi_2' dx \\ \int EA\,\phi_2'\phi_1' dx & \int EA\,\phi_2'\phi_2' dx \end{pmatrix} \begin{Bmatrix} u_{1A} \\ u_{1B} \end{Bmatrix} = \begin{Bmatrix} \int p(x)\,\phi_1 dx \\ \int p(x)\,\phi_2\, dx \end{Bmatrix} \qquad \text{……...…...…….(24)}$$

For concentrated loads, p(x) is a delta function of x. If loads X_A and X_B are applied at joints A and B, they can be added to the right side of Eq. (24). Eq. (24) can be rewritten as,

$$\mathbf{K_m}\,\mathbf{u_m} = \mathbf{f_m} \qquad \text{………...………(25)}$$

where $\mathbf{K_m}$ is the stiffness matrix of the member, while $\mathbf{u_m}$ and $\mathbf{f_m}$ are the member displacement and load vectors. They can be formed once the shape functions ϕ_1 and ϕ_2 are known or assumed. One-dimensional two-noded elements with linear interpolation functions are typically chosen in such cases, so that the shape functions ϕ_1 and ϕ_2 for axially loaded members are

$$\phi_1(x) = 1 - x/L, \text{ and } \phi_2(x) = x/L \qquad \text{..…………(26)}$$

Therefore, elements of the member stiffness matrices are $K_{mij} = \int EA\,\phi_i'\,\phi_j'\,dx$ ..…………(27)

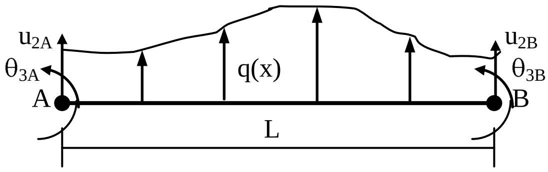
Shape functions $\phi_1(x)$ and $\phi_2(x)$

Transversely Loaded Beam

Applying the method of virtual work to beams subjected to flexural load of q(x) per unit length

$\Rightarrow \int u'' E\, I\, \delta u''\, dx = \int q(x)\, dx\, \delta u$ ………………(13)

Following the same type of formulation as for axial members, the member equations for flexural members subjected to transverse load of q(x) per unit length (shown below) can be written in matrix form like Eq. (24), but the member matrices are different here.



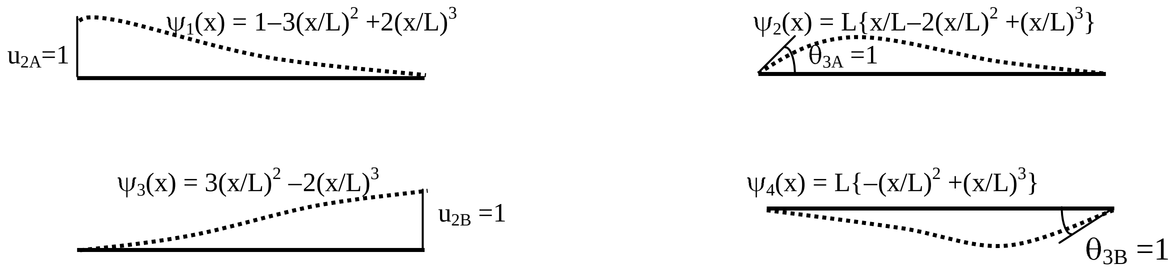
Transversely Loaded Member

Two-noded elements with cubic interpolation functions for u_{2A}, θ_{3A}, u_{2B} and θ_{3B} are typically chosen in such cases, so that

$u(x) = u_{2A}\,\psi_1 + \theta_{3A}\,\psi_2 + u_{2B}\,\psi_3 + \theta_{3B}\,\psi_4$ ……………………(28)

where $\psi_1(x) = 1-3(x/L)^2 + 2(x/L)^3$, $\psi_2(x) = x\{1-(x/L)\}^2$

$\psi_3(x) = 3(x/L)^2 - 2(x/L)^3$, $\psi_4(x) = (x-L)(x/L)^2$ ……………………(29)



Shape functions $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$ and $\psi_4(x)$

The size of the stiffness matrix is (4×4) here, due to transverse joint displacements (u_{2A}, u_{2B}) joint rotations (θ_{3A}, θ_{3B}) and its elements are given by

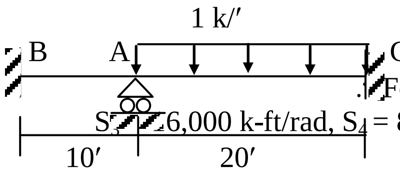
$K_{mij} = \int EI\, \psi_i''\, \psi_j''\, dx$ ……………………(30)

The equations of the stiffness matrix for axial members [Eq. (27)] as well as flexural members [Eq. (30)] guarantee that for linear problems

(i) The stiffness matrices are symmetric [i.e., element (i,j) = element (j,i)],

(ii) The diagonal elements of the matrices are positive [as element (i,i) involves square].

1.



$EI = 40,000$ k-ft^2

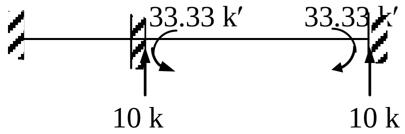
$\therefore$ For BA, $S_1 = 480$ k/ft, $S_2 = 2400$ k/rad

$S_3 = 16,000$ k-ft/rad, $S_4 = 8,000$ k-ft/rad

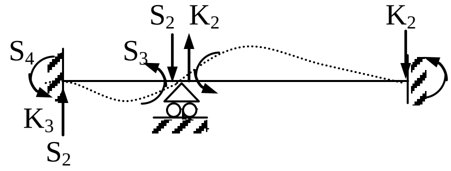
$\therefore$ For AC, $K_1 = 60$ k/ft, $K_2 = 600$ k/rad

$K_3 = 8,000$ k-ft/rad, $K_4 = 4,000$ k-ft/rad

d.o.k.i. = 1 (u_1 is rotation at A)



Case 0 (FER)



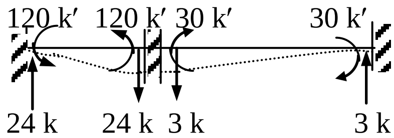
Case 1 ($u_1 = 1$)

$\sum M_{z(A)} = 0 \Rightarrow 33.33 + S_3\,u_1 + K_3\,u_1 = 0 \Rightarrow 24\times10^3\,u_1 = -33.33 \Rightarrow u_1 = -1.389\times10^{-3}$ rad

$BM_{(B)} = 0 + S_4\,u_1 = -11.11$ k′, $BM_{(A)/BA} = 0 + S_3\,u_1 = -22.22$ k′,

$BM_{(A)/AC} = 33.33 + K_3\,u_1 = 22.22$ k′, $BM_{(C)} = -33.33 + K_4\,u_1 = -38.89$ k′

2. The only difference from Problem 1 is the additional FER due to support settlement.



For BA, $6EI\Delta/L^2 = 6\times40,000\times0.05/10^2 = 120$ k-ft

$12EI\Delta/L^3 = 12\times40,000\times0.05/10^3 = 24$ k

For AC, $6EI\Delta/L^2 = 6\times40,000\times0.05/20^2 = 30$ k-ft

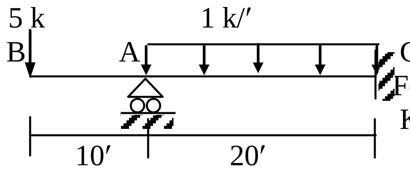
$12EI\Delta/L^3 = 12\times40,000\times0.05/20^3 = 3$ k

$\sum M_{z(A)} = 0 \Rightarrow 33.33 + 120 - 30 + S_3\,u_1 + K_3\,u_1 = 0 \Rightarrow 24\times10^3\,u_1 = -123.33 \Rightarrow u_1 = -5.138\times10^{-3}$ rad

$BM_{(B)} = 120 + S_4\,u_1 = 78.89$ k′, $BM_{(A)/BA} = 120 + S_3\,u_1 = 37.78$ k′,

$BM_{(A)/AC} = 33.33 - 30 + K_3\,u_1 = -37.78$ k′, $BM_{(C)} = -33.33 - 30 + K_4\,u_1 = -83.89$ k′

3.

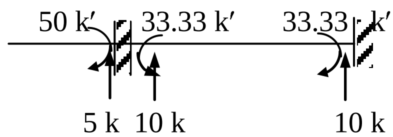


Stiffness of BA = 0

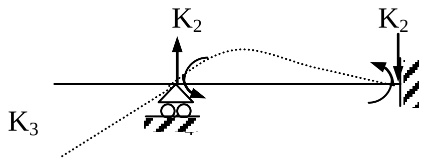
For AC, $K_1 = 60$ k/ft, $K_2 = 600$ k/rad

$K_3 = 8,000$ k-ft/rad, $K_4 = 4,000$ k-ft/rad

d.o.k.i. = 1 (u_1 is rotation at A)



Case 0 (FER)



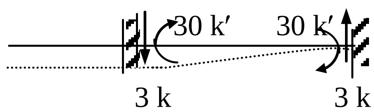
Case 1 ($u_1 = 1$)

$\sum M_{z(A)} = 0 \Rightarrow -50 + 33.33 + K_3\,u_1 = 0 \Rightarrow 8\times10^3\,u_1 = 16.67 \Rightarrow u_1 = 2.083\times10^{-3}$ rad

$BM_{(B)} = 0 + 0 = 0$, $BM_{(A)/BA} = -50 + 0 = -50$ k′,

$BM_{(A)/AC} = 33.33 + K_3\,u_1 = 50$ k′, $BM_{(C)} = -33.33 + K_4\,u_1 = -16.67$ k′

4. The only difference from Problem 3 is the additional FER due to support settlement.



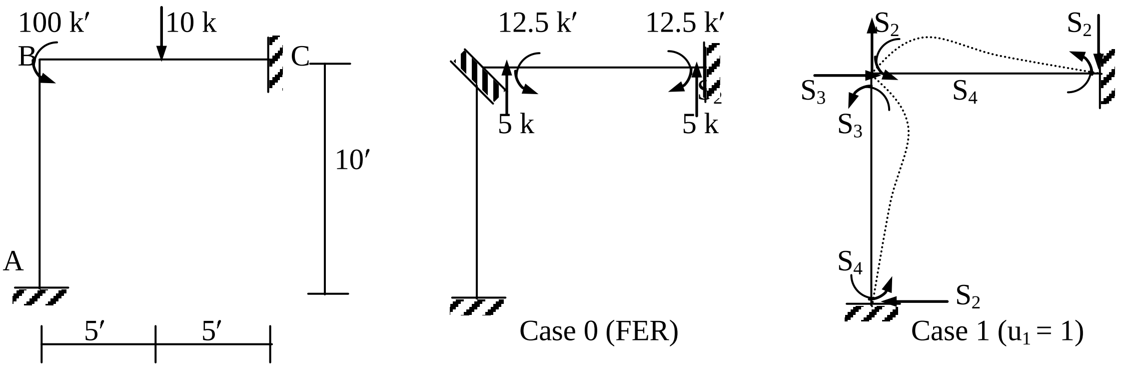
$\sum M_{z(A)} = 0 \Rightarrow -50 + 33.33 - 30 + K_3\,u_1 = 0$

$\Rightarrow u_1 = 2.667\times10^{-3}$ rad

$BM_{(B)} = 0$, $BM_{(A)/BA} = -50$ k′,

$BM_{(A)/AC} = 50$ k′, $BM_{(C)} = -63.33 + K_4\,u_1 = -52.67$ k′

5. Both members have the same stiffness (e.g., S_3 = 16,000 k-ft/rad), and d.o.k.i. = 1

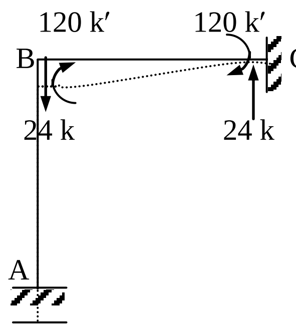


$\sum M_{z(B)} = 0 \Rightarrow 12.5 + S_3\, u_1 + S_3\, u_1 = 100 \Rightarrow 32\times10^3\, u_1 = 87.5 \Rightarrow u_1 = 2.734\times10^{-3}$ rad

$BM_{(A)} = 0 + S_4\, u_1 = 21.88$ k′, $BM_{(B)/AB} = 0 + S_3\, u_1 = 43.75$ k′,

$BM_{(B)/BC} = 12.5 + S_3\, u_1 = 56.25$ k′, $BM_{(C)} = -12.5 + S_4\, u_1 = 9.38$ k′

6. The only difference from Problem 5 is the additional FER due to support settlement.



$\sum M_{z(B)} = 0 \Rightarrow 12.5 - 120 + S_3\, u_1 + S_3\, u_1 = 100$

$\Rightarrow u_1 = 6.484\times10^{-3}$ rad

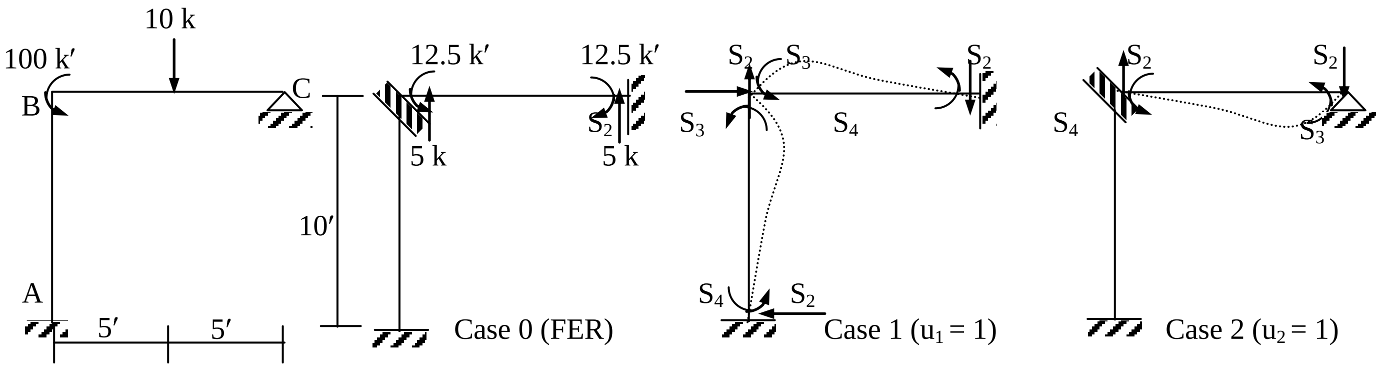
$BM_{(A)} = 0 + S_4\, u_1 = 51.88$ k′

$BM_{(B)/AB} = 0 + S_3\, u_1 = 103.75$ k′

$BM_{(B)/BC} = 12.5 - 120 + S_3\, u_1 = -3.75$ k′

$BM_{(C)} = -12.5 - 120 + S_4\, u_1 = -81.62$ k′

7. Here d.o.k.i. = 2 (rotations at B, C); for both members S_3 = 16,000 k-ft/rad, S_4 = 8,000 k-ft/rad



$\sum M_{z(B)} = 0 \Rightarrow 12.5 + 2S_3\, u_1 + S_4\, u_2 = 100 \Rightarrow 2S_3\, u_1 + S_4\, u_2 = 87.5$

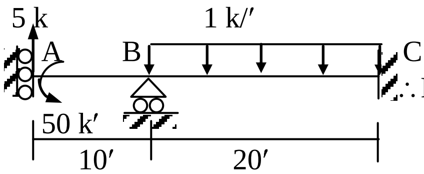
$\sum M_{z(C)} = 0 \Rightarrow -12.5 + S_4\, u_1 + S_3\, u_2 = 0 \Rightarrow S_4\, u_1 + S_3\, u_2 = 12.5$

$\Rightarrow u_1 = 2.902\times10^{-3}$ rad, $u_2 = -0.670\times10^{-3}$ rad

$BM_{(A)} = 0 + S_4\, u_1 + 0 = 23.21$ k′, $BM_{(B)/AB} = 0 + S_3\, u_1 = 46.43$ k′,

$BM_{(B)/BC} = 12.5 + S_3\, u_1 + S_4\, u_2 = 53.57$ k′, $BM_{(C)} = -12.5 + S_4\, u_1 + S_3\, u_2 = 0$

11.



EI = 40,000 k-ft^2

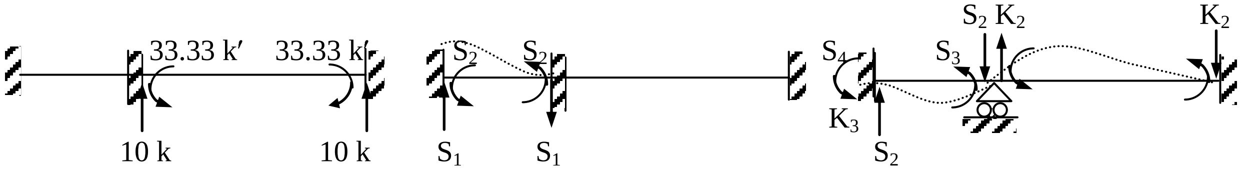
$\therefore$ For AB, S_1 = 480 k/ft, S_2 = 2400 k/rad

S_3 = 16,000 k-ft/rad, S_4 = 8,000 k-ft/rad

$\therefore$ For BC, K_1 = 60 k/ft, K_2 = 600 k/rad

K_3 = 8,000 k-ft/rad, K_4 = 4,000 k-ft/rad

d.o.k.i. = 2 (u_1 is deflection at A, u_2 is rotation at B)



Case 0 (FER) Case 1 ($u_1 = 1$) Case 2 ($u_2 = 1$)

$\sum F_{y(A)} = 0 \Rightarrow 0 + S_1 u_1 + S_2 u_2 = 5 \Rightarrow 480\, u_1 + 2400\, u_2 = 5$

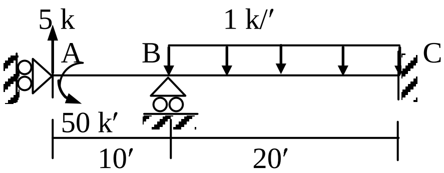
$\sum M_{z(B)} = 0 \Rightarrow 33.33 + S_2 u_1 + S_3 u_2 + K_3 u_2 = 0 \Rightarrow 2400\, u_1 + 24000\, u_2 = -33.33$

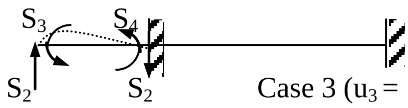
$\Rightarrow u_1 = 34.72\times10^{-3}$ ft, $u_2 = -4.861\times10^{-3}$ rad

$BM_{(A)} = 0 + S_2 u_1 + S_4 u_2 = 44.44$ k′, $BM_{(B)/AB} = 0 + S_2 u_1 + S_3 u_2 = 5.56$ k′,

$BM_{(B)/BC} = 33.33 + K_3 u_2 = -5.56$ k′, $BM_{(C)} = -33.33 + K_4 u_2 = -52.78$ k′

12. The only difference from Problem 11 is the presence of another rotation at A $\Rightarrow$ d.o.k.i. = 3





Case 3 ($u_3 = 1$)

$\sum F_{y(A)} = 0 \Rightarrow 0 + S_1 u_1 + S_2 u_2 + S_2 u_3 = 5 \Rightarrow 480\, u_1 + 2400\, u_2 + 2400\, u_3 = 5$

$\sum M_{z(B)} = 0 \Rightarrow 33.33 + S_2 u_1 + S_3 u_2 + K_3 u_2 + S_4 u_3 = 0 \Rightarrow 2400\, u_1 + 24000\, u_2 + 8000\, u_2 = -33.33$

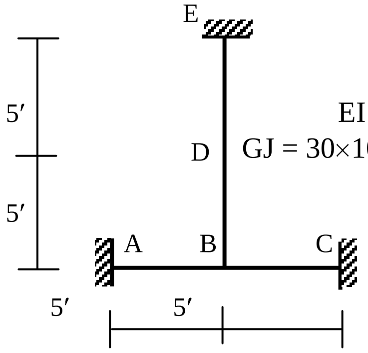
$\sum M_{z(A)} = 0 \Rightarrow 0 + S_2 u_1 + S_4 u_2 + S_3 u_3 = 50 \Rightarrow 2400\, u_1 + 8000\, u_2 + 16000\, u_3 = 50$

$\Rightarrow u_1 = 20.83\times10^{-3}$ ft, $u_2 = -4.167\times10^{-3}$ rad, $u_3 = 2.083\times10^{-3}$ rad

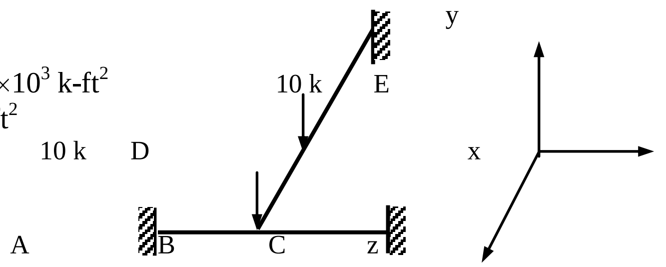
$BM_{(A)} = S_2 u_1 + S_3 u_2 + S_4 u_3 = 50$ k′, $BM_{(B)/AB} = 0 + S_2 u_1 + S_3 u_2 + S_4 u_3 = 0$,

$BM_{(B)/BC} = 33.33 + K_3 u_2 = 0$, $BM_{(C)} = -33.33 + K_4 u_2 = -50$ k′

Stiffness Method for Grids



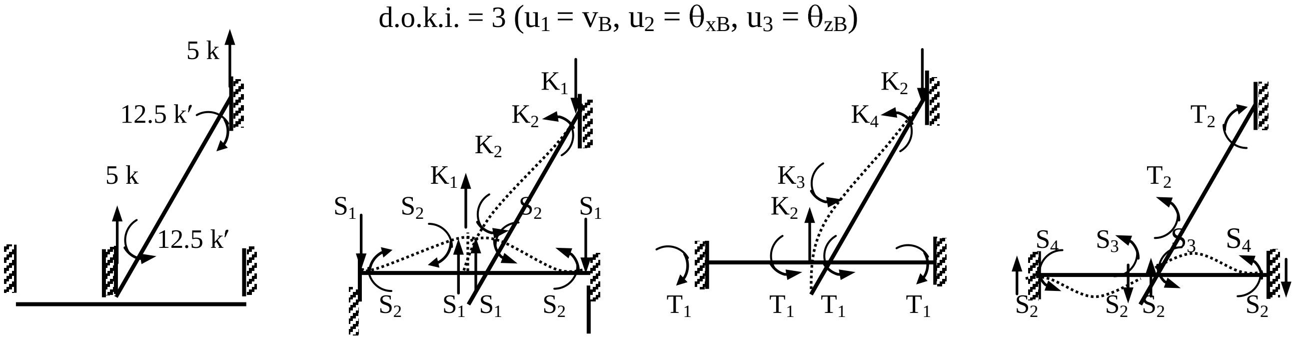
$EI = 40\times10^3$ k-ft^2
$GJ = 30\times10^3$ k-ft^2



Top View

Isometric View

d.o.k.i. = 3 ($u_1 = v_B$, $u_2 = \theta_{xB}$, $u_3 = \theta_{zB}$)



Case 0 (FER)

Case 1 ($u_1 = 1$)

Case 2 ($u_2 = 1$)

Case 3 ($u_3 = 1$)

Here, $S_1 = 12\times40\times10^3/5^3 = 3840$ k/ft, $S_2 = 6\times40\times10^3/5^2 = 9600$ k/rad,

$S_3 = 4\times40\times10^3/5 = 32000$ k-ft/rad, $S_4 = 2\times40\times10^3/5 = 16000$ k-ft/rad

$K_1 = 12\times40\times10^3/10^3 = 480$ k/ft, $K_2 = 6\times40\times10^3/10^2 = 2400$ k/rad,

$K_3 = 4\times40\times10^3/10 = 16000$ k-ft/rad, $K_4 = 2\times40\times10^3/10 = 8000$ k-ft/rad

$T_1 = 30\times10^3/5 = 6000$ k-ft/rad, $T_2 = 30\times10^3/10 = 3000$ k-ft/rad

$\sum F_{y(B)} = 0 \Rightarrow 5 + (2S_1+K_1)\, u_1 + K_2\, u_2 + (S_2-S_2)\, u_3 = -10 \Rightarrow 8160\, u_1 + 2400\, u_2 + 0 = -15$

$\sum M_{x(B)} = 0 \Rightarrow 12.5 + K_2\, u_1 + (K_3+2T_1)\, u_2 + 0 = 0 \Rightarrow 2400\, u_1 + 28000\, u_2 + 0 = -12.5$

$\sum M_{z(B)} = 0 \Rightarrow 0 + (S_2-S_2)\, u_1 + 0 + (2S_3+T_2)\, u_3 = 0 \Rightarrow 0 + 0 + 67000\, u_3 = 0$

$\Rightarrow u_1 = -1.751\times10^{-3}$ ft, $u_2 = -0.296\times10^{-3}$ rad, $u_3 = 0$

$SF_{(A)} = 0 - S_1\, u_1 + 0 + S_2\, u_3 = 6.72$ k, $SF_{(B)/AB} = 0 + S_1\, u_1 + 0 - S_2\, u_3 = -6.72$ k,

$SF_{(B)/BC} = 0 + S_1\, u_1 + 0 + S_2\, u_3 = -6.72$ k, $SF_{(C)} = 0 - S_1\, u_1 + 0 - S_2\, u_3 = 6.72$ k,

$SF_{(B)/BE} = 5 + K_1\, u_1 + K_2\, u_2 + 0 = 3.44$ k, $SF_{(E)/BE} = 5 - K_1\, u_1 - K_2\, u_2 + 0 = 6.55$ k

$BM_{(A)} = 0 - S_2\, u_1 + 0 + S_4\, u_3 = 16.81$ k′, $BM_{(B)/AB} = 0 - S_2\, u_1 + 0 + S_3\, u_3 = 16.81$ k′,

$BM_{(B)/BC} = 0 + S_2\, u_1 + 0 + S_3\, u_3 = -16.81$ k′, $BM_{(C)} = 0 + S_2\, u_1 + 0 + S_4\, u_3 = -16.81$ k′,

$BM_{(B)/BE} = 12.5 + K_2\, u_1 + K_3\, u_2 + 0 = 3.56$ k′, $BM_{(E)/BE} = -12.5 + K_2\, u_1 + K_4\, u_2 + 0 = -19.07$ k′

$T_{(A)} = 0 + 0 - T_1\, u_2 + 0 = 1.78$ k′, $T_{(B)/AB} = 0 + 0 + T_1\, u_2 + 0 = -1.78$ k′,

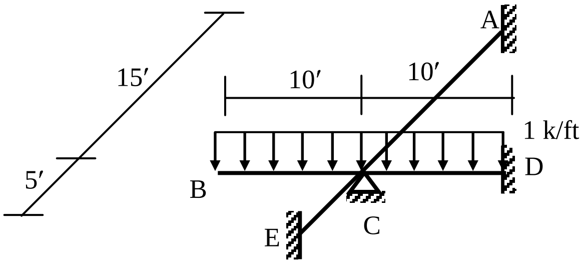
$T_{(B)/BC} = 0 + 0 + T_1\, u_2 + 0 = -1.78$ k′, $T_{(C)} = 0 + 0 - T_1\, u_2 + 0 = 1.78$ k′,

$T_{(B)/BE} = 0 + 0 + 0 + T_2\, u_3 = 0$, $T_{(E)/BE} = 0 + 0 + 0 - T_2\, u_3 = 0$

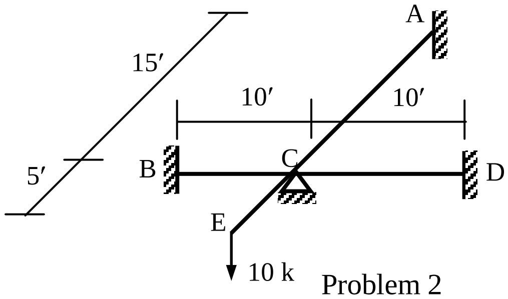
Problems on Stiffness Method for Grids

Given: EI = 40×10^3 k-ft^2, GJ = 30×10^3 k-ft^2 for all the problems.

1,2. Use the Stiffness Method to calculate the rotations at joint C for the grids shown below.

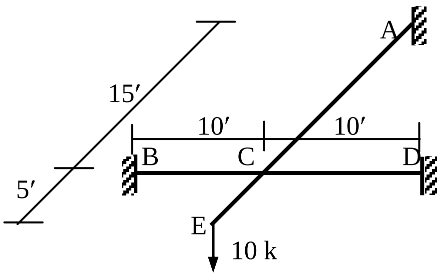


Problem 1

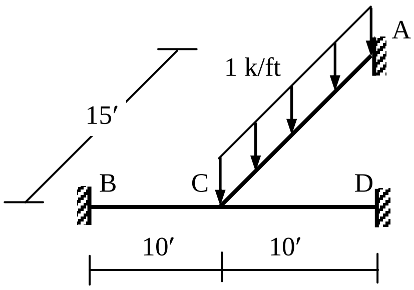


Problem 2

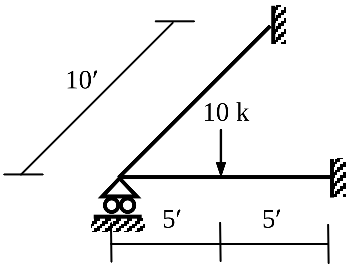
3. Using the Stiffness Method, calculate the deflection and rotation at joint C for the grid shown in the figure below.



4, 5. Formulate the stiffness matrix and load vector for the grids shown in the figures below.



Problem 4



Problem 5