

## Structural Stability and Determinacy

Stability is an essential precondition for a structure to be able to carry the loads it is subjected to, and therefore being suitable for structural analysis. Since structural analysis is based on solving the unknown forces (or displacements) within a structure using some equations, it is essentially the comparison of the equations and unknowns that determine the stability of a structural system.

Statical determinacy of a structure is a concept closely related to its stability. Once a structure is determined to be stable, it is important to determine whether it remains in equilibrium; i.e., if it can be analyzed by the concepts of statics alone, particularly for hand calculation. Although this information is not essential in the context of computer-based structural analysis, there are important differences between structures that are solvable by statics alone and those requiring additional information (usually from kinematics).

The number of external reactions is often the simplest means to determine the stability of a structure. They must be greater than the number of equations available for the structure to remain in static equilibrium. The number of equations for two-dimensional (planar) structures (e.g., 2D trusses and 2D frames) is three (i.e.,  $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum M_z = 0$ ), while it is six (i.e.,  $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum F_z = 0$ ,  $\sum M_x = 0$ ,  $\sum M_y = 0$ ,  $\sum M_z = 0$ ) for three-dimensional (non-coplanar) structures (e.g., 3D trusses and 3D frames).

The number of equations of static equilibrium may be increased for structures with internal hinges ( $h$ ), each providing an additional equation for  $BM = 0$ . Therefore stability requires the number of equations to be greater than (The number of equations of statics +  $h$ ); e.g.,  $(3 + h)$  for 2D frames and  $(6 + h)$  for 3D frames. This condition is not applicable for trusses though, because truss members are axially loaded only and have no bending moment.

However, structures can be unstable despite having adequate number of external reactions; i.e., they can be internally unstable. In general, the static stability of a structure depends on the number of unknown forces and the equations of statics available to determine these forces. This requires

- \* The number of structural members =  $m$ , e.g., each having one unknown (axial force) for trusses, three (axial force, shear force, bending moment) for 2D frames and six (axial force, two shear forces, torsional moment, two bending moments) for 3D frames
- \* The number of external reactions =  $r$
- \* The number of joints =  $j$ , e.g., each having two equations of equilibrium for 2D trusses ( $\sum F_x = 0$ ,  $\sum F_y = 0$ ), three for 2D frames ( $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum M_z = 0$ ), three for 3D trusses ( $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum F_z = 0$ ) and six for 3D frames ( $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum F_z = 0$ ,  $\sum M_x = 0$ ,  $\sum M_y = 0$ ,  $\sum M_z = 0$ ).

Eventually, the term ‘Degree of Statical Indeterminacy (*dosi*)’ is used to denote the difference between the available equations of static equilibrium and the number of unknown forces. The structure is classified as statically unstable, determinate or indeterminate depending on whether *dosi* is  $< 0$ ,  $= 0$  or  $> 0$ . Table 1 shows the conditions of static stability and determinacy of 2D and 3D trusses and frames.

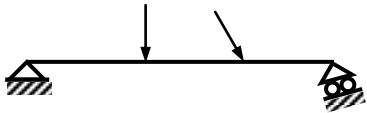
**Table 1: Statical Stability and Determinacy of Trusses and Frames**

Structure	Unknown Forces for		Equations at		Stability		Dosi
	Member	Reaction	Joint	Internal Hinge	Reaction	Dosi	
2D Truss	$m$	$r$	$2j$	*	$r \geq 3$	$Dosi \geq 0$	$m + r - 2j$
2D Frame	$3m$	$r$	$3j$	$h$	$r \geq 3 + h$		$3m + r - 3j - h$
3D Truss	$m$	$r$	$3j$	*	$r \geq 6$		$m + r - 3j$
3D Frame	$6m$	$r$	$6j$	$h$	$r \geq 6 + h$		$6m + r - 6j - h$

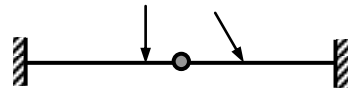
## Problems on Structural Stability and Determinacy

Determine the static/geometric stability and statical indeterminacy of the following structures.

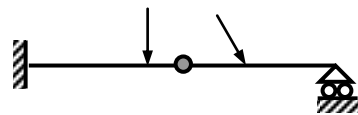
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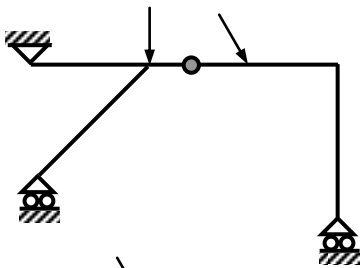
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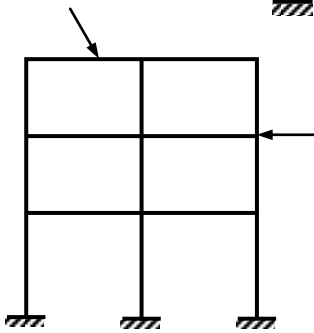
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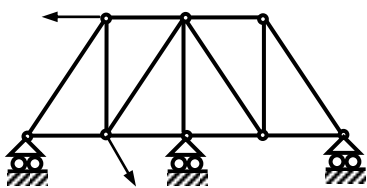
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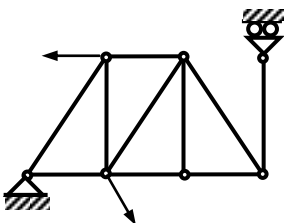
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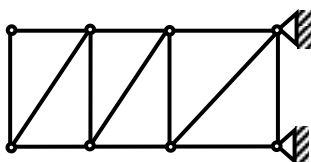
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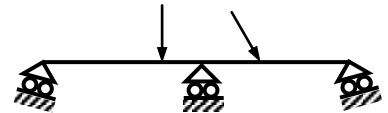
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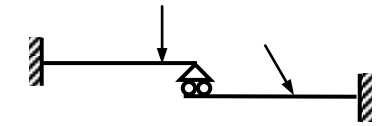
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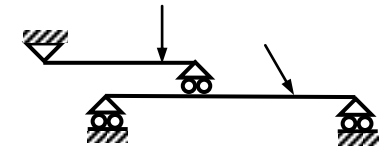
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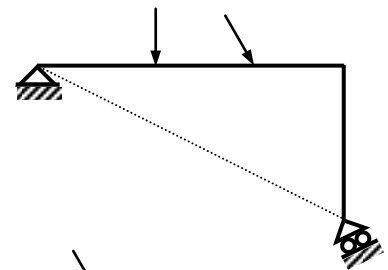
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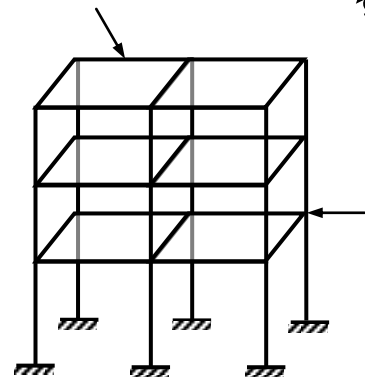
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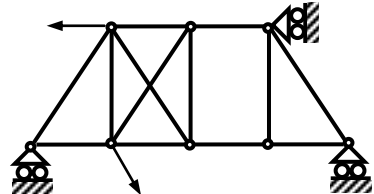
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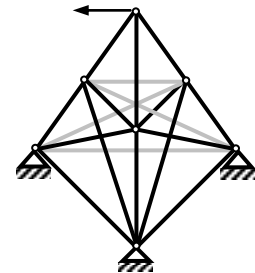
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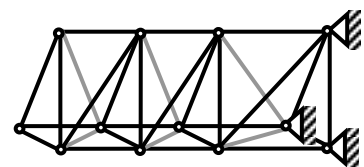
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14.



16.

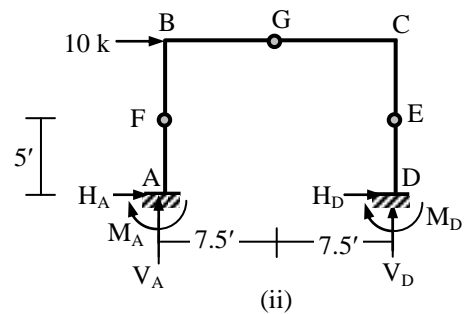
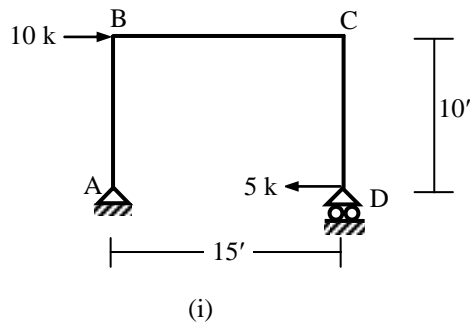


## Axial Force, Shear Force and Bending Moment Diagram of Frames

Frame is an assembly of several flexural members oriented in different directions and connected by rigid joints. Therefore, the axial force, shear force and bending moment diagrams of frame consist of drawing the individual AFD, SFD and BMD for each member (similar to beams) and assembling the diagrams for entire the frame, using the free-body diagram of each member. The equilibrium (i.e.,  $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum M_z = 0$ ) of each joint must be considered while drawing the member free-bodies.

### Example 2.1

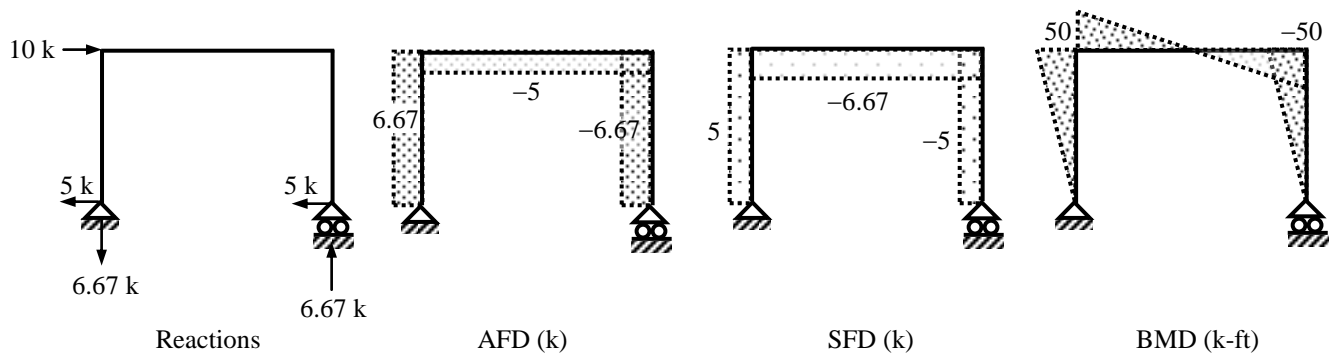
Draw the axial force, shear force and bending moment diagrams of the frames loaded as shown below.



### Solution

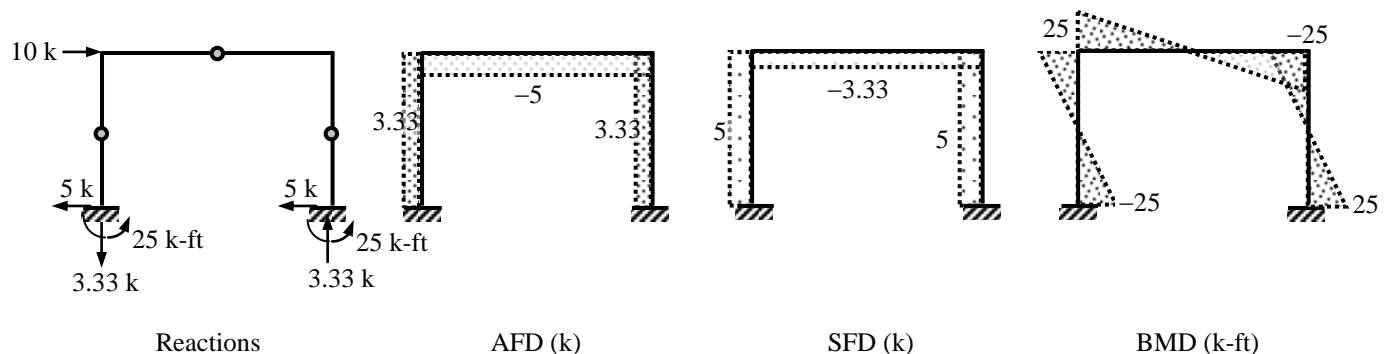
(i) For this frame,  $\text{dosi} = 3 \times 3 + 3 - 3 \times 4 = 0$ ; i.e., It is statically determinate

$$\begin{aligned} \therefore \sum F_x = 0 &\Rightarrow 10 + H_A - 5 = 0 \Rightarrow H_A = -5 \text{ k} \\ \therefore \sum M_A = 0 &\Rightarrow 10 \times 10 - V_D \times 15 = 0 \Rightarrow V_D = 6.67 \text{ k} \\ \therefore \sum F_y = 0 &\Rightarrow V_A + V_D = 0 \Rightarrow V_A = -6.67 \text{ k} \end{aligned}$$



(ii)  $\text{dosi} = 3 \times 6 + 6 - 3 \times 7 - 3 = 0$ ; i.e., It is statically determinate

$$\begin{aligned} \therefore \text{BM}_F = 0 &\Rightarrow -H_A \times 5 + M_A = 0 \Rightarrow M_A = 5H_A; \text{ Similarly } \text{BM}_E = 0 \Rightarrow M_D = 5H_D \\ \text{BM}_G = 0 &\Rightarrow -H_A \times 10 + V_A \times 7.5 + M_A = 0 \Rightarrow 5H_A - 10H_A + 7.5V_D = 0 \Rightarrow H_A = 1.5V_A \\ \text{And also } &\Rightarrow -H_D \times 10 - V_D \times 7.5 + M_D = 0 \Rightarrow 5H_D - 10H_D - 7.5V_D = 0 \Rightarrow H_D = -1.5V_D \\ \therefore \sum F_y = 0 &\Rightarrow V_A + V_D = 0 \Rightarrow V_D = -V_A \\ \text{and } \sum F_x = 0 &\Rightarrow H_A + H_D + 10 = 0 \Rightarrow 1.5V_A - 1.5V_D = -10 \Rightarrow 3V_A = -10 \Rightarrow V_A = -3.33 \text{ k} \\ &\Rightarrow V_D = -V_A = 3.33 \text{ k} \\ \therefore H_A = 1.5V_A &= -5 \text{ k} \text{ and } H_D = -1.5V_D = -5 \text{ k} \end{aligned}$$



(iii)  $\text{dosi} = 3 \times 3 + 6 - 3 \times 4 = 3$ ;  $\therefore$  Assume internal hinges at E, F, G

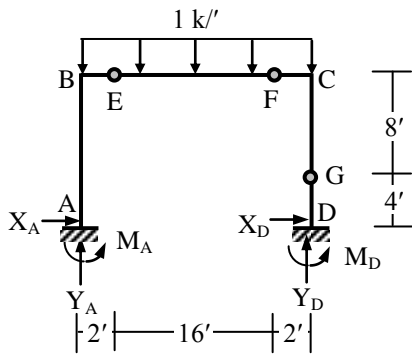
$\therefore$  Free-body of Member EF  $\Rightarrow Y_E = Y_F = 1 \times 16/2 = 8 \text{ k}$

$\therefore$  Member FCG  $\Rightarrow Y_G = Y_F + 1 \times 2 = 10 \text{ k}$ ,  $X_G = X_F$ , and  $-Y_F \times 2 - 1 \times 2 \times 2/2 + X_F \times 8 = 0 \Rightarrow X_F = 2.25 \text{ k}$   
 $\Rightarrow X_G = X_F = 2.25 \text{ k}$

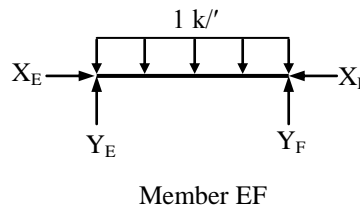
$\therefore$  Member GD  $\Rightarrow Y_D = Y_G = 10 \text{ k}$ ,  $X_D = -X_G = -2.25 \text{ k}$ , and  $-M_D + X_G \times 4 = 0 \Rightarrow M_D = 9 \text{ k-ft}$

$\therefore$  Overall  $\Sigma F_x = 0 \Rightarrow X_A + X_D = 0 \Rightarrow X_A = 2.25 \text{ k}$ ,

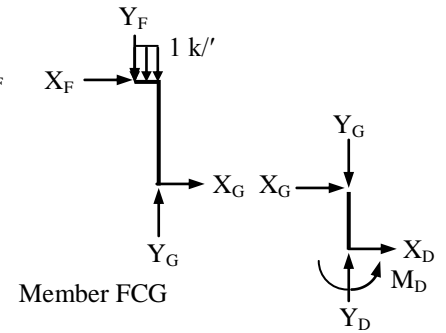
$\Sigma F_y = 0 \Rightarrow Y_A = 1 \times 20 - 10 = 10 \text{ k}$ , and  $\Sigma M_A = 0 \Rightarrow -M_A - M_D = 0 \Rightarrow M_A = -9 \text{ k-ft}$



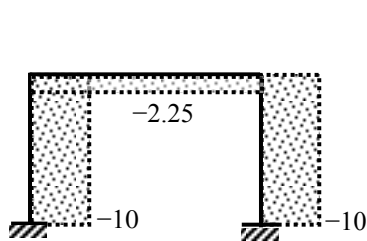
Entire Frame



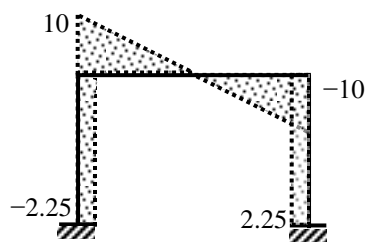
Member EF



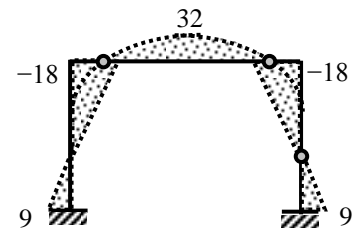
Member FCG  
Member GD



AFD (k)



SFD (k)

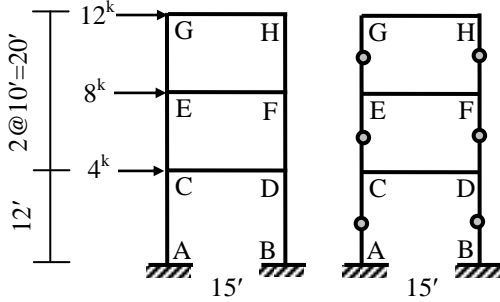


BMD (k-ft)

## Axial Force, Shear Force and Bending Moment Diagram of Multi-Storied Frames

### Example 2.2

Draw the axial force, shear force and bending moment diagrams of the three-storied frame loaded as shown below, assuming (i) equal share of story shear forces between columns, (ii) internal hinge at column midspans.



$$m = 9, r = 6, j = 8 \Rightarrow \text{dosi} = 3 \times 9 + 6 - 3 \times 8 = 9$$

$\therefore$  Nine assumptions needed for statical determinacy

Equal shear among story columns  $\Rightarrow$

$$V_{EG} = V_{FH} = 12/2 = 6^k, V_{CE} = V_{DF} = (12 + 8)/2 = 20/2 = 10^k$$

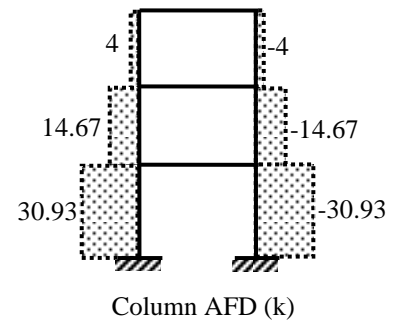
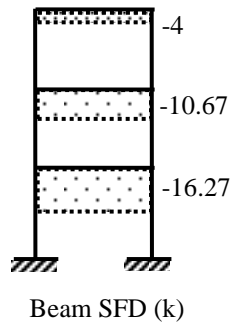
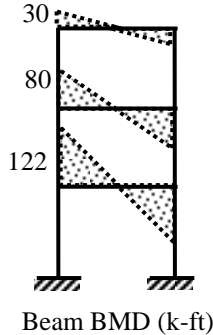
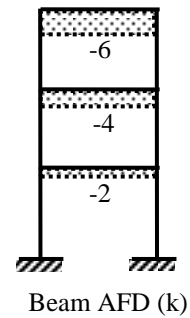
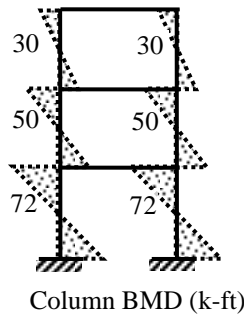
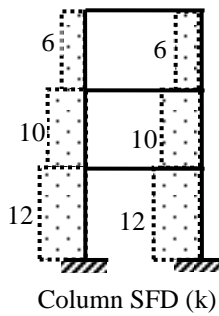
$$V_{AC} = V_{BD} = (12 + 8 + 4)/2 = 24/2 = 12^k$$

$\therefore$  End bending moments in columns are

$$M_{EG} = M_{FH} = 6 \times 10/2 = 30^k, M_{CE} = M_{DF} = 10 \times 10/2 = 50^k$$

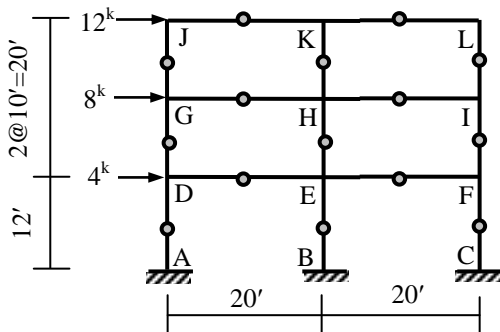
$$M_{AC} = M_{BD} = 12 \times 12/2 = 72^k$$

The rest of the calculations follow from the free-body diagrams



### Example 2.3

Draw the AFD, SFD and BMD of the three-storied, two-bay frame loaded as shown below, assuming (i) internal hinge at the midspan of each column and beam, (ii) no axial force at middle columns.



$$m = 15, r = 9, j = 12 \Rightarrow \text{dosi} = 3 \times 15 + 9 - 3 \times 12 = 18$$

$\therefore$  18 assumptions needed for statical determinacy

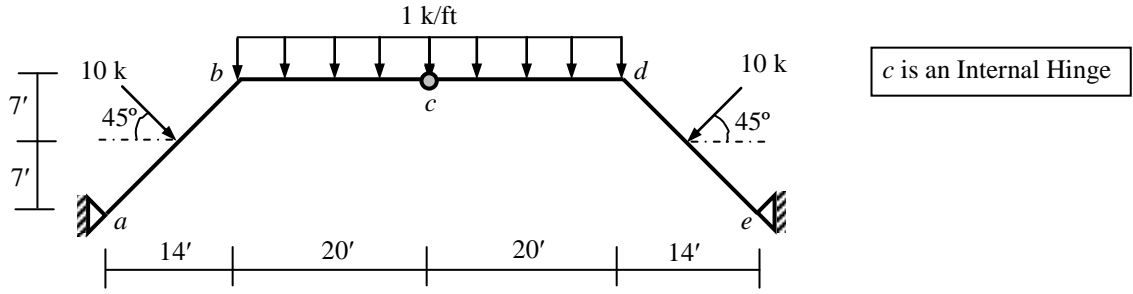
Internal hinges  $\Rightarrow$  BM = 0, at midspan of 6 beams and 9 columns

No axial force at mid columns  $\Rightarrow X_{BE} = X_{EH} = X_{HK} = 0$

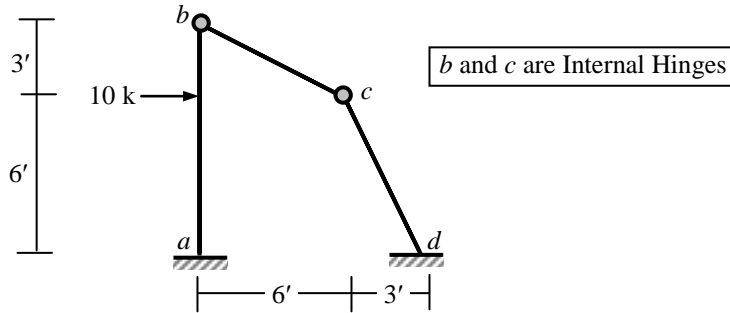
The rest of the calculations follow from the free-body diagrams

### Problems on AFD, SFD, BMD of Frames

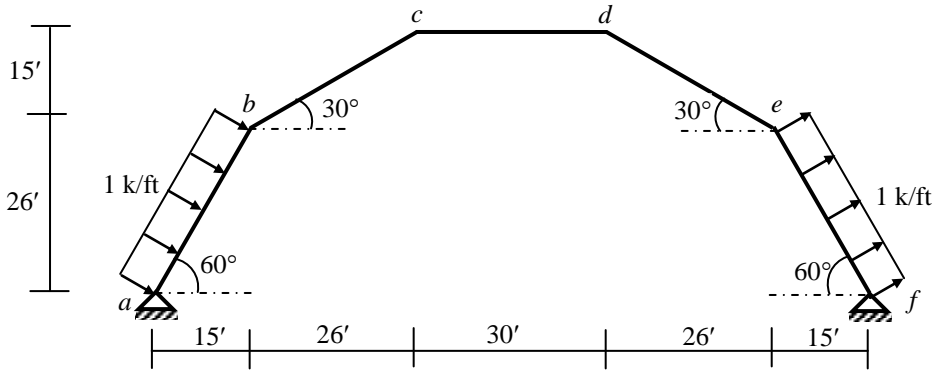
1. Draw the AFD, SFD and BMD of the beam *bcd* in the frame *abcde* loaded as shown below.



2. Determine the degree of statical indeterminacy (dosi) of the frame *abcd* shown below. Also draw the Axial Force, Shear Force and Bending Moment diagram of the member *ab*, assuming the horizontal reactions at support *a* and *d* are equal.



3. Determine the degree of statical indeterminacy (dosi) of the frame shown below. Also draw the Axial Force, Shear Force and Bending Moment diagram of the member *ab*, assuming the horizontal reactions at support *a* and *f* are equal.



4. Figure (a) below shows the column shear forces (kips) in a 2-storied frame.  
 (A) Determine the degree of statical indeterminacy (dosi) of the frame.  
 (B) Calculate the applied loads  $F_1$ ,  $F_2$  and draw the (i) beam AFD, (ii) column BMD, (iii) beam BMD (assuming internal hinges at member midspans).

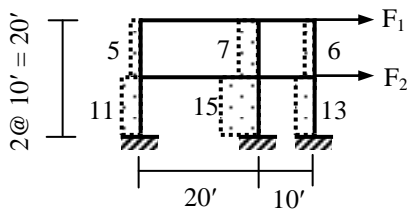


Fig. (a)

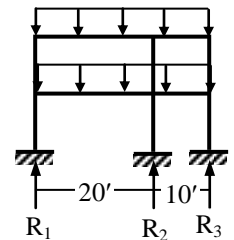


Fig. (b)

5. The support reaction  $R_1$  for the 2-storied frame (loaded by equal UDLs on beams only) in Fig. (b) [shown above] is 50 kips. Calculate the reactions  $R_2$ ,  $R_3$  and draw the (i) column AFD, (ii) beam SFD, (iii) beam BMD and (iv) column BMD (making appropriate assumptions).

## Live Loads and Influence Lines

### Live Loads

Live Loads, either moving or movable, produce varying effects in a structure, depending on the location of the part being considered, the force function being considered (i.e., reaction, shear, bending moment, etc), and the position of the loads producing the effect. It is necessary to determine the critical position of the loading system which will produce the greatest force (i.e., reaction, shear or bending moment) and to calculate that force after having found the critical position.

### Influence Lines

An influence line is a diagram showing the variation of a particular force (i.e., reaction, shear, bending moment at a section, stress at a point or other direct function) due to a unit load moving across the structure. Influence Line can be defined to be a curve the ordinate of which at any point equals the value of some particular function due to a unit load (say 1-lb load) acting at that point, and is constructed by plotting directly under the point where the unit load is placed an ordinate the height of which represents the value of the particular function being studied when the load is in that position. Influence Lines are often useful in studying the effect of a system of moving loads across a structure.

### Example 3.1

Draw the influence lines of  $R_A$ ,  $V_C$  and  $M_C$  for the simply supported beam AB shown below in Fig. 3.1.

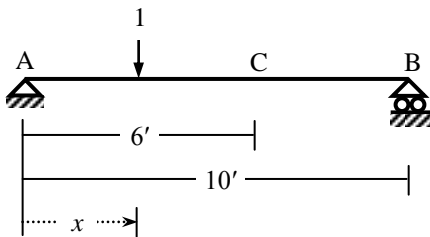


Fig. 3.1

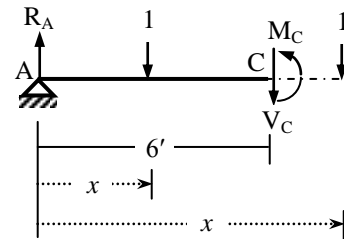


Fig. 3.2

The forces  $R_A$ ,  $V_C$  and  $M_C$  are shown in Fig. 3.2, which is the free-body diagram of AC. Table 3.1 shows the values of  $R_A$ ,  $V_C$  and  $M_C$  calculated for various values of the distance  $x$ , while Fig. 3.3 shows the respective influence lines.

**Table 3.1: Calculated Forces**

$x$ (ft)	$R_A$	$V_C$	$M_C$ (ft)
0	1.0	0.0	0.0
1	0.9	-0.1	0.4
2	0.8	-0.2	0.8
3	0.7	-0.3	1.2
4	0.6	-0.4	1.6
5	0.5	-0.5	2.0
6	0.4	-0.6, 0.4	2.4
7	0.3	0.3	1.8
8	0.2	0.2	1.2
9	0.1	0.1	0.6
10	0.0	0.0	0.0

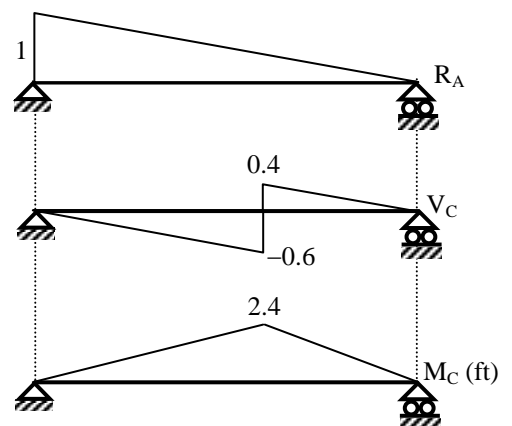


Fig. 3.3: Influence Lines of  $R_A$ ,  $V_C$ ,  $M_C$

## Equations of Influence Lines using Singularity Functions

### Influence Lines and Singularity Functions

The calculations shown before for influence lines can be carried out more conveniently by deriving their general equations based on singularity functions. The following examples illustrate this method for two simple cases; i.e., a simply supported beam and a cantilever beam.

#### Example 3.2

Derive the equations for the influence lines of  $R_A$ ,  $V_C$  and  $M_C$  for the simply supported beam AB shown below.

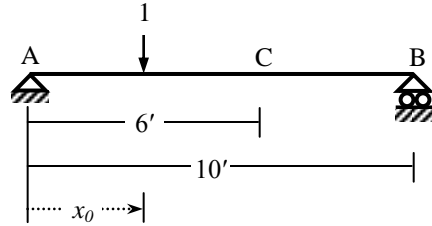


Fig. 3.4: Simply Supported Beam with Unit Load

$$w(x) = -1 \langle x - x_0 \rangle_*^{-1} \quad \dots\dots\dots(3.1)$$

$$\Rightarrow V(x) = -1 \langle x - x_0 \rangle^0 + C_1 \quad \dots\dots\dots(3.2)$$

$$\Rightarrow M(x) = -1 \langle x - x_0 \rangle^1 + C_1 x + C_2 \quad \dots\dots\dots(3.3)$$

Boundary conditions:  $M(0) = 0 \Rightarrow C_2 = 0$ ,  $M(10) = 0 \Rightarrow C_1 = 1 - x_0/10$

$$\begin{aligned} \therefore V(x) &= -1 \langle x - x_0 \rangle^0 + (1 - x_0/10) \\ M(x) &= -1 \langle x - x_0 \rangle^1 + (1 - x_0/10) x \\ \Rightarrow R_A &= C_1 = 1 - x_0/10 \quad \dots\dots\dots(3.4) \end{aligned}$$

$$V_C = V(6) = -1 \langle 6 - x_0 \rangle^0 + (1 - x_0/10) \quad \dots\dots\dots(3.5)$$

$$M_C = M(6) = -1 \langle 6 - x_0 \rangle^1 + (1 - x_0/10) 6 \quad \dots\dots\dots(3.6)$$

#### Example 3.3

Derive the equations for the influence lines of  $R_B$ ,  $V_C$  and  $M_C$  for the cantilever beam AB shown below.

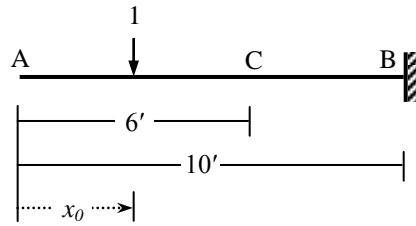


Fig. 3.5: Cantilever Beam with Unit Load

$$w(x) = -1 \langle x - x_0 \rangle_*^{-1} \quad \dots\dots\dots(3.7)$$

$$\Rightarrow V(x) = -1 \langle x - x_0 \rangle^0 + C_1 \quad \dots\dots\dots(3.8)$$

$$\Rightarrow M(x) = -1 \langle x - x_0 \rangle^1 + C_1 x + C_2 \quad \dots\dots\dots(3.9)$$

Boundary conditions:  $V(0) = 0 \Rightarrow C_1 = 0$ ,  $M(0) = 0 \Rightarrow C_2 = 0$

$$\begin{aligned} \therefore V(x) &= -1 \langle x - x_0 \rangle^0 \\ M(x) &= -1 \langle x - x_0 \rangle^1 \\ \Rightarrow R_B &= 1 - C_1 = 1 \quad \dots\dots\dots(3.10) \end{aligned}$$

$$V_C = V(6) = -1 \langle 6 - x_0 \rangle^0 \quad \dots\dots\dots(3.11)$$

$$M_C = M(6) = -1 \langle 6 - x_0 \rangle^1 \quad \dots\dots\dots(3.12)$$



## Influence Lines of Beams using Müller-Breslau's Principle

Although it is possible to derive the equations of influence lines using Singularity Functions, the method is still quite laborious and too mathematical to be used for practical purpose. Therefore faster and less complicated methods are desirable. The *Müller-Breslau's Principle*, based on deflected shapes of modified structures, is a widely used method particularly for drawing qualitative influence lines of beams.

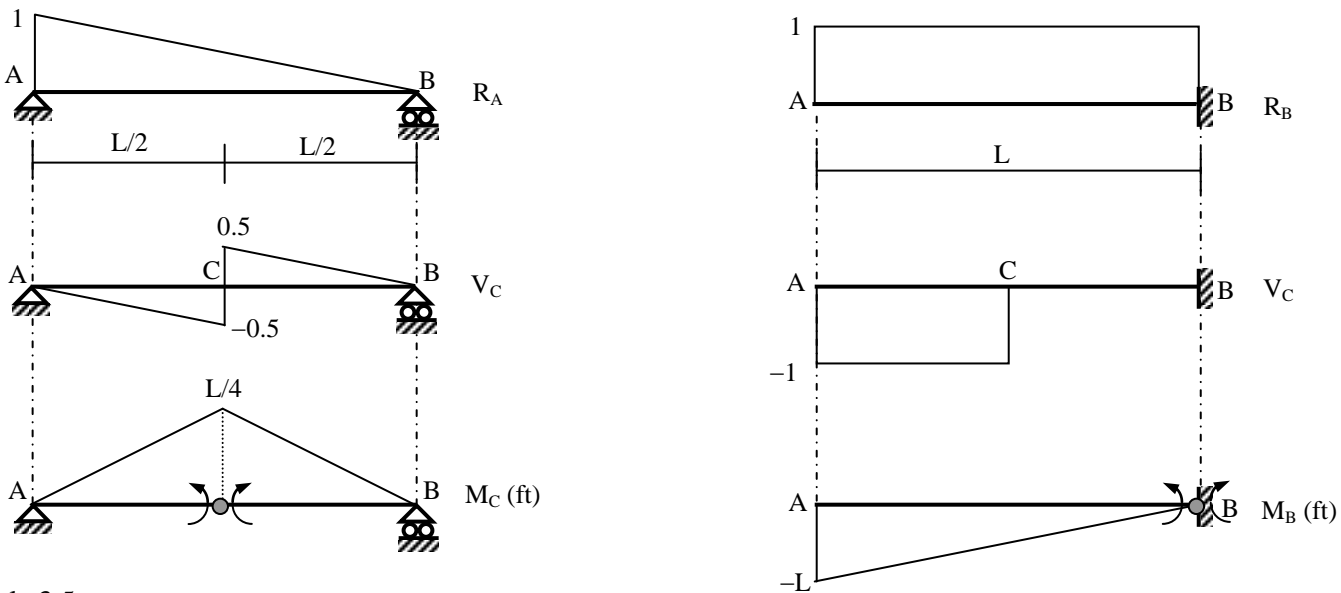
### Müller-Breslau's Principle

The ordinates of the influence line for any force ( i.e., reaction, shear force, bending moment) of any structure are equal to those of the deflected shape obtained by removing the restraint corresponding to that element from the structure and introducing in its place a corresponding unit deformation into the modified structure.

Therefore, a unit deflection of support is used for the influence line of support reaction (after removing the support), unit discontinuity of section is used for influence line of its shear force (after inserting a sectional shear cut) while a unit rotation is used for influence line of bending moment (after inserting an internal hinge).

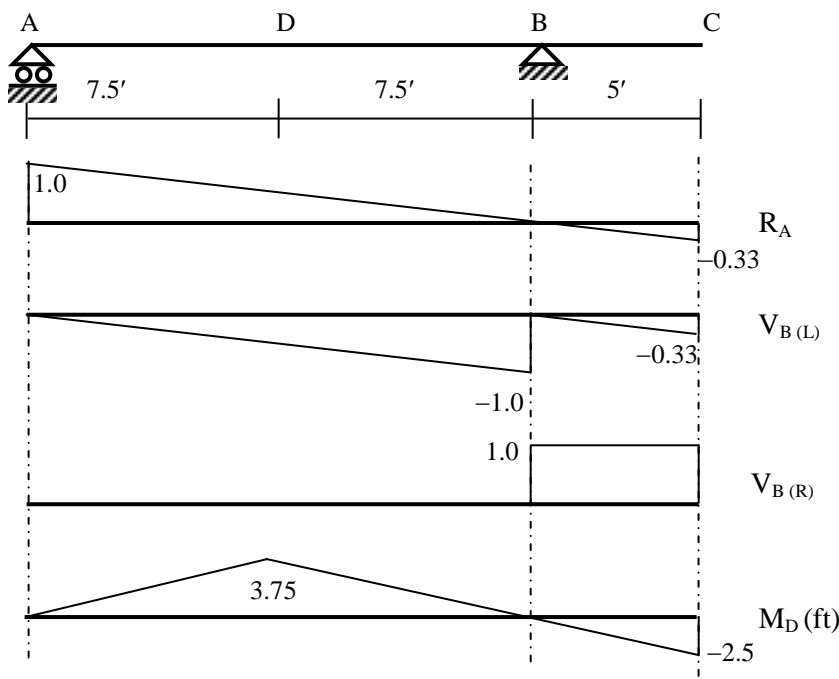
### Example 3.4

Draw the influence lines (as mentioned) for the simply supported beam and cantilever beam shown below.



### Example 3.5

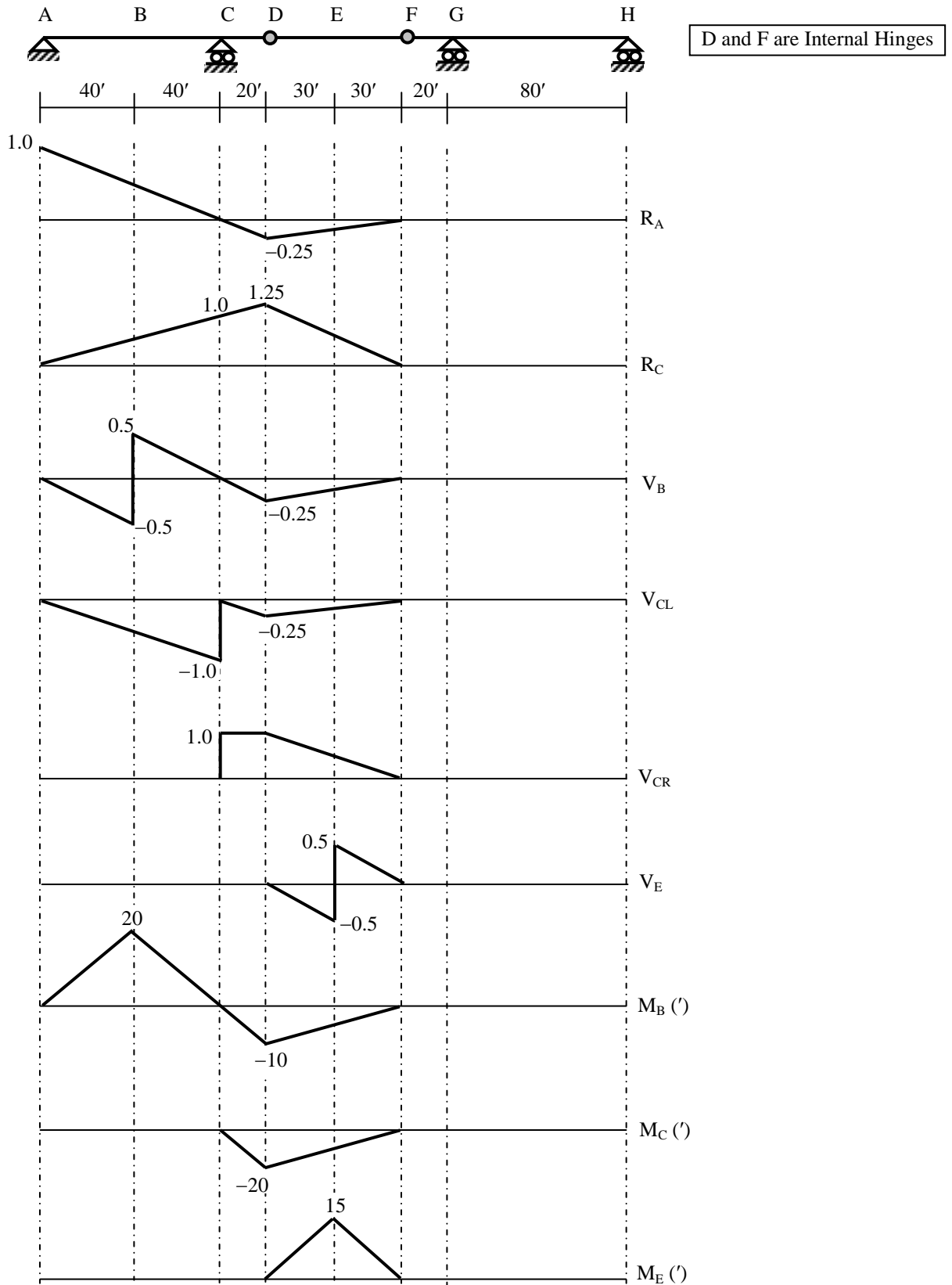
Draw the influence lines  $R_A$ ,  $V_{B(L)}$ ,  $V_{B(R)}$  and  $M_D$  and  $M_B$  the beam shown below.



**Example 3.6**

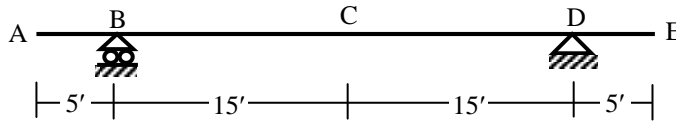
For the beam shown below, draw the influence lines for

- (i)  $R_A, R_C$ , (ii)  $V_B, V_{CL}, V_{CR}, V_E$ , (iii)  $M_B, M_C, M_E$ .

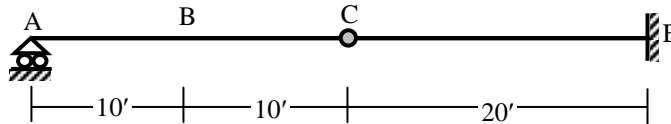


### Problems on Influence Lines of Beams

1. For the beam shown below, draw the influence lines for  
 (i)  $R_B$ , (ii)  $V_{BL}$ ,  $V_{BR}$ ,  $V_C$ , (iii)  $M_B$ ,  $M_C$ .

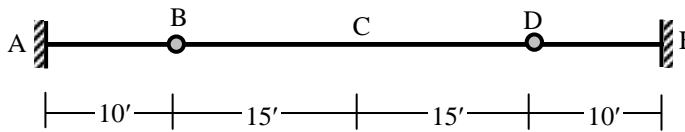


2. For the beam shown below, draw the influence lines for  
 (i)  $R_A$ ,  $R_E$ , (ii)  $V_B$ ,  $V_{CL}$ ,  $V_{CR}$ , (iii)  $M_B$ ,  $M_E$ .



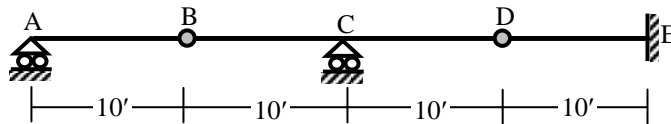
C is an Internal Hinge

3. For the beam shown below, draw the influence lines for  
 (i)  $R_A$ , (ii)  $V_{BL}$ ,  $V_{BR}$ ,  $V_C$ , (iii)  $M_A$ ,  $M_C$ .



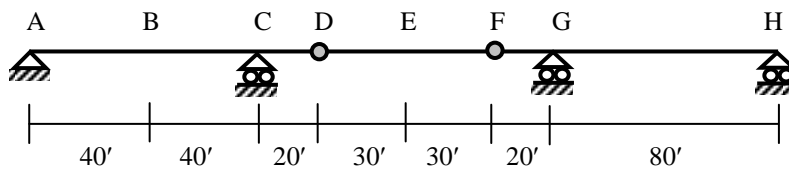
B and D are Internal Hinges

4. For the beam shown below, draw the influence lines for  
 (i)  $R_A$ ,  $R_C$ ,  $R_E$ , (ii)  $V_B$ ,  $V_{CL}$ ,  $V_{CR}$ ,  $V_D$ , (iii)  $M_C$ ,  $M_E$ .



B and D are Internal Hinges

5. For the beam shown below, draw the influence lines for  
 (i)  $R_A$ ,  $R_C$ , (ii)  $V_B$ ,  $V_{CL}$ ,  $V_{CR}$ ,  $V_E$ , (iii)  $M_B$ ,  $M_C$ ,  $M_E$ .



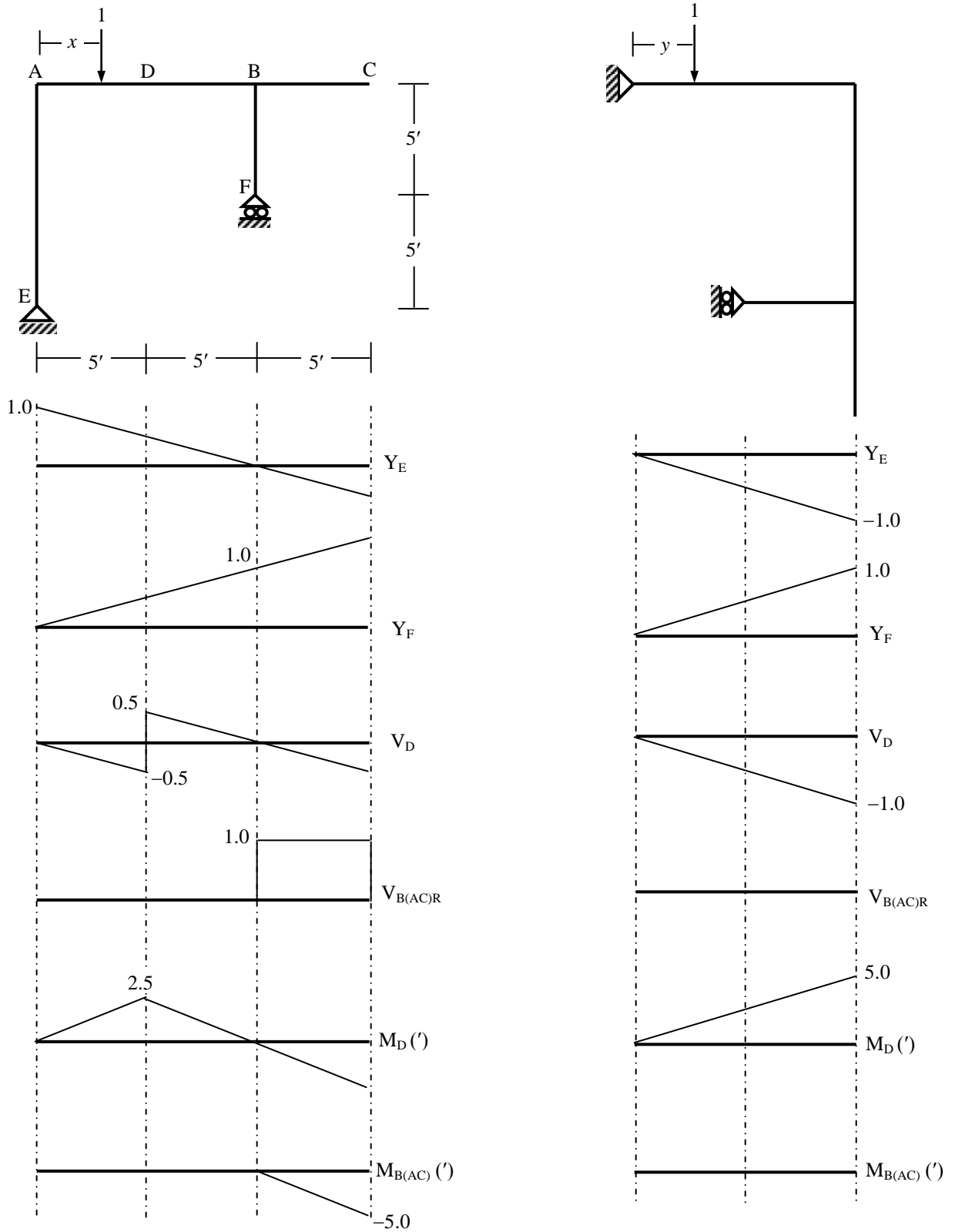
D and F are Internal Hinges

## Influence Lines of Frames

### Example 3.7

For the frame shown below, draw the influence lines for

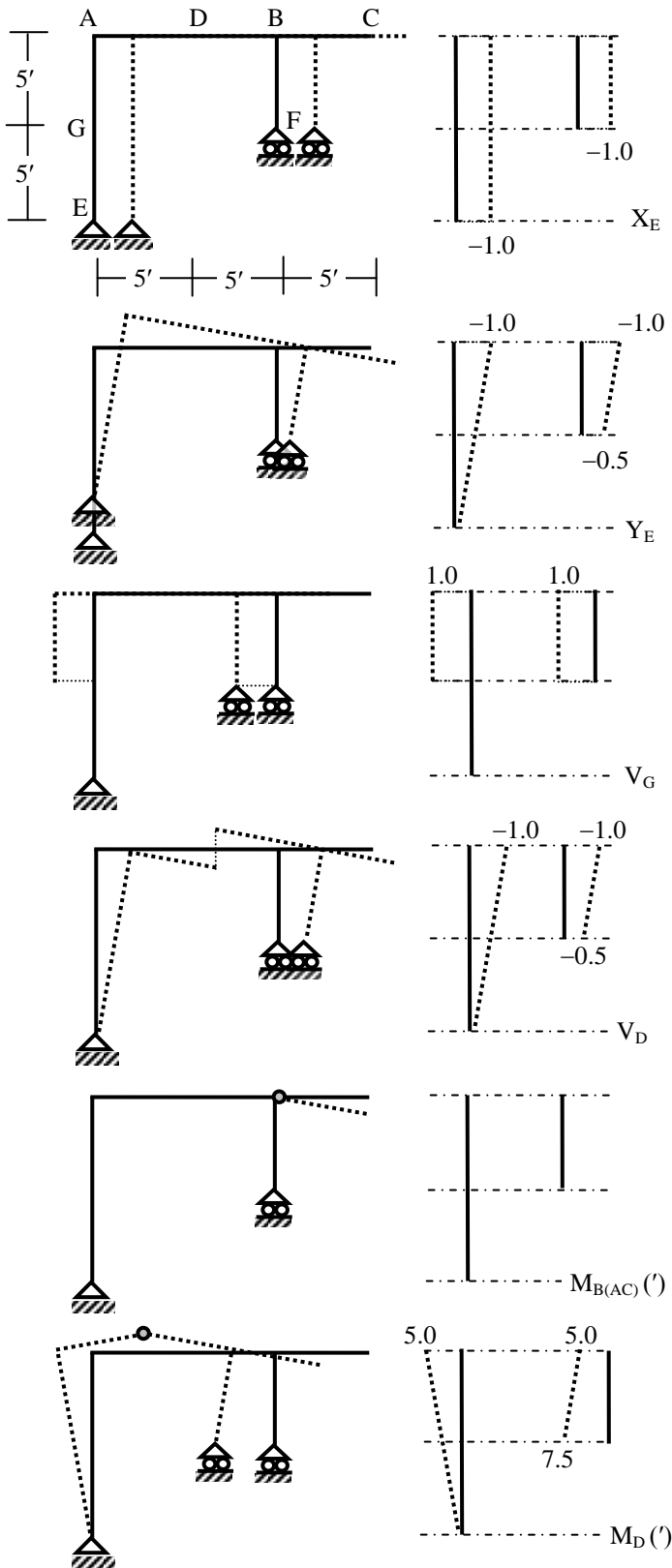
- (i)  $Y_E$ ,  $Y_F$ , (ii)  $V_D$ ,  $V_{B(AC)R}$ , (iii)  $M_D$ ,  $M_{B(AC)}$ , if the unit load moves over  
 (a) beam AC, (b) column EA.



## Influence Lines of Frames using Müller-Breslau's Principle

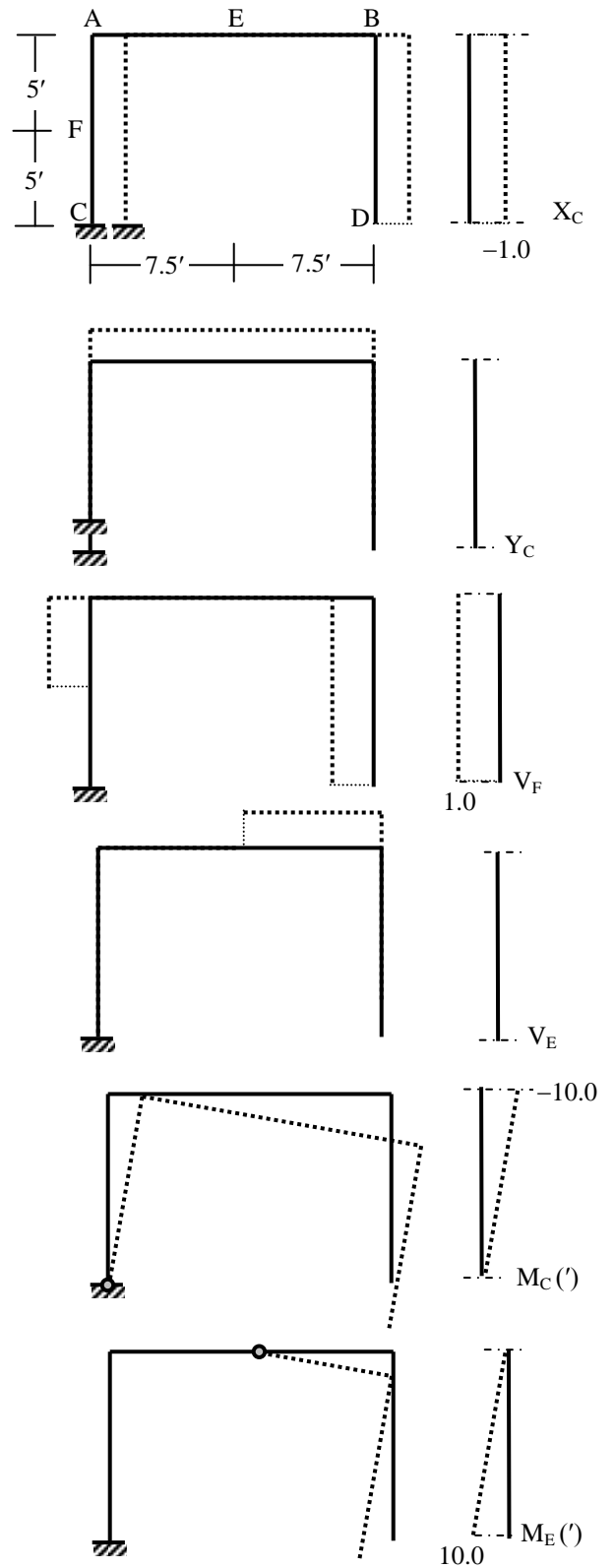
### Example 3.8

For the frame shown below, draw the influence lines for  
 (i)  $X_E$ ,  $Y_E$ , (ii)  $V_G$ ,  $V_D$ , (iii)  $M_{B(AC)}$ ,  $M_D$ ,  
 if the unit load moves over (a) column EA, (b) column FB.



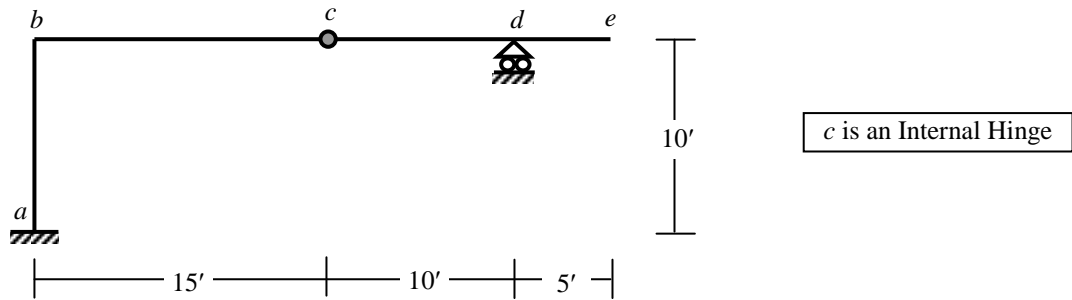
### Example 3.9

For the frame shown below, draw the influence lines for  
 (i)  $X_C$ ,  $Y_C$ , (ii)  $V_F$ ,  $V_E$ , (iii)  $M_C$ ,  $M_E$ ,  
 if the unit load moves over column DB.

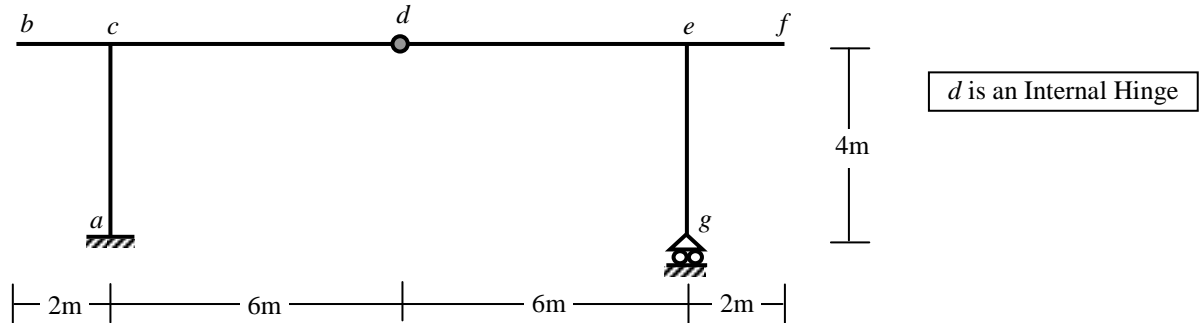


### Problems on Influence Lines of Frames

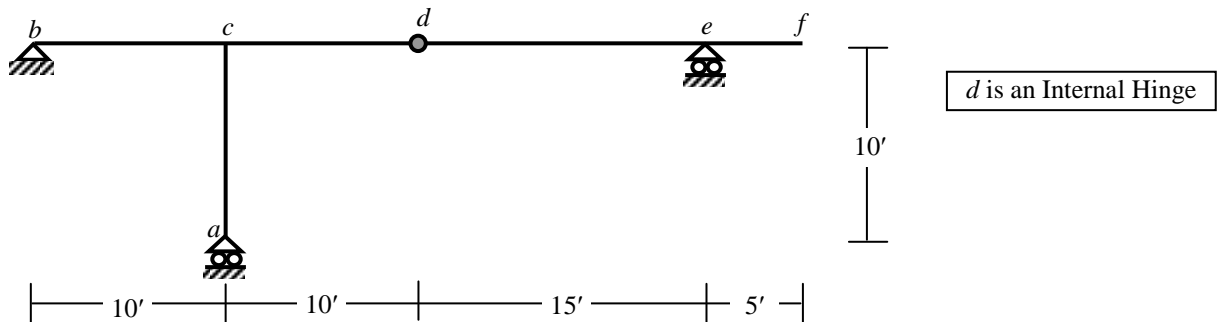
1. Determine the degree of statical indeterminacy (dosi) of the frame  $abcde$  shown below, and draw the influence lines of  $X_a$ ,  $Y_a$ ,  $V_c$ ,  $M_a$  and  $M_{b(be)}$ , if the unit load moves over (i) beam  $be$ , (ii) column  $ab$ .



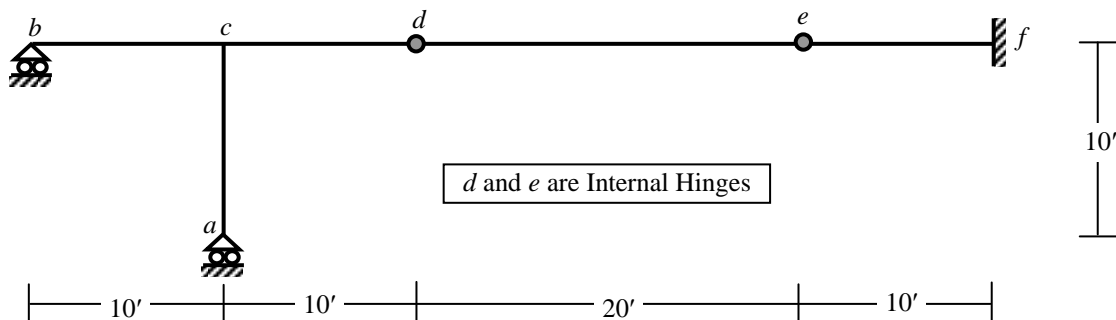
2. Determine the degree of statical indeterminacy (dosi) of the frame  $abcdefg$  shown below, and draw the influence lines of  $X_a$ ,  $Y_a$ ,  $V_{e(L)}$ ,  $M_a$  and  $M_{c(ac)}$ , if the unit load moves over (i) beam  $bf$ , (ii) column  $eg$ .



3. Determine the degree of statical indeterminacy (dosi) of the frame  $abcdef$  shown below, and draw the influence lines of  $Y_a$ ,  $X_b$ ,  $V_{c(R)}$  and  $M_{c(bf)}$ , if the unit load moves over (i) beam  $bf$ , (ii) column  $ac$ .



4. Determine the degree of statical indeterminacy (dosi) of the frame  $abcdef$  shown below, and draw the influence lines of  $X_f$ ,  $Y_b$ ,  $V_d$  and  $M_{c(ac)}$ , if the unit load moves over (i) beam  $bf$ , (ii) column  $ac$ .



## Influence Lines of Girders with Floor Beams

The loads to which a beam or girder is subjected are not often applied to it directly but to a secondary framing system which is supported by the beam or girder. A typical construction of this kind is shown below in Fig. 1(a). In such a structure, the loads are applied to the longitudinal members  $S$ , which are called *stringers*. These are supported by the transverse members  $FB$ , called *floor beams*. The floor beams are in turn supported by *girders*  $G$ . Therefore, no matter where the loads are applied to the stringers as a uniformly distributed load or as some system of concentrated loads, their effect on the girder is that of concentrated loads applied by the floor beams at points  $a, b, c, d$  and  $e$ .

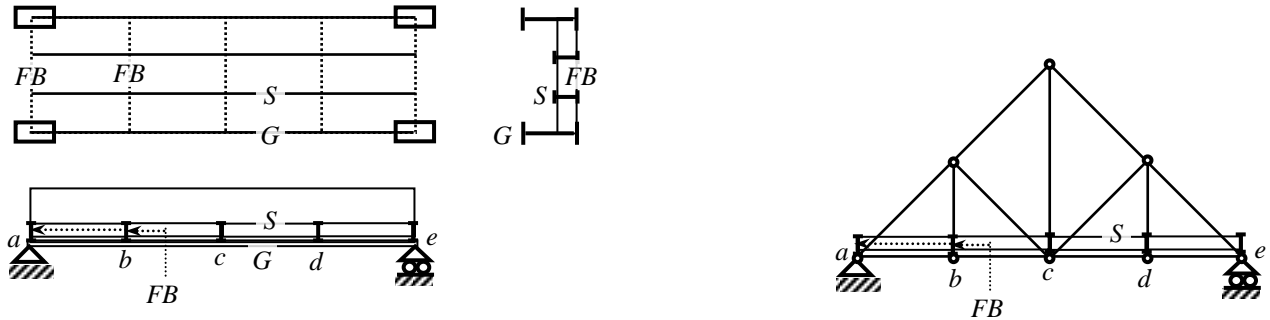
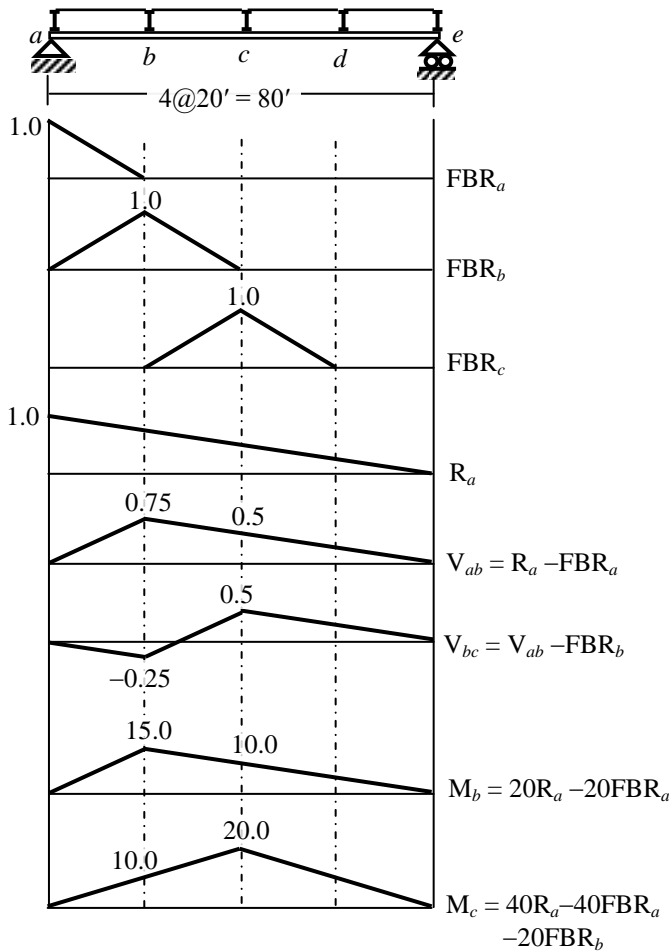


Fig. 1: Floor Beam System for (a) Plate Girder Bridge, (b) Truss Bridge

Fig. 1(b) shows another example of such a support system for a bridge truss. This will be discussed in more detail in subsequent lectures.

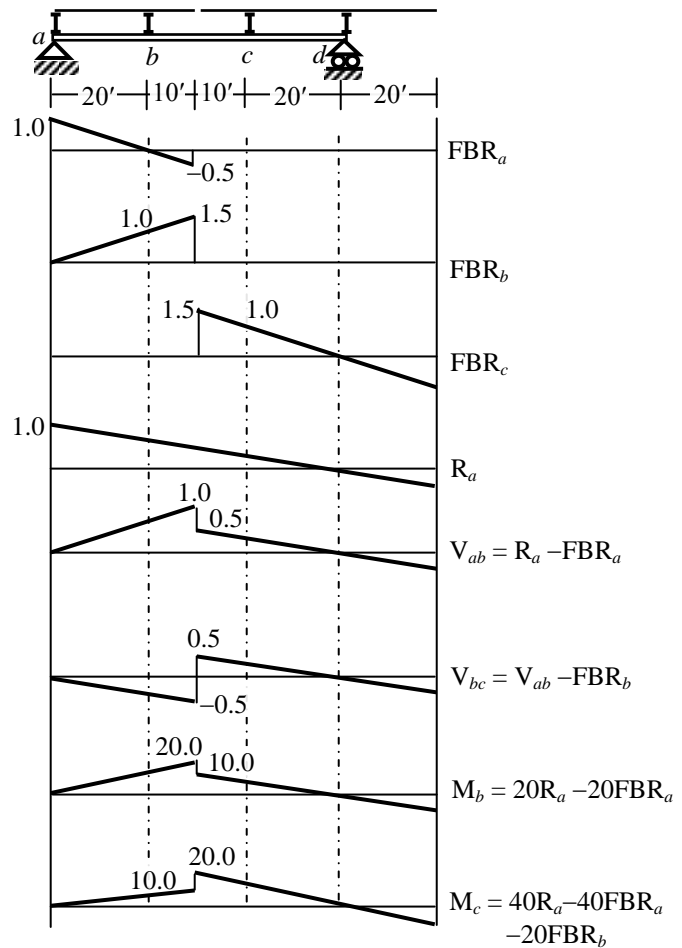
### Example 3.10

For the floor beam system shown, draw influence lines for (i)  $FBR_a, FBR_b, FBR_c$ , (ii)  $R_a$ , (iii)  $V_{ab}, V_{bc}$ , (iv)  $M_b, M_c$



### Example 3.11

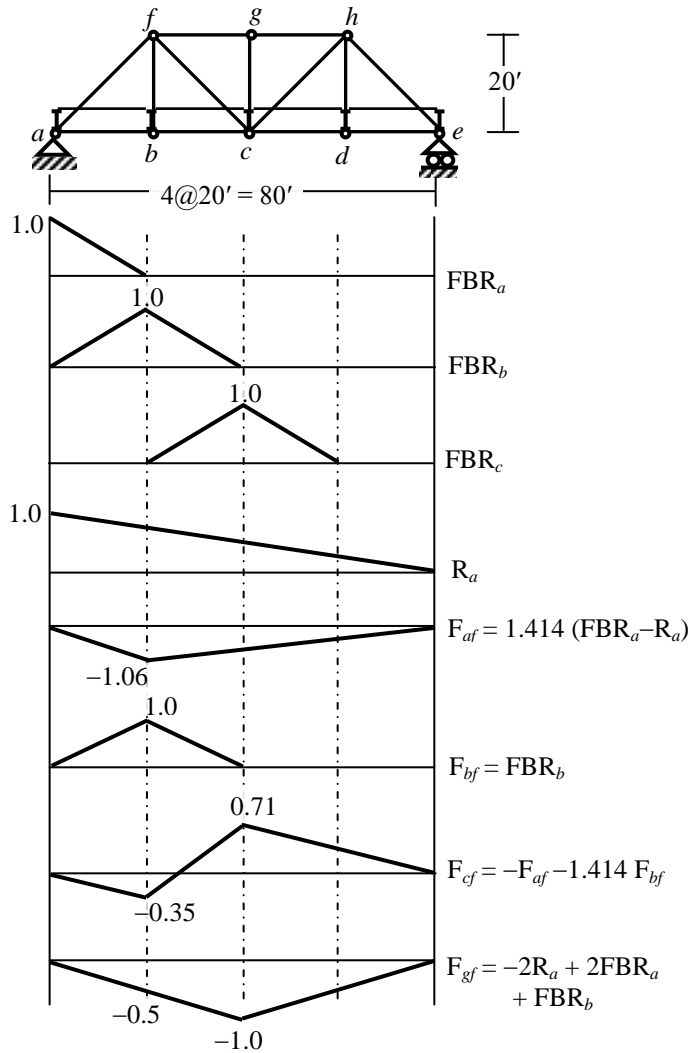
For the floor beam system shown, draw influence lines for (i)  $FBR_a, FBR_b, FBR_c$ , (ii)  $R_a$ , (iii)  $V_{ab}, V_{bc}$ , (iv)  $M_b, M_c$



## Influence Lines of Trusses

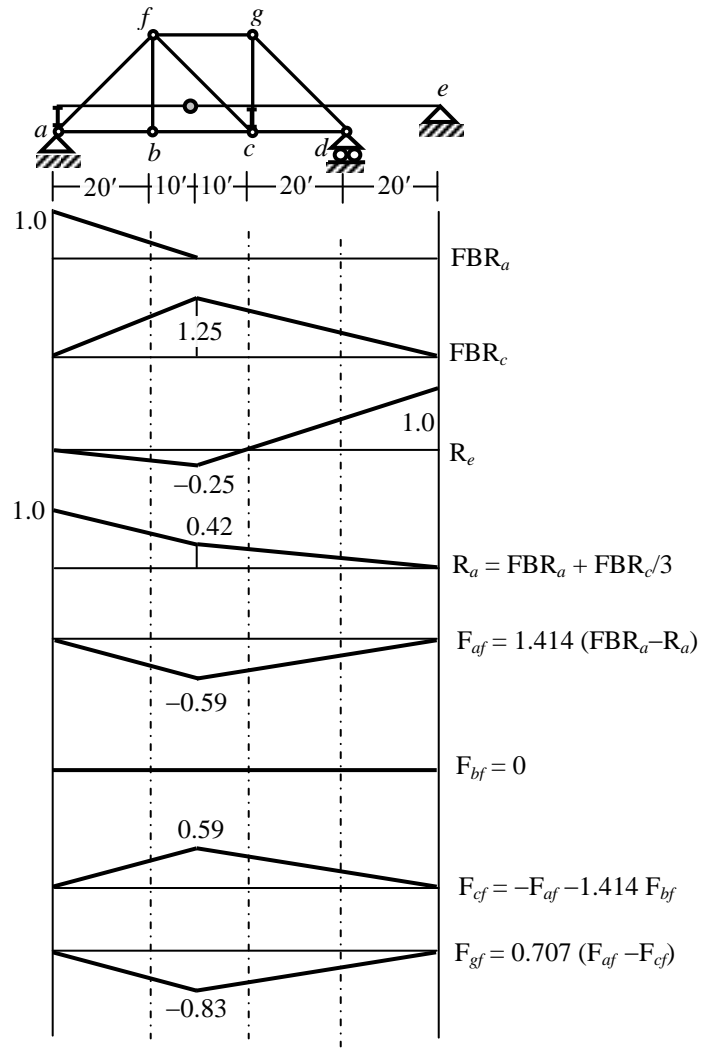
### Example 3.12

For the truss shown, draw influence lines for  
 (i)  $FBR_a, FBR_b, FBR_c$ , (ii)  $R_a$ , (iii)  $F_{af}, F_{bf}, F_{cf}, F_{gf}$



### Example 3.13

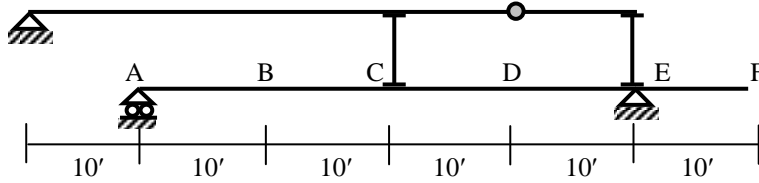
For the truss shown, draw influence lines for  
 (i)  $FBR_a, FBR_c$ , (ii)  $R_e, R_a$ , (iii)  $F_{af}, F_{bf}, F_{cf}, F_{gf}$



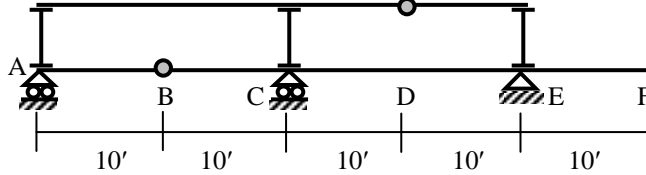


### Problems on Influence Lines of Plate Girders and Trusses

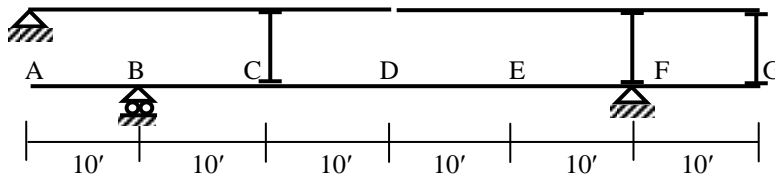
1. For the plate girder shown below, draw the influence lines for  $R_A$ ,  $R_E$ ,  $V_D$  and  $M_C$ .



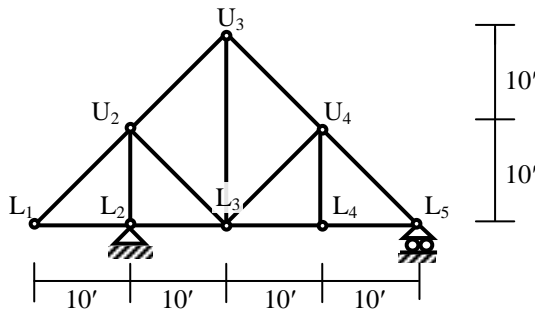
2. For the plate girder shown below, draw the influence lines for  $R_A$ ,  $R_E$ ,  $V_D$  and  $M_C$ .



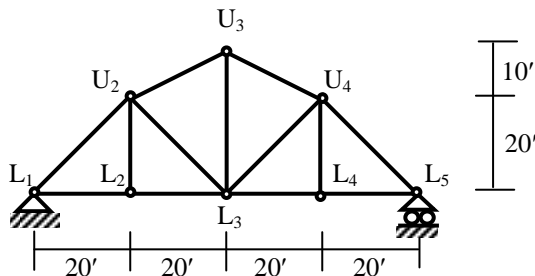
3. For the plate girder shown below, draw the influence lines for  $R_B$ ,  $R_F$ ,  $V_{C(R)}$  and  $M_D$ .



4. For the truss shown below, draw the influence lines for forces in members  $U_3U_4$ ,  $L_3U_4$  and  $L_3L_4$  [Note: There are floor-beams over the bottom-cords].



5. For the truss shown below, draw the influence lines for forces in members  $U_2U_3$ ,  $U_2L_3$  and  $L_2L_3$  [Note: There are floor-beams over the bottom-cords].



## **Force Calculation using Influence Lines**

### Application of Influence Lines

#### Force Calculation for Concentrated Loads

.....Eq. (3.1)

#### Force Calculation for Uniformly Distributed Loads

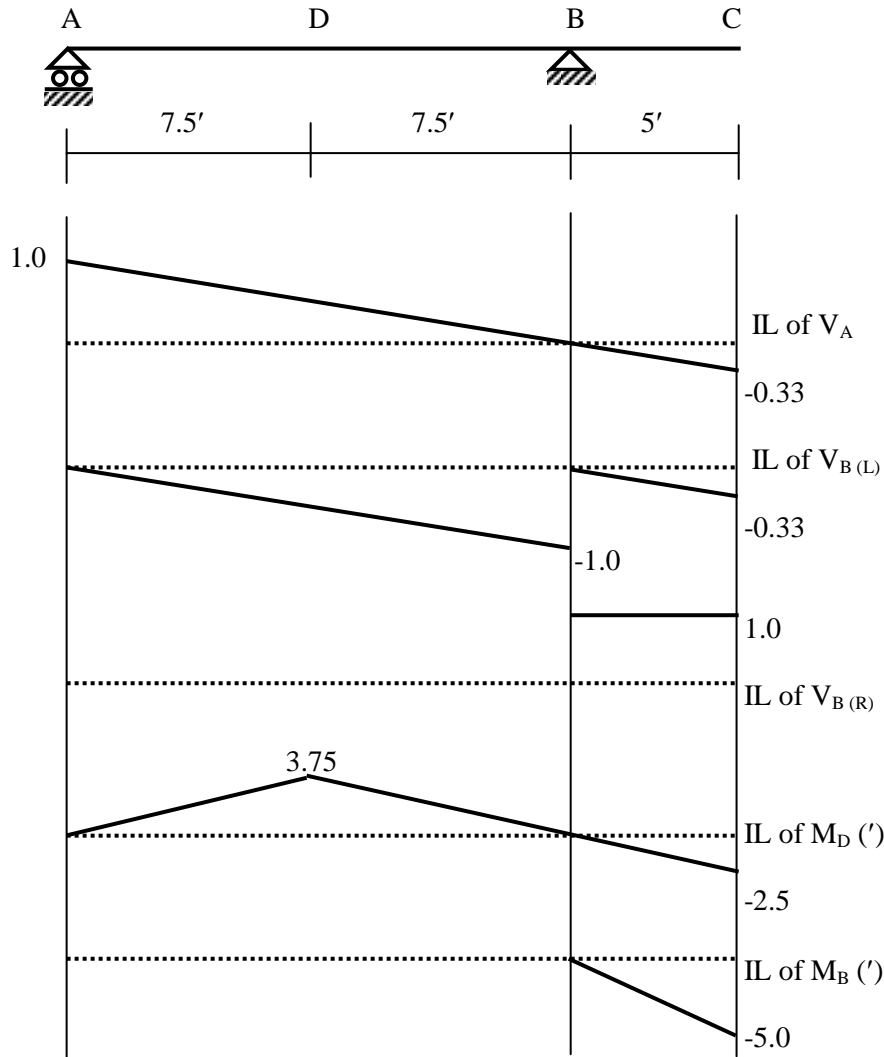
.....Eq. (3.2)

#### Example 3.14

## Maximum Force and Design Force Diagrams for Moving Loads

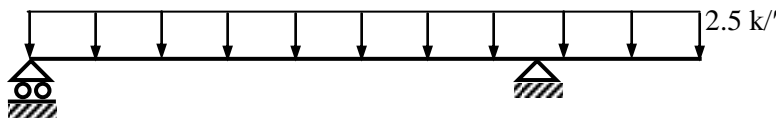
### Example 3.15

- (i) Calculate the maximum shear force at A, B and bending moment at D, B for the beam shown below, for a Dead Load of 1.5 k', and moving Live Load of 1.0 k'.



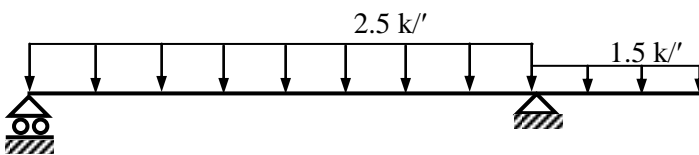
Loading Cases [DL = 1.5 k', moving LL = 1.0 k']

1. DL + LL throughout the beam



<p>Maximum (+ve or -ve) values of</p> $V_{B(L)} = -(1 \times 15/2 + 0.33 \times 5/2) \times 2.5$ $= -20.83 \text{ k}$ $V_{B(R)} = 1 \times 5 \times 2.5 = 12.5 \text{ k}$ $M_B = -5 \times 5/2 \times 2.5 = -31.25 \text{ k-ft}$
--

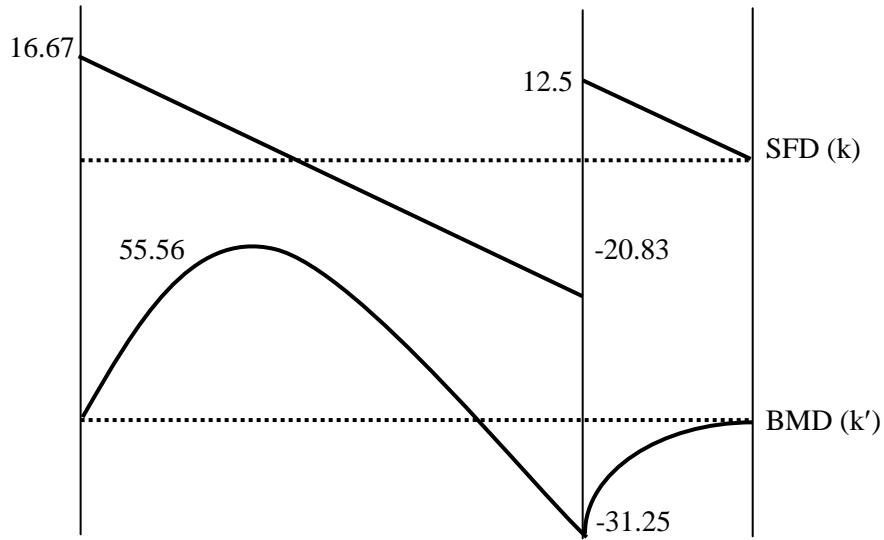
2. DL + LL on ADB, DL on BC



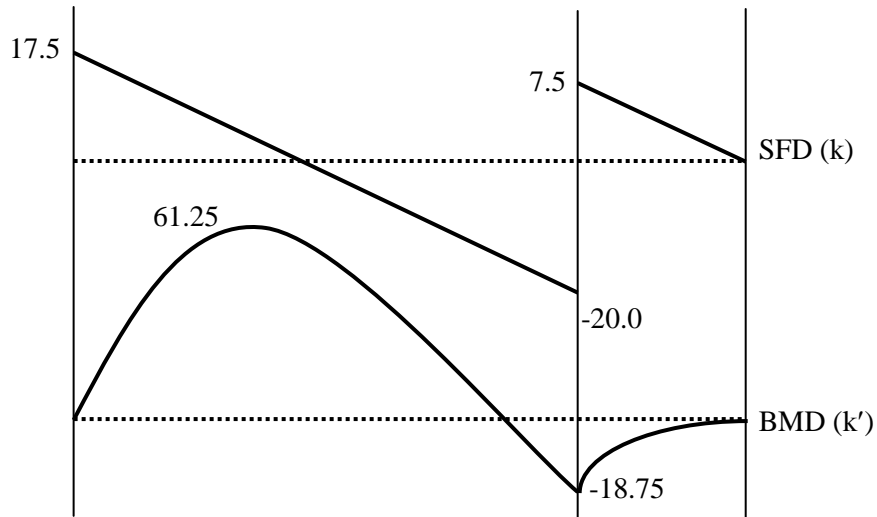
<p>Maximum values of</p> $V_A = (1 \times 15/2) \times 2.5 - (0.33 \times 5/2) \times 1.5$ $= 17.5 \text{ k}$ $M_D = (3.75 \times 15/2) \times 2.5 - (2.5 \times 5/2) \times 1.5$ $= 60.94 \text{ k-ft}$
--

(ii) Draw the design Shear Force and Bending Moment Diagrams of the beam loaded as described before.

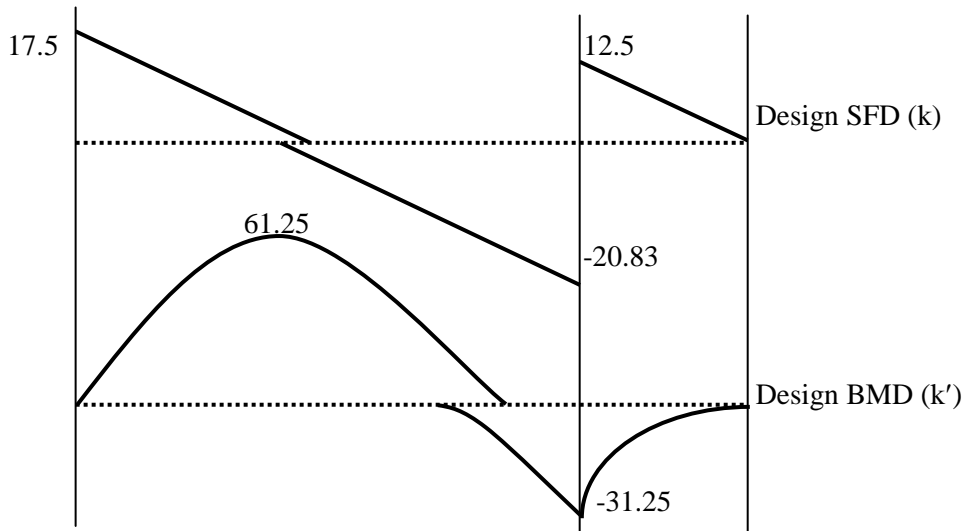
Case 1



Case 2



Design SFD and BMD



### Maximum 'Support Reaction' due to Wheel Loads

Consider the simply supported beam AB of length  $L$ , being subjected to the wheel load arrangement as shown in Fig. 1. The maximum reaction at support A will obviously be due to placement of one of the wheel loads directly on the support itself.

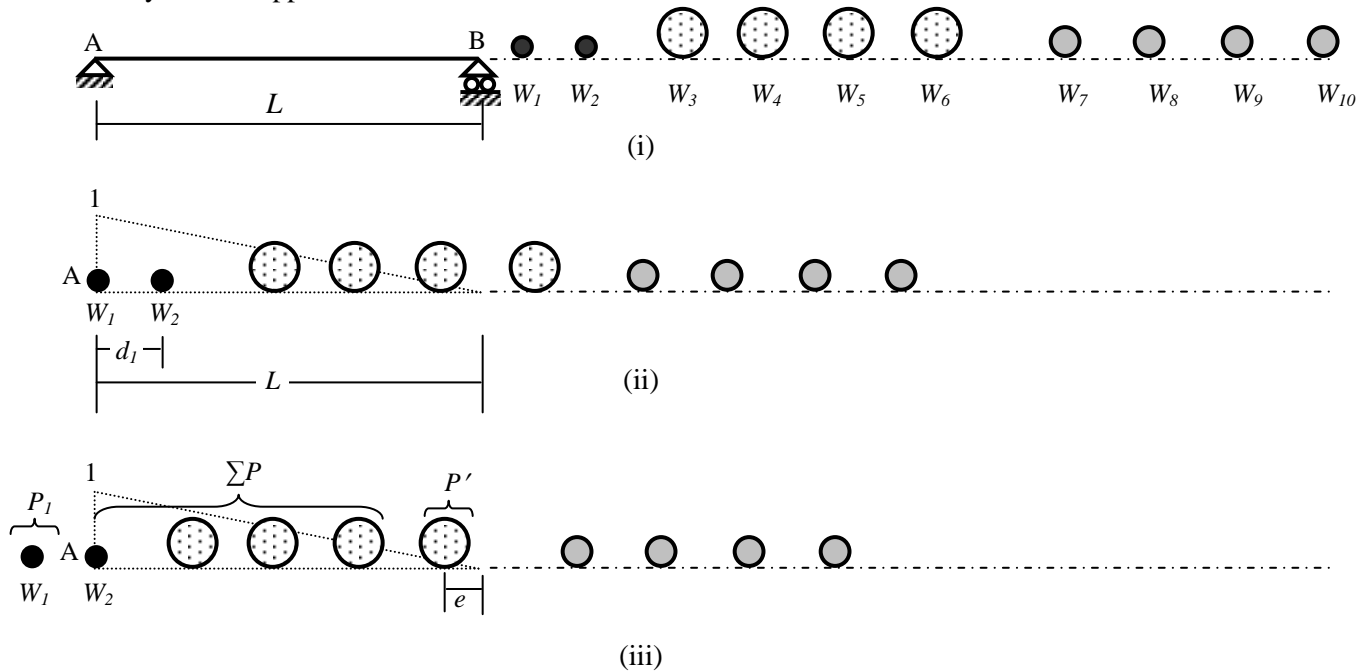


Fig. 1: Wheel Loads with 'Reaction Type' Influence Line

Considering the difference of support reaction at A ( $\Delta R$ ) between cases with wheel  $W_1$  at A [(ii) in Fig. 1] and wheel  $W_2$  at A [(iii) in Fig. 1], the increase in support reaction is due to the shift  $d_1$  of load  $\Sigma P$ ; i.e., an increase of ordinate by an amount  $d_1/L$ . Moreover, there is an additional increase due to the new load  $P'$  moving a distance  $e$  within the influence line (ordinate increases  $e/L$ ). However, since the load  $P_1$  has moved out of the influence line; i.e., its ordinate decreases by 1, there is a further decrease of  $P_1$  in the support reaction.

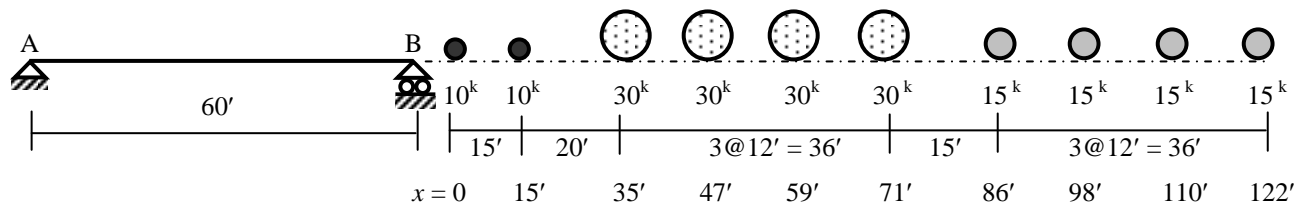
Therefore, the overall change of reaction between (ii) and (iii) is given by

$$\Delta R = \{(\Sigma P) d_1 + P' e\}/L - P_1 \quad \dots\dots\dots(3.3)$$

Since derivation of (3.3) is based on the shape of influence line, it is valid for all influence lines of similar shape.

**Example 3.16**

Calculate the maximum value of  $R_A$  for the wheel load arrangement shown below.



Between  $W_1$  and  $W_2$ ,  $\Sigma P = 10 + 3 \times 30 = 100$  k,  $d_1 = 15'$ ,  $P' = 30$  k,  $e = 4'$ ,  $P_1 = 10$  k

$$\therefore \Delta R_{12} = \{100 \times 15 + 30 \times 4\}/60 - 10 = 17$$

Between  $W_2$  and  $W_3$ ,  $\Sigma P = 4 \times 30 = 120$  k,  $d_1 = 20'$ ,  $P' = 15$  k,  $e = 9'$ ,  $P_1 = 10$  k

$$\therefore \Delta R_{23} = \{120 \times 20 + 15 \times 9\}/60 - 10 = 32.25$$

Between  $W_3$  and  $W_4$ ,  $\Sigma P = 3 \times 30 + 15 = 105$  k,  $d_1 = 12'$ ,  $P' = 15$  k,  $e = 9'$ ,  $P_1 = 30$  k

$$\therefore \Delta R_{34} = \{105 \times 12 + 15 \times 9\}/60 - 30 = -6.75$$

$$\therefore R_A \text{ is maximum when } W_3 \text{ is at A} \Rightarrow R_{A(\text{Max})} = 30 \times (60 + 48 + 36 + 24)/60 + 15 \times 9/60 = 86.25 \text{ kips}$$

### Maximum 'Shear Force' due to Wheel Loads

Using similar arguments as mentioned for 'support reactions', the change in shear force between successive wheel arrivals at a section is given by

$$\Delta V = \{(\sum P) d_1 + P' e + P_0 e_0\} / L - P_1 \quad \dots\dots\dots(3.4)$$

where  $\sum P$  = Load remaining on the influence line throughout the wheel movement,  $d_1$  = Shift of the wheels,  $P'$  = New load moving a distance  $e$  within the influence line,  $P_1$  = Load which shifted off the section, and  $P_0$  = Load moving off the influence line from a distance  $e_0$  inside.

These terms are illustrated in (i) and (ii) of Fig. 1, demonstrating transition from  $W_2$  to  $W_3$  at a critical section.

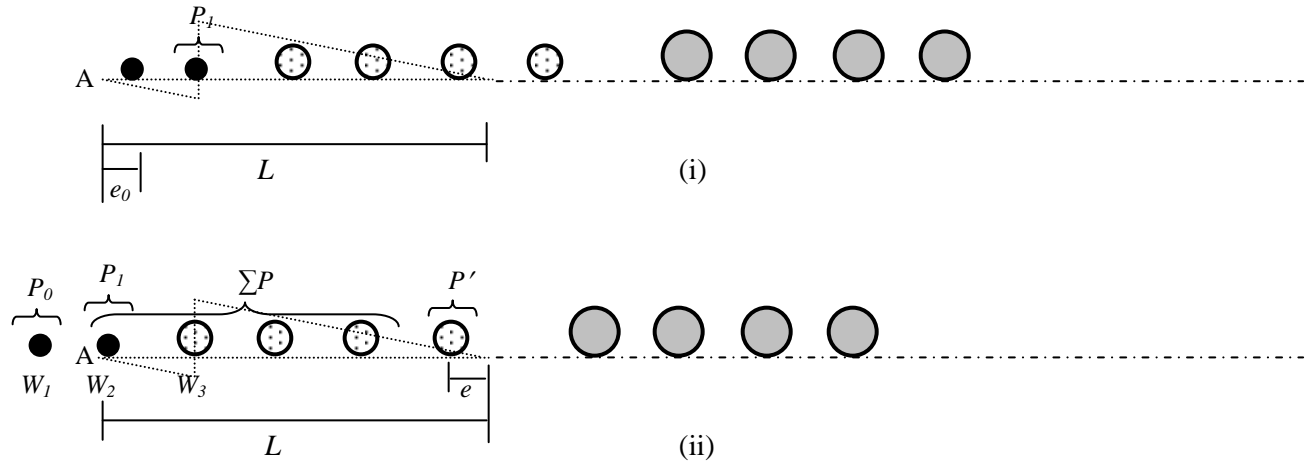
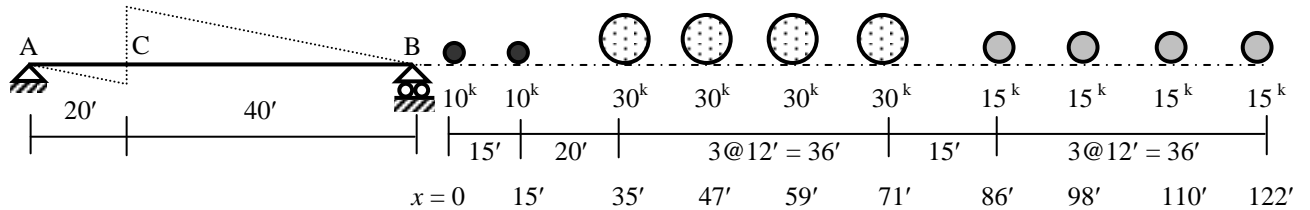


Fig. 1: Wheel Loads with 'Shear Type' Influence Line

**Example 3.17**

Calculate the maximum value of  $V_C$  for the wheel load arrangement shown below.



Between  $W_1$  and  $W_2$ ,  $\sum P = 2 \times 10 + 30 = 50$  k,  $d_1 = 15'$ ,  $P' = 30$  k,  $e = 8'$ ,  $P_0 = 0$ ,  $P_1 = 10$  k

$$\therefore \Delta V_{12} = \{50 \times 15 + 30 \times 8\} / 60 - 10 = 6.5 \text{ k}$$

Between  $W_2$  and  $W_3$ ,  $\sum P = 10 + 2 \times 30 = 70$  k,  $d_1 = 20'$ ,  $P' = 30$  k,  $30$  k,  $e = 16'$ ,  $4'$ ,  $P_0 = 10$  k,  $e_0 = 5'$ ,  $P_1 = 10$  k

$$\therefore \Delta V_{23} = \{70 \times 20 + 30 \times 16 + 30 \times 4 + 10 \times 5\} / 60 - 10 = 24.17 \text{ k}$$

Between  $W_3$  and  $W_4$ ,  $\sum P = 4 \times 30 = 120$  k,  $d_1 = 12'$ ,  $P' = 15$  k,  $e = 1'$ ,  $P_0 = 10$  k,  $e_0 = 0$ ,  $P_1 = 30$  k

$$\therefore \Delta V_{34} = \{120 \times 12 + 15 \times 1 + 10 \times 0\} / 60 - 30 = -5.75 \text{ k}$$

$$\therefore V_C \text{ is maximum when } W_3 \text{ is at } C \Rightarrow V_{C(\text{Max})} = 30 \times (40 + 28 + 16 + 4) / 60 - 15 \times 0 / 60 = 44 \text{ kips}$$

## Maximum 'Moment' and Greatest Maximum Moment

### Maximum 'Moment' for Wheel Loads

Fig. 1 (i) and (ii) demonstrate transition from  $W_2$  to  $W_3$  to obtain the maximum moment at a section that splits the influence line into two lengths,  $a$  and  $b$ . If  $i$  = maximum ordinate of the influence line, the change in bending moment due to the transition can be obtained as

$$\Delta M = (P_2 d_1 + P' e) (i/b) - (P_1 d_1 + P_0 e_0) (i/a) \quad \dots\dots\dots(3.5)$$

where  $P_2$  = Load remaining on the right (increasing) portion during wheel movement,  
 $P_1$  = Load remaining on the left (decreasing) portion during wheel movement,  $d_1$  = Shift of the wheels,  
 $P'$  = New load moving a distance  $e$  within the influence line,  
 and  $P_0$  = Load moving off the influence line from a distance  $e_0$  inside.

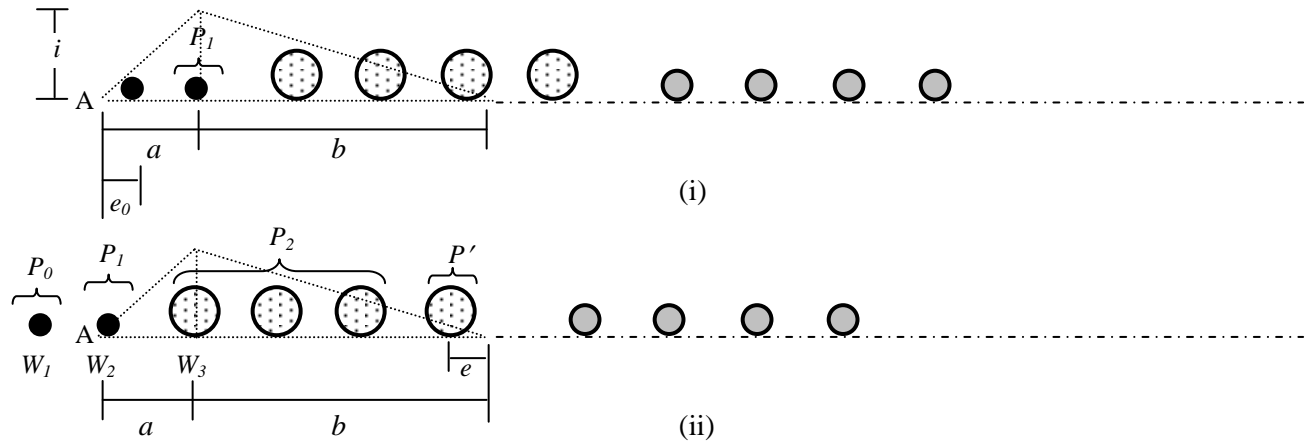


Fig. 1: Wheel Loads with 'Moment Type' Influence Line

### Greatest Maximum Moment

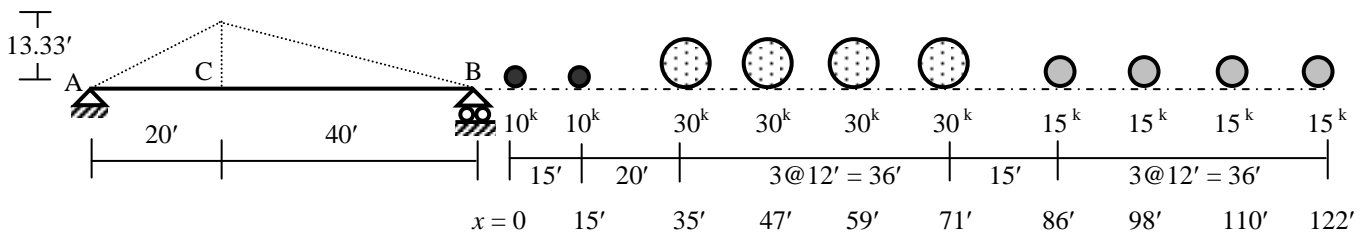
For a given group of wheel loads, it is often required to obtain the location of the maximum possible moment in a beam and also its value. This is called the 'Greatest Maximum Moment' (GMM) of the beam for the given loads, and for a simple span  $L$  under total load  $\sum P$ , it is possible to derive the value of GMM as

$$M_{(Max)} = (\sum P/L) (L/2 - a/2)^2 - P b, \text{ located } a/2 \text{ from the beam midspan} \quad \dots\dots\dots(3.6)$$

where  $a$  = Distance of the centroid of all loads from 'critical' load (often the load closest to centroid of loads),  
 $P$  = Load on the shorter side of the critical section,  $b$  = Distance of the centroid of load  $P$  from the 'critical' load.

### Example 3.18

Calculate the maximum value of  $M_C$  for the wheel load arrangement shown below.

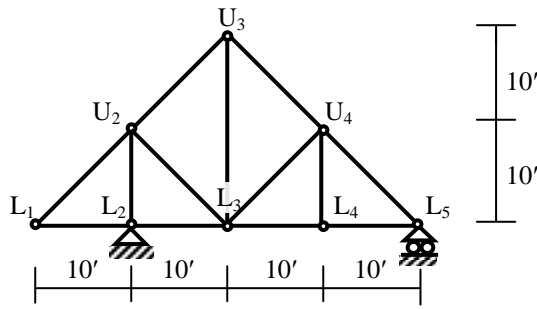


Between  $W_3$  and  $W_4$ ,  $P_1 = 30$  k,  $P_2 = 3 \times 30 = 90$  k,  $d_1 = 12'$ ,  $P' = 15$  k,  $e = 1'$ ,  $P_0 = 10$  k,  $e_0 = 0$   
 $\Delta M_{34} = i \{ (90 \times 12 + 15 \times 1)/40 - (30 \times 12 + 10 \times 0)/20 \} = i \{ 27.375 - 18 \} = +ve$   
 Between  $W_4$  and  $W_5$ ,  $P_1 = 30$  k,  $P_2 = 2 \times 30 + 15 = 75$  k,  $d_1 = 12'$ ,  $P' = 15$  k,  $e = 1'$ ,  $P_0 = 30$  k,  $e_0 = 8'$   
 $\Delta M_{45} = i \{ (75 \times 12 + 15 \times 1)/40 - (30 \times 12 + 30 \times 8)/20 \} = i \{ 22.875 - 30 \} = -ve$   
 $\therefore M_C$  is maximum when  $W_4$  is at C, with  $i = 13.33'$   
 $\Rightarrow M_{C(Max)} = 13.33 [30 \times 8/20 + \{30 \times (40 + 28 + 16) + 15 \times 1\}/40] = 13.33 [12 + 63.375] = 1005$  k-ft

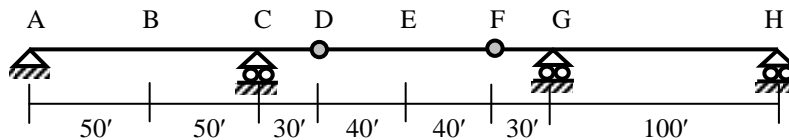
Assuming  $W_3$  to  $W_7$  to be on the beam for greatest maximum moment under  $W_5$ , their CG is at a distance  
 $a = [120 \times 6 - 15 \times 27]/135 = 2.33'$  from  $W_5$   
 $\therefore M_{(Max)} = (135/60) [(60 - 2.33)/2]^2 - 30 \times 12 - 15 \times 27 = 1105.56$  k-ft, at  $(60 - 2.33)/2 = 28.83'$  from either side.

### Problems on Load Calculation using Influence Lines

- For the truss shown below, calculate the maximum tension and compression in members  $U_4L_5$ ,  $U_3U_4$ ,  $L_3U_4$  and  $L_3L_4$  for a uniformly distributed dead load of 1 k/ft and a concentrated live load of 20 k [Note: Stringers are simply supported on floor-beams at bottom-cord joints].

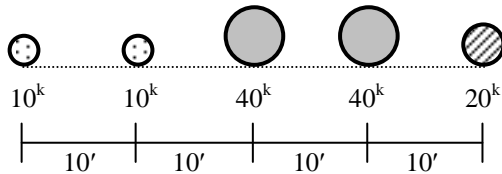


- For the beam shown below, calculate the maximum and minimum values of (i)  $R_A$ ,  $R_C$ , (ii)  $V_B$ ,  $V_{CL}$ ,  $V_{CR}$ ,  $V_E$ , (iii)  $M_B$ ,  $M_C$ ,  $M_E$  for a uniformly distributed dead load of 1 k/ft and a live load of 2 k/ft.

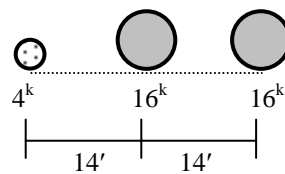


D and F are Internal Hinges

- For the beam loaded as described in Question 2, use the influence lines of (i)  $V_A$ ,  $V_{C(L)}$  to draw the Design SFD, and (ii)  $M_B$ ,  $M_C$  to draw the Design BMD.
- Calculate the maximum values of  $R_C$ ,  $V_E$ ,  $M_C$  and  $M_E$  for the beam described in Question 2, using both the wheel loads shown below.

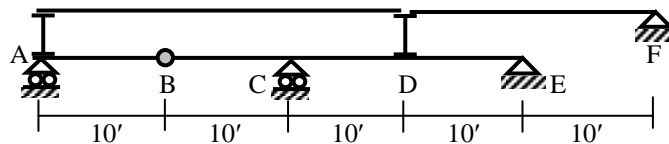


Wheel Loads 1

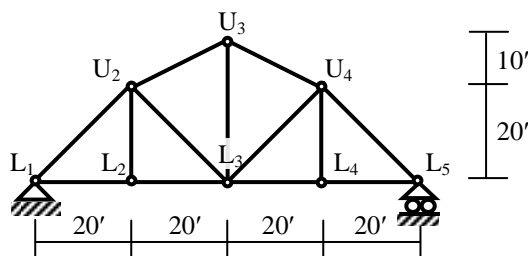


Wheel Loads 2

- Use both the wheel loads described in Question 4 to calculate the greatest maximum moment within the span DF of the beam described in Question 2.
- For the Wheel Loads1 shown in Question 4, calculate the maximum values of  $FBR_A$ ,  $R_C$ ,  $V_{D(R)}$  and  $M_D$  for the plate girder shown below.



- For the truss shown below, calculate the maximum reaction at support  $L_5$  and maximum forces in members  $U_2U_3$  and  $L_2L_3$  for the (i) distributed loads described in Question 1, (ii) Wheel Loads1 shown in Question 4 [Note: Stringers are simply supported on floor-beams at bottom-cord joints].





## Wind Pressure and Coefficients

### Basic Wind Pressure

The basic wind pressure on a surface is given by  $q_b = \rho_{\text{air}} V_b^2/2$  .....(1)

where  $\rho_{\text{air}} = \text{Density of air} = 0.0765/32.2 = 23.76 \times 10^{-4} \text{ slug/ft}^3$

$V_b = \text{Basic wind speed, ft/sec} = 1.467 \times \text{Basic wind speed, mph}$

$\therefore \text{Eq. (1)} \Rightarrow q_b = 23.76 \times 10^{-4} \times (1.467 V_b)^2/2 = 0.00256 V_b^2$  .....(2)

where  $q_b$  is in psf (lb/ft<sup>2</sup>) and  $V_b$  is in mph (mile/hr).

The basic wind speeds at different important locations of Bangladesh are given below. A more detailed map for the entire country is available in BNBC 1993.

Location	$V_b$ (mph)
Dhaka	130
Chittagong	160
Rajshahi	95
Khulna	150

### Sustained Wind Pressure

The wind velocity (and pressure) increases from zero at the base of the structure and is also a function of the exposure (i.e., open terrain or congested area). Moreover one has to account for the importance of the structure; i.e., design the sensitive structures more conservatively.

The sustained wind pressure on a building surface at any height  $z$  above ground is given by

$$q_z = 0.00256 C_1 C_z V_b^2 \quad \text{.....(3)}$$

where  $C_1 = \text{Structural importance coefficient}$ ,  $C_z = \text{Height and exposure coefficient}$ .

Category	$C_1$
Essential facilities	1.25
Hazardous facilities	1.25
Special occupancy	1.00
Standard occupancy	1.00
Low-risk structure	0.80

Height $z$ (ft)	$C_z$		
	Exp A	Exp B	Exp C
0~15	0.368	0.801	1.196
50	0.624	1.125	1.517
100	0.849	1.371	1.743
150	1.017	1.539	1.890
200	1.155	1.671	2.002
300	1.383	1.876	2.171
400	1.572	2.037	2.299
500	1.736	2.171	2.404
650	1.973	2.357	2.547
1000	2.362	2.595	2.724

### Design Wind Pressure

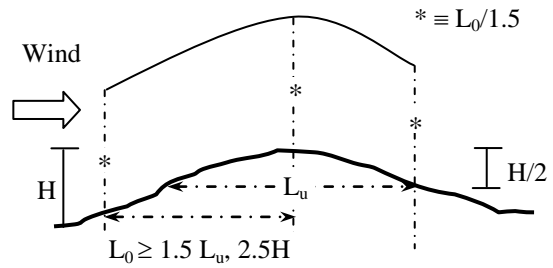
The design wind pressure can be calculated by multiplying the sustained wind pressure by appropriate pressure coefficients due to wind gust and turbulence as well as local topography.

The design wind pressure on a surface at any height  $z$  above ground is given by

$$p_z = C_G C_l C_p q_z \quad \text{.....(4)}$$

where  $C_G = \text{Wind gust coefficient}$ ,  $C_l = \text{Local topography coefficient}$ ,  $C_p = \text{Pressure coefficient}$ .

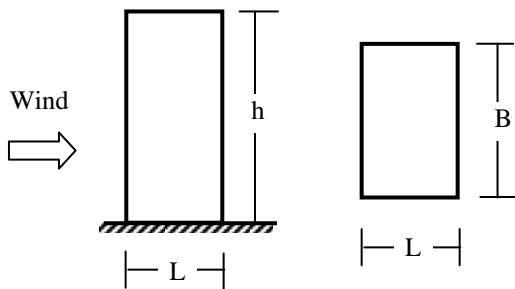
Height z (ft)	C <sub>G</sub> (for non-slender structures)		
	Exp A	Exp B	Exp C
0~15	1.654	1.321	1.154
50	1.418	1.215	1.097
100	1.309	1.162	1.067
150	1.252	1.133	1.051
200	1.215	1.114	1.039
300	1.166	1.087	1.024
400	1.134	1.070	1.013
500	1.111	1.057	1.005
650	1.082	1.040	1.000
1000	1.045	1.018	1.000



H/2L <sub>u</sub>	C <sub>t</sub>
0.05	1.19
0.10	1.39
0.20	1.85
0.30	2.37

The value of C<sub>G</sub> for slender structures (height > 5 times the minimum width) would be determined by dynamic analysis. Although code-based formulae are available, it is unlikely to exceed 2.0.

The pressure coefficient C<sub>p</sub> for rectangular buildings with flat roofs may be obtained as follows



h/B	L/B					
	0.1	0.5	0.65	1.0	2.0	≥ 3.0
≤ 0.5	1.40	1.45	1.55	1.40	1.15	1.10
1.0	1.55	1.85	2.00	1.70	1.30	1.15
2.0	1.80	2.25	2.55	2.00	1.40	1.20
≥ 4.0	1.95	2.50	2.80	2.20	1.60	1.25

The pressure coefficient C<sub>p</sub> for the windward surfaces of trusses or inclined surfaces are approximated by

$$C_p = -0.7, \text{ for } 0 \leq \alpha \leq 20^\circ$$

$$C_p = (0.07\alpha - 2.1), \text{ for } 20^\circ \leq \alpha \leq 30^\circ$$

$$C_p = (0.03\alpha - 0.9), \text{ for } 30^\circ \leq \alpha \leq 60^\circ$$

$$C_p = 0.9, \text{ for } 60^\circ \leq \alpha \leq 90^\circ$$

.....(5(a)-(d))

For leeward surface, C<sub>p</sub> = -0.7, for any value of α

.....(5(e))

### Vortex Induced Vibration (VIV)

This phenomenon, has been (and still is) extensively studied in various branches of structural as well as fluid dynamics. The pressure difference around a bluff body in flowing fluid may result in separated flow and shear layers over a large portion of its surface. The outermost shear layers (in contact with the fluid) move faster than the innermost layers, which are in contact with the structure. If the fluid velocity is large enough, this causes the shear layers to roll into the near wake and form periodic vortices. The interaction of the structure with these vortices causes it to vibrate transverse to the flow direction, and this vibration is called VIV.

The frequency of vortex-shedding is called Strouhal frequency (after Strouhal 1878) and is given by the simple equation

$$f_s = SU/D$$

.....(6)

where U = Fluid Velocity, D = Transverse dimension of the structure, S = Strouhal number, which is a function of Reynolds number and the geometry of the structure. For circular cylinders, S ≈ 0.20.

## Graphs for Wind Coefficients

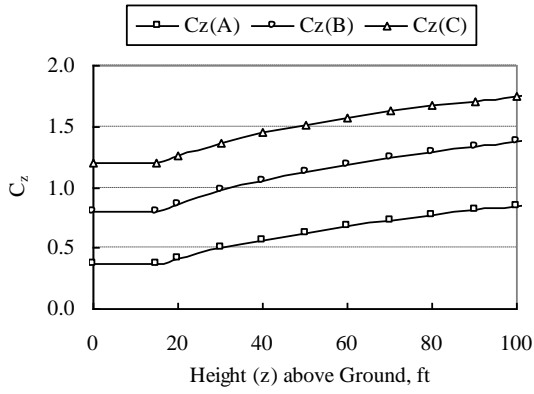


Fig. 1.1: Height and Exposure Coefficient,  $C_z$

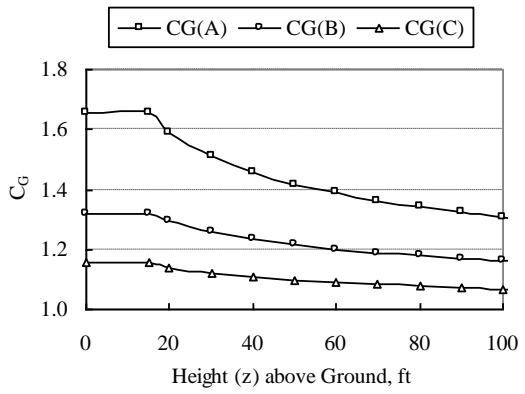


Fig. 2.1: Gust Response Factor,  $C_G$

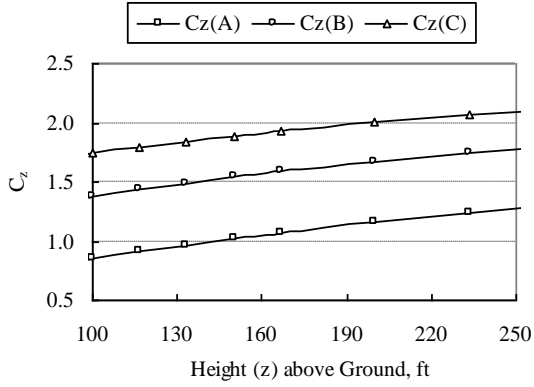


Fig. 1.2: Height and Exposure Coefficient,  $C_z$

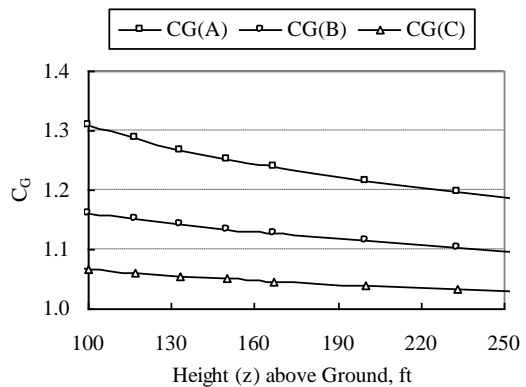


Fig. 2.2: Gust Response Factor,  $C_G$

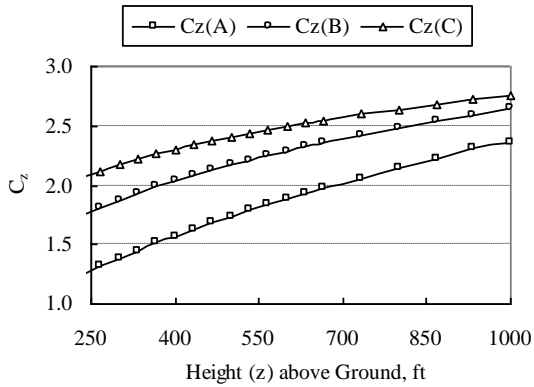


Fig. 1.3: Height and Exposure Coefficient,  $C_z$

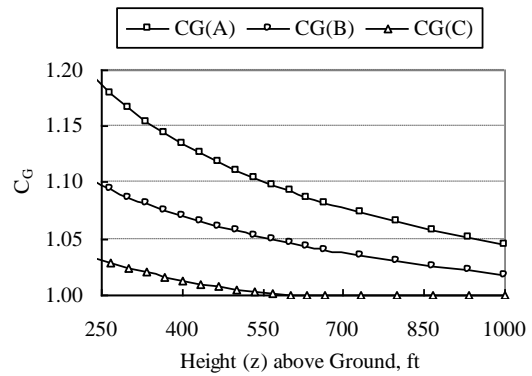


Fig. 2.3: Gust Response Factors,  $C_G$

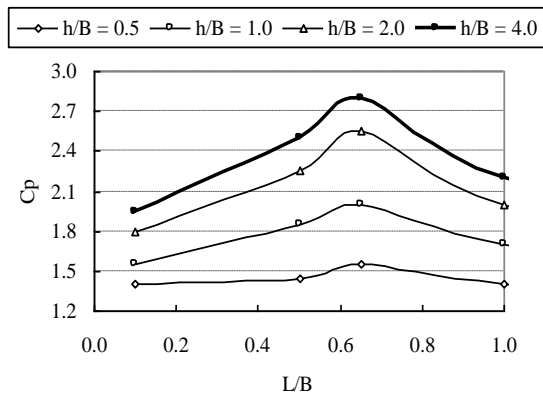


Fig. 3.1: Overall Pressure Coefficient,  $C_p$

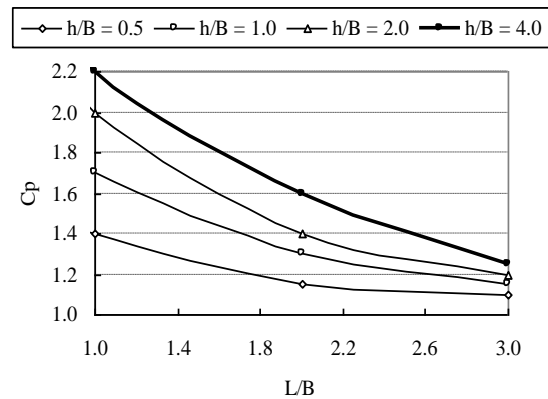
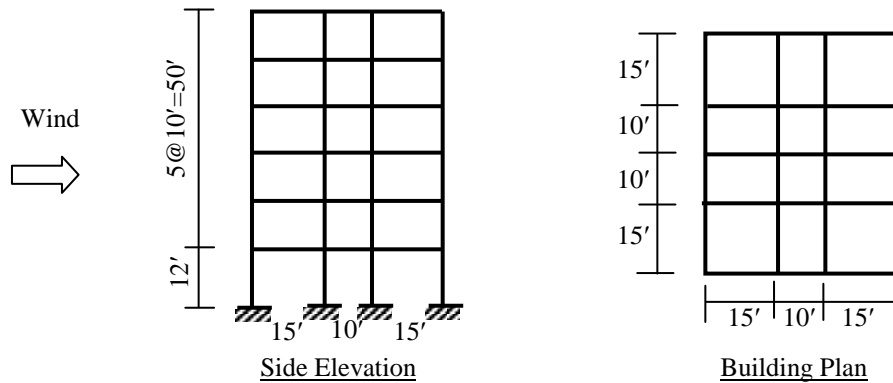


Fig. 3.2: Overall Pressure Coefficient,  $C_p$

## Calculation of Wind Load

### Wind Load on a Building

Calculate the wind load at each story of a six-storied hospital building (shown below) located at a flat terrain in Dhaka. Assume the structure to be subjected to Exposure B.



### Solution

The design wind pressure at a height  $z$  is given by  $p_z = 0.00256 C_1 C_z C_G C_t C_p V_b^2$

Since the building is located in Dhaka, the basic wind speed  $V_b = 130$  mph

For the hospital building (essential facility), Structural importance coefficient  $C_1 = 1.25$

In plane terrain, Local topography coefficient  $C_t = 1.0$

Building height  $h = 62'$ , dimensions  $L = 40'$  and  $B = 50'$ ; i.e.,  $h/B = 1.24$  and  $L/B = 0.80 \Rightarrow C_p \cong 1.98$

$\therefore p_z = 0.00256 \times 1.25 \times C_z \times C_G \times 1.00 \times 1.98 \times (130)^2 = 107.08 C_z C_G$

$\therefore$  The corresponding force  $F_z = B h_{\text{eff}} p_z = 50 h_{\text{eff}} p_z$ ; where  $h_{\text{eff}} =$  Effective height of the tributary area  
 $h_{\text{eff}} = 6' + 5' = 11'$  at 1<sup>st</sup> floor,  $(5' + 5' =) 10'$  between 2<sup>nd</sup> and 5<sup>th</sup> floor and  $5'$  at 6<sup>th</sup> floor

The coefficients  $C_z$ ,  $C_G$  and the design wind pressure  $p_z$  and force  $F_z$  at different heights are shown below.

Story	z (ft)	$C_z$	$C_G$	$p_z$ (psf)	$F_z$ (kips)	$F_{\text{frames}}$ (kips)				
						9.35	15.58	12.46	15.58	9.35
1	12	0.801	1.321	113.30	62.32	9.35	15.58	12.46	15.58	9.35
2	22	0.866	1.300	120.55	60.27	9.04	15.07	12.05	15.07	9.04
3	32	0.958	1.270	130.28	65.14	9.77	16.29	13.03	16.29	9.77
4	42	1.051	1.239	139.44	69.72	10.46	17.43	13.94	17.43	10.46
5	52	1.135	1.213	147.42	73.71	11.06	18.43	14.74	18.43	11.06
6	62	1.184	1.202	152.39	38.10	5.72	9.53	7.62	9.53	5.72

### Wind Load on a Truss

Calculate the wind load at each joint of the industrial truss (30' separated) located at a hilly terrain in Chittagong (with  $H = 20'$ ,  $L_u = 100'$ ). Assume the structure subjected to Exposure C.

### Solution

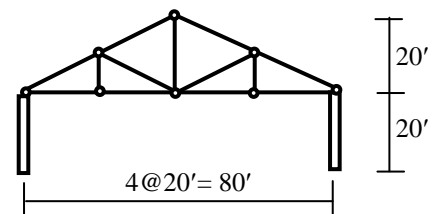
Angle  $\alpha = \tan^{-1}(20/40) = 26.6^\circ$

$\therefore C_p = (0.07 \times 26.6 - 2.1) = -0.24$  (windward)

$C_p = -0.70$  (leeward)

Using  $V_b = 160$  mph,  $C_1 = 1.25$ ,  $H/2L_u = 0.10 \Rightarrow C_t = 1.39$

and assuming  $C_z = 1.30$ ,  $C_G = 1.10$  (uniform)



$\therefore p_z$  (windward)  $= 0.00256 \times 1.25 \times 1.30 \times 1.10 \times 1.39 \times (-0.24) \times (160)^2 = -39.08$  psf

$\therefore p_z$  (leeward)  $= 0.00256 \times 1.25 \times 1.30 \times 1.10 \times 1.39 \times (-0.70) \times (160)^2 = -113.98$  psf

$\therefore F$  (windward, horizontal)  $= -39.08 \times 30 \times 5/1000 = -5.86$  k,  $-11.72$  k and  $-5.86$  k

$F$  (windward, vertical)  $= -39.08 \times 30 \times 10/1000 = -11.72$  k,  $-23.45$  k and  $-11.72$  k

$\therefore F$  (leeward, horizontal)  $= -113.98 \times 30 \times 5/1000 = -17.10$  k,  $-34.20$  k and  $-17.10$  k

$F$  (leeward, vertical)  $= -113.98 \times 30 \times 10/1000 = -34.20$  k,  $-68.40$  k and  $-34.20$  k

## Seismic Vibration and Structural Response

Earthquakes have been responsible for millions of deaths and an incalculable amount of damage to property. While they inspired dread and superstitious awe since ancient times, little was understood about them until the 20<sup>th</sup> century. Seismology, which involves the scientific study of all aspects of earthquakes, has yielded plausible answers to such long-standing questions as why and how earthquakes occur.

### Cause of Earthquake

According to the *Elastic Rebound Theory* (Reid 1906), earthquakes are caused by pieces of the crust of the earth that suddenly shift relative to each other. The most common cause of earthquakes is faulting; i.e., a break in the earth's crust along which movement occurs. Most earthquakes occur in narrow belts along the boundaries of crustal plates, particularly where the plates push together or slide past each other. At times, the plates are locked together, unable to release the accumulating energy. When this energy grows strong enough, the plates break free. When two pieces that are next to each other get pushed in different directions, they will stick together for many years, but eventually the forces pushing on them will cause them to break apart and move. This sudden shift in the rock shakes the ground around it.

### Earthquake Terminology

The point beneath the earth's surface where the rocks break and move is called the focus of the earthquake. The *focus* is the underground point of origin of an earthquake. Directly above the focus, on earth's surface, is the *epicenter*. Earthquake waves reach the epicenter first. During an earthquake, the most violent shaking is found at the epicenter.

Earthquakes release the strain energy stored within the crustal plates through 'seismic waves'. There are three main types of *seismic waves*. Primary or *P-waves* vibrate particles along the direction of wave, Secondary or *S-waves* that vibrate particles perpendicular to the direction of wave while Rayleigh or *R-waves* and Love or *L-waves* move along the surface.

### Earthquake Magnitude

A number of measures of earthquake 'size' are used for different purposes. From a seismologic point of view, the most important measure of size is the amount of strain energy released at the source, indicated quantitatively as the *earthquake magnitude*. Charles F. Richter introduced the concept of magnitude, which is the logarithm of the maximum amplitude measured in micrometers ( $10^{-6}$  m) of the earthquake record obtained by a standard short-period seismograph, corrected to a distance of 100 km; i.e.,

$$M_L = \log_{10} (A/A_0) \dots\dots\dots(1)$$

where  $A$  is the maximum trace amplitude in micrometers recorded on a seismometer and  $A_0$  is a correction factor as a function of distance. *Earthquake intensity* is another well-known measure of earthquake severity at a point, most notable of which is the Modified Mercalli (MM) scale.

### Nature of Earthquake Vibration

Earthquake involves vibration of the ground typically for durations of 10~40 seconds, which increases gradually to the peak amplitude and then decays. It is primarily a horizontal vibration, although some vertical movement is also present. Since the vibrations are time-dependent, earthquake is essentially a dynamic problem and the only way to deal with it properly is through dynamic analysis of the structure. Figs. 1~4 show the temporal variation of ground accelerations recorded during some of the best known and widely studied earthquakes of the 20<sup>th</sup> century. The *El Centro* earthquake (USA, 1940) data has over the last sixty years been the most used seismic data. However, Figs. 1 and 2 show that the ground accelerations recorded during this earthquake were different at different stations. It is about  $6.61 \text{ ft/sec}^2$  for the first station and  $9.92 \text{ ft/sec}^2$  for the second, which shows that the location of the recording station should be mentioned while citing the peak acceleration in an earthquake. The earthquake magnitudes calculated from these data are also different.

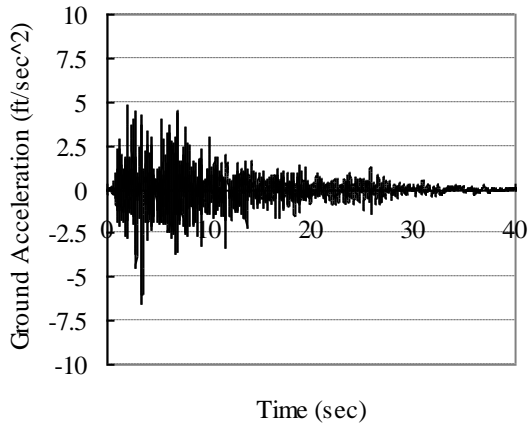


Fig. 1: El Centro1 Ground Acceleration

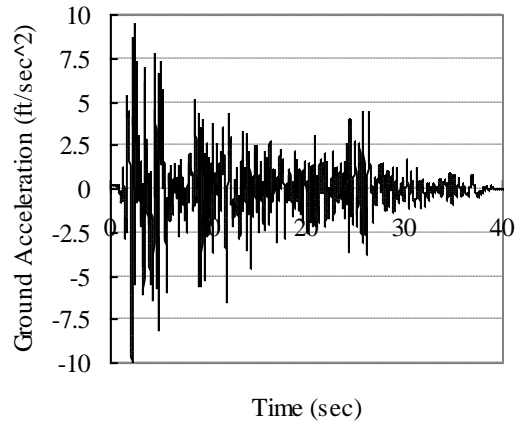


Fig. 2: El Centro2 Ground Acceleration

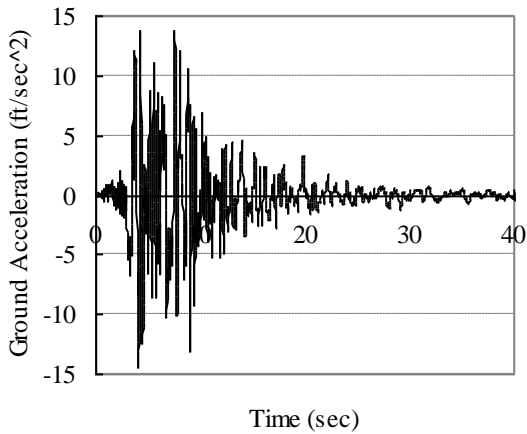


Fig. 3: Northridge Ground Acceleration

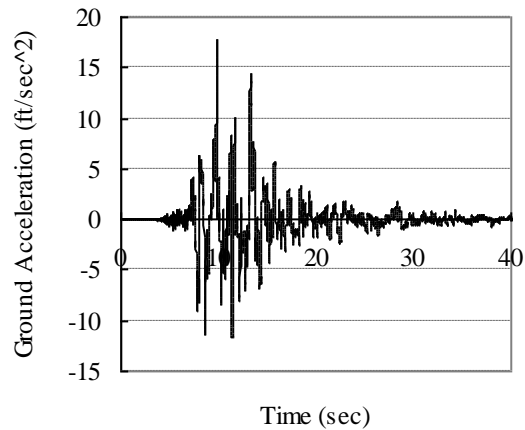


Fig. 4: Kobe Ground Acceleration

Figs. 3 and 4 show the ground acceleration from the Northridge (1994) and Kobe (1995) earthquake, both of which caused major destructions in the recent past in two of the better-prepared nations. The maximum ground accelerations they represent can only provide a rough estimate of their nature. The Fourier amplitude spectra need to be obtained and studied in order to gain better insight into their nature. Fourier amplitude spectra for the El Centro2 and Kobe earthquake ground acceleration are shown in Figs. 5 and 6.

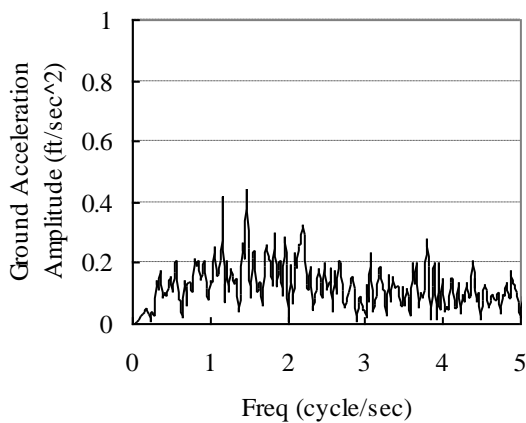


Fig. 5: El Centro Ground Acceleration Spectrum

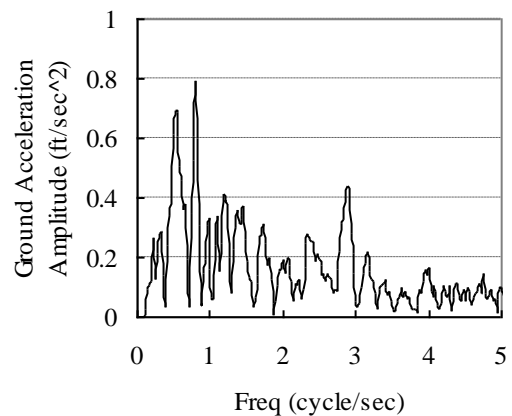


Fig. 6: Kobe Ground Acceleration Spectrum

### Frequency of Earthquakes Worldwide

A rough idea of frequency of occurrence of large earthquakes is given by the following tables (Table 1 and Table 2). These are collected from Internet sources as data reported by the National Earthquake Information Center (NEIC) of the United States Geological Survey (USGS).

**Table 1: Frequency of Occurrence of Earthquakes (Based on Observations since 1900)**

Descriptor	Magnitude	Average Annually
Great	8 and higher	1
Major	7 - 7.9	18
Strong	6 - 6.9	120
Moderate	5 - 5.9	800
Light	4 - 4.9	6,200 (estimated)
Minor	3 - 3.9	49,000 (estimated)
Very Minor	< 3.0	Magnitude 2 - 3: about 1,000 per day Magnitude 1 - 2: about 8,000 per day

**Table 2: The Number of Earthquakes Worldwide for 1992 - 2001 (Located by the USGS-NEIC)**

Magnitude	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
8.0 and higher	0	1	2	3	1	0	2	0	4	1
7.0 - 7.9	23	15	13	22	21	20	14	23	14	6
6.0 - 6.9	104	141	161	185	160	125	113	123	157	45
5.0 - 5.9	1541	1449	1542	1327	1223	1118	979	1106	1318	382
4.0 - 4.9	5196	5034	4544	8140	8794	7938	7303	7042	8114	2127
3.0 - 3.9	4643	4263	5000	5002	4869	4467	5945	5521	4741	1624
2.0 - 2.9	3068	5390	5369	3838	2388	2397	4091	4201	3728	1319
1.0 - 1.9	887	1177	779	645	295	388	805	715	1028	225
0.1 - 0.9	2	9	17	19	1	4	10	5	6	0
Less than 0.1	4084	3997	1944	1826	2186	3415	2426	2096	3199	749
Total	19548	21476	19371	21007	19938	19872	21688	20832	22309	6478
Estimated Deaths	3814	10036	1038	7949	419	2907	8928	22711	231	14923

### History of Earthquakes in Bangladesh

During the last 150 years, seven major earthquakes (with  $M > 7.0$ ) have affected the zone that is now within the geographical borders of Bangladesh. Out of these, three had epicenters within Bangladesh. The earthquakes and their effects are described in Table 3.

**Table 3: List of major Earthquakes affecting Bangladesh**

Date	Name of Earthquake	Magnitude (Richter)	Epicentral distance from Dhaka (km)	Affected zone
10 <sup>th</sup> Jan, 1869	Cachar Earthquake	7.5	250	Tremor mainly in Sylhet
14 <sup>th</sup> July, 1885	Bengal Earthquake	7.0	170	Damage in Jamalpur Sherpuur, Bogra
12 <sup>th</sup> June, 1897	Great Indian Earthquake	8.7	230	Damage in Sylhet, Mymensingh
8 <sup>th</sup> July, 1918	Srimongal Earthquake	7.6	150	Tremor in Sylhet
2 <sup>nd</sup> July, 1930	Dhubri Earthquake	7.1	250	Damage in Eastern part of Rangpur
15 <sup>th</sup> Jan, 1934	Bihar-Nepal Earthquake	8.3	510	None
15 <sup>th</sup> Aug, 1950	Assam Earthquake	8.5	780	Tremor throughout the country

## Response Spectrum Analysis

The main objective of seismic design methods is to conveniently calculate the peak displacements and forces resulting from a particular design ground motion. The Response Spectrum Analysis (RSA) is an approximate method of dynamic analysis that can be readily used for a reasonably accurate prediction of dynamic response. The governing equation of motion for a single-degree-of-freedom (SDOF) system subjected to ground motion  $u_g(t)$  is given by

$$m \frac{d^2 u_r}{dt^2} + c \frac{du_r}{dt} + k u_r = -m \frac{d^2 u_g}{dt^2} \quad \dots\dots\dots(2)$$

Since the loads themselves (on the right side of the equations) are proportional to the structural properties, each of these equations can be normalized in terms of the system properties (natural frequency  $\omega_n$  and damping ratio  $\xi$ ) and the ground motion (acceleration or displacement and velocity).

$$\frac{d^2 u_r}{dt^2} + 2\omega_n \xi \frac{du_r}{dt} + \omega_n^2 u_r = -\frac{d^2 u_g}{dt^2} \quad \dots\dots\dots(3)$$

For a specified ground motion data (e.g., the El Centro2 or Kobe data shown in Fig. 2 or 4) the temporal variation of structural displacement, velocity and acceleration depends only on its natural frequency  $\omega_n$  and the damping ratio  $\xi$ . From the time series thus obtained, the maximum parameters can be identified easily as the maximum design criteria for that particular structure (and that particular ground motion). Such maximum values can be similarly obtained for structures with different natural frequency (or period) and damping ratio. Since natural period ( $T_n$ ) is a more familiar concept than  $\omega_n$ , the peak responses can be represented as functions of  $T_n$  and  $\xi$  for the ground motion under consideration.

If a ‘standard’ ground motion data can be chosen for the design of all SDOF structures, the maximum responses thus obtained will depend on the two structural properties only. A plot of the peak value of the response quantity as a function of natural  $T_n$  and  $\xi$  is called the response spectrum of that particular quantity. If such curves can be obtained for a family of damping ratios ( $\xi$ ), they can provide convenient curves for seismic analysis of SDOF systems. For example, the peak responses for the relative acceleration ( $a$ ) are called the response spectra for the acceleration; i.e.,

$$a_0(T_n, \xi) = \text{Max} |a(t, T_n, \xi)| \quad \dots\dots\dots(4)$$

Such response spectra have long been used as useful tools for the seismic analysis of SDOF and MDOF (multi-degree-of-freedom) systems, which can be decomposed into several SDOF systems by Modal Analysis. Once the peak responses for all the modes are calculated from the response spectra, they can be combined statistically to obtain the approximate maximum response for the whole structure.

In order to account for the amplification of waves while propagating through soft soils, some simplified wave propagation analyses can be performed. Such works, performed statistically for a variety of soil conditions, provide the acceleration response spectra shown in Fig. 7 and code-specified spectra in Fig. 8.

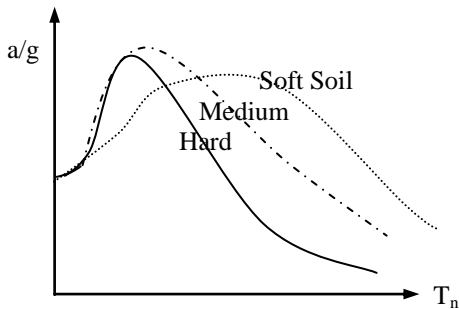


Fig. 7: Response Spectra for different sites

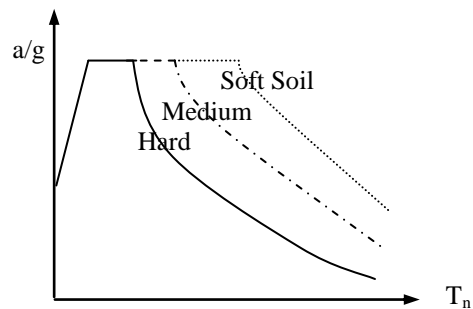


Fig. 8: Code Specified Response Spectra

The analogy of this formulation to the ‘Equivalent Static Force Method’ is added later.



## Equivalent Static Force Method

The 'Equivalent' Static Analysis of seismic vibration is based on the concept of replacing the inertia forces at various 'lumped masses' (i.e., story levels) by equivalent horizontal forces that are proportional the weight of the body (therefore its mass) and its displacement (therefore its acceleration). The summation of these concentrated forces is balanced by a 'base shear' at the base of the structure.

This method may be used for calculation of seismic lateral forces for all structures specified in the building codes. The following provisions are taken from the Bangladesh National Building Code (BNBC, 1993) for most part.

### (1) Design Base Shear

The total design base shear in a given direction is determined from the following relation:

$$V = (ZIC/R) W \quad \dots\dots\dots(5)$$

where, Z = Seismic zone coefficient given in Table 6.2.22 and Fig. 6.2.10 (BNBC 1993);

i.e., Z = 0.075, 0.15 and 0.25 for Seismic Zones 1, 2 and 3 respectively

I = Structure importance coefficient given in Table 6.2.23, similar to C<sub>1</sub> for wind

R = Response modification coefficient for structural systems given in Table 4;

i.e., 6.2.24 in BNBC

W = The total seismic dead load

The 'Seismic Dead Load' is not only the dead load (including permanent partitions) of the structure but also has to include some live loads as and when they superimpose on the dead loads.

C = Numerical coefficient given by the relation:

$$C = 1.25 S/T^{2/3} \quad \dots\dots\dots(6)$$

S = Site coefficient for soil characteristics as provided in Table 5; i.e., 6.2.25 in BNBC

T = Fundamental period of vibration of the structure in the direction considered (in sec)

The value of C need not exceed 2.75. Except for those requirements where Code prescribed forces are scaled up by 0.375R, the minimum value of the ratio C/R is 0.075.

### (2) Structural Period

The value of T may be approximated by the equation  $T = C_t (h_n)^{3/4}$  .....(7)

where, C<sub>t</sub> = 0.083 for steel moment resisting frames

= 0.073 for RCC moment resisting frames, and eccentric braced steel frames

= 0.049 for all other structural systems

h<sub>n</sub> = Height (in meters) above the base to level n.

There are alternative ways of calculating T and C<sub>t</sub>.

### (3) Vertical Distribution of Lateral Forces

In the absence of a more rigorous procedure, the total lateral force which is the base shear V, is distributed along the height of the structure in accordance with Eqs. (8)~(10).

$$V = F_t + \sum F_i \quad \dots\dots\dots(8)$$

where, F<sub>i</sub> = Lateral force applied at storey level i and

F<sub>t</sub> = Additional concentrated lateral force considered at the top of the building

The concentrated force, F<sub>t</sub> acting at the top of the building is determined as follows:

$$F_t = 0.07 TV \leq 0.25V \text{ when } T > 0.7 \text{ second, and } = 0, \text{ when } T \leq 0.7 \text{ second} \quad \dots\dots\dots(9)$$

The remaining portion of the base shear (V-F<sub>t</sub>), is distributed over the height of the building, including level n, according to the relation

$$F_j = (V-F_t) [w_j h_j / \sum w_i h_i] \quad \dots\dots\dots(10)$$

The design story shear V<sub>x</sub> in any story x is the sum of the forces F<sub>x</sub> and F<sub>t</sub> above that story.

**Table 4: Response Modification Coefficient, R for Structural Systems**

Basic Structural System	Description Of Lateral Force Resisting System	R
(a) Bearing Wall System	Light framed walls with shear panels	6~8
	Shear walls	6
	Light steel framed bearing walls with tension only bracing	4
	Braced frames where bracing carries gravity loads	4~6
(b) Building Frame System	Steel eccentric braced frame (EBF)	10
	Light framed walls with shear panels	7~9
	Shear walls	8
	Concentric braced frames (CBF)	8
(c) Moment Resisting Frame System	Special moment resisting frames (SMRF)	
	(i) Steel	12
	(ii) Concrete	12
	Intermediate moment resisting frames (IMRF), concrete	8
	Ordinary moment resisting frames (OMRF)	
	(i) Steel	6
(ii) Concrete	5	
(d) Dual System	Shear walls	7~12
	Steel EBF	6~12
	Concentric braced frame (CBF)	6~10
(e) Special Structural Systems	According to Sec 1.3.2, 1.3.3, 1.3.5 of BNBC	

**Table 5: Site Coefficient, S for Seismic Lateral Forces**

Site Soil Characteristics		Coefficient, S
Type	Description	
S <sub>1</sub>	A soil profile with either: A rock-like material characterized by a shear-wave velocity greater than 762 m/s or by other suitable means of classification, or Stiff or dense soil condition where the soil depth exceeds 61 meters	1.0
S <sub>2</sub>	A soil profile with dense or stiff soil conditions, where the soil depth exceeds 61 meters	1.2
S <sub>3</sub>	A soil profile 21 meters or more in depth and containing more than 6 meters of soft to medium stiff clay but not more than 12 meters of soft clay	1.5
S <sub>4</sub>	A soil profile containing more than 12 meters of soft clay characterized by a shear wave velocity less than 152 m/s	2.0

### Structural Dynamics in Building Codes

The Equivalent Static Force Method (ESFM) tries to model the dynamic aspects of seismic loads in an approximate manner. Therefore it is natural that the ESFM includes several equations that are derived from Structural Dynamics. The following are worth noting, as formulated in the ‘Response Spectrum Analysis’.

Peak ground acceleration due to earthquake =  $a_g$ .

Maximum acceleration  $a_{max}$  of the structure due to this force =  $a_g \times$  Ordinate of Response Spectrum (C)

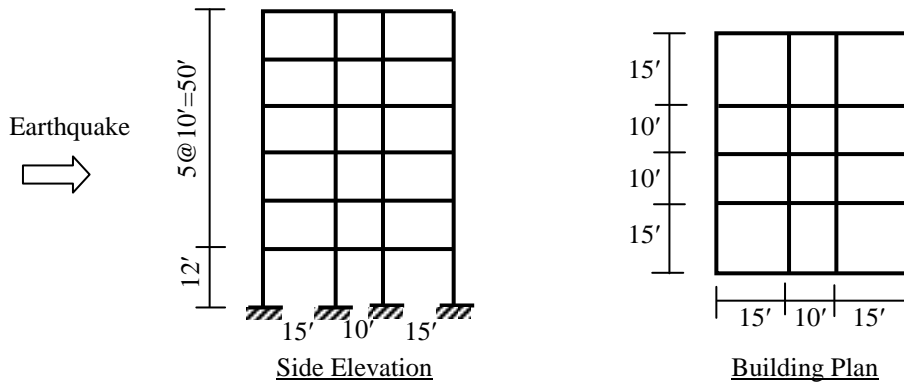
$\therefore$  Maximum ‘earthquake force’ on the structure =  $m a_{max} = (W/g) (a_g \times C) = (a_g/g) (C) W = Z C W$

- (1) The zone factor Z can be interpreted as the ratio of the maximum ground acceleration and g (=  $a_g/g$ ), while the factor C is the ordinate of the Response Spectrum. Including I (the importance factor for different structures), the  $V_e = (ZIC) W$  gives the maximum elastic force on the building. Therefore the factor R is the building resistance factor that accounts for the ductility of the building, i.e., its ability to withstand inelastic deformations and thereby reduce the elastic force.
- (2) The distribution of story shear [Eq. (10)] in proportion to the mass and height of the story is an approximation of the 1<sup>st</sup> modal shape, which is almost linear for shorter buildings but tends to be parabolic to include higher modes of vibration. Therefore, a concentrated load is added at the top to approximately add the 2<sup>nd</sup> mode of vibration for taller buildings.
- (3) Factor S is introduced in the factor C to account for amplification of seismic waves in soft soils.
- (4) The equation of the natural frequency [Eq.(7)] is very similar to the equation of natural frequency of continuous dynamic systems.

## Calculation of Seismic Load

### Seismic Load on a Building

Use the Equivalent Static Force Method to calculate the seismic load at each story of a six-storied hospital building (shown below) located in Dhaka. Assume the structure to be an Ordinary Moment Resisting Frame (OMRF) built on soil condition  $S_2$ , carrying a Dead Load of  $150 \text{ lb/ft}^2$  and Live Load  $40 \text{ lb/ft}^2$ .



### Solution

#### (1) Base shear

The total design base shear  $V = (ZIC/R) W$

where,  $Z$  = Seismic zone coefficient in Dhaka = 0.15

$I$  = Structure importance coefficient for hospital building = 1.25

$R$  = Response modification coefficient for OMRF (concrete) = 5.0

$w$  = Total seismic DL pressure =  $DL + 25\% \text{ of } LL = 150 + 0.25 \times 40 = 160 \text{ lb/ft}^2 = 0.16 \text{ k/ft}^2$

$W = 0.16 \times \text{Total Floor Area} = 0.16 \times 6 \times 40 \times 50 = 1920 \text{ kips}$

Numerical Coefficient  $C = 1.25 S/T^{2/3}$

where,  $S$  = Site coefficient for soil type  $S_2 = 1.2$

$T$  = Fundamental period of vibration =  $C_t (h_n)^{3/4} = 0.073 (62/3.28)^{3/4} = 0.661 \text{ sec}$

$\therefore C = 1.25 S/T^{2/3} = 1.25 \times 1.2/(0.661)^{2/3} = 1.975$ , which is  $< 2.75$ ; i.e., OK

$\therefore$  Total design base shear,  $V = (ZIC/R)W = (0.15 \times 1.25 \times 1.975/5) \times 1920 = 0.074 \times 1920 = 142.22 \text{ kips}$

#### (2) Vertical Distribution of Lateral Forces

Since  $T = 0.661 \text{ sec} < 0.7 \text{ sec}$ ,  $F_t = 0$

$$\therefore F_j = (V - F_t) [w_j h_j / \sum w_i h_i] = (142.2 - 0) \times [320 h_j / \{320 (12 + 22 + 32 + 42 + 52 + 62)\}] = 0.641 h_j$$

The design story shear  $V_j$  in any story  $j$  is the sum of the forces  $F_j$  and  $F_t$  above that story.

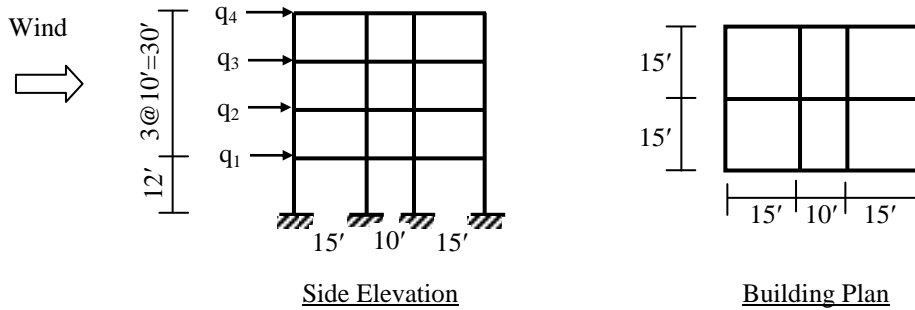
Story	$h_j$ (ft)	$w_j$ (kips)	$F_j$ (kips)	$V_j$ (kips)	$F_{\text{frames}}$ (kips)				
1	12	320	7.69	142.22	1.15	1.92	1.54	1.92	1.15
2	22	320	14.09	134.53	2.11	3.52	2.82	3.52	2.11
3	32	320	20.50	120.44	3.07	5.12	4.10	5.12	3.07
4	42	320	26.91	99.94	4.04	6.73	5.38	6.73	4.04
5	52	320	33.31	73.03	5.00	8.33	6.66	8.33	5.00
6	62	320	39.72	39.72	5.96	9.93	7.94	9.93	5.96

### Problems on Calculation of Lateral Loads

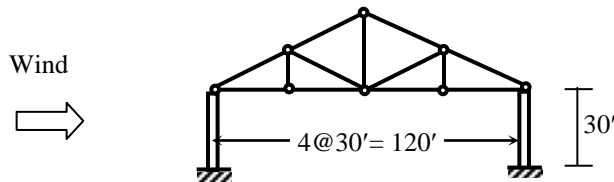
1. The basic wind pressure on the surface of a (10' × 10' × 10') overhead water tank (essential facility) shown below and located at a flat terrain with Exposure B is 30 psf. Calculate the (i) sustained wind pressure, (ii) design wind pressure, (iii) design wind force on the tank.



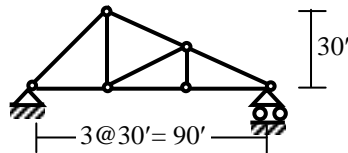
2. Calculate the (i) sustained wind pressure, (ii) sustained wind force at each story of the four-storied residential building (shown below) subjected to a basic wind pressure is 30 psf with Exposure B.



3. Calculate the (i) design wind pressure, (ii) design wind force at each story of the four-storied residential building (shown in Question 2) if the sustained wind pressure  $q_1$  at the first story is 35 psf. Assume the structure to be located at a hilly terrain (with  $H = 10'$ ,  $L_u = 100'$ ).
4. Calculate the design wind pressures on an industrial truss located at a hilly terrain (with  $H = 10'$ ,  $L_u = 100'$ ) under Exposure C if the windward pressure is (i) zero, (ii) equal to the suction pressure at the leeward surface [Given: Sustained wind pressure = 40 psf].



5. Calculate the design wind pressures on the industrial truss shown below if it is located at a flat terrain in Khulna at Exposure B and the wind blows (i) from left, (ii) from right.



6. Calculate seismic base shear force for the structure shown in Question 1, if it is located on medium stiff soil (type  $S_3$ ) in Dhaka, has a natural frequency of 3 Hz and has a response modification factor  $R = 4$ . Neglect the weight of the column and assume the entire weight of the structure to be concentrated at the tank, which is filled with water (unit weight  $62.5 \text{ lb/ft}^3$ ).
7. Calculate the (i) seismic base shear force, (ii) seismic force at each story of the four-storied residential building shown in Question 2, if the seismic force at the first story is 5 kips.

## Solution of Problems on Calculation of Lateral Loads

1. Basic wind pressure  $q_b = 30$  psf

(i) Sustained wind pressure  $q_z = C_1 C_z q_b$

where  $C_1 = 1.25$ ,  $C_z$  at height  $(30 + 10/2 = 35') = 1.05$

$\therefore$  The sustained wind pressure  $q_z = 1.25 \times 1.05 \times 30 = 39.38$  psf

(ii) Design wind pressure  $p_z = C_G C_t C_p q_z$

where  $C_G$  at  $35' = 1.25$ ,  $C_t$  at flat terrain = 1.0,  $C_p$  (for  $h/B = 10/10 = 1.0$ ,  $L/B = 10/10 = 1.0$ ) = 1.7

$\therefore$  The design wind pressure  $p_z = 1.25 \times 1.0 \times 1.7 \times 39.38 = 83.67$  psf

(iii) Design wind force  $F_z = B h_0 p_z$ ; where  $B = 10'$ ,  $h_0 = 10'$

$\therefore$  The design wind force  $F_z = 10 \times 10 \times 83.67 = 8367$  lb = 8.37 kips

2. Basic wind pressure  $q_b = 30$  psf

(i) Sustained wind pressure  $q_z = C_1 C_z q_b$

where  $C_1 = 1.0$ ,  $C_z$  changes with height as shown below

(ii) Sustained wind force  $F_z = q_z \times$  Tributary Area =  $q_z \times$  Effective height  $h_0 \times$  Effective width B

Story	z (ft)	$C_1$	$C_z$	$q_z$ (psf)	$h_{0z}$ (ft)	B (ft)	$Q_z$ (kips)
1	12	1.0	0.801	24.03	11	30	7.93
2	22		0.866	25.98	10		7.79
3	32		0.958	28.74	10		8.62
4	42		1.051	31.53	5		4.73

3. Design wind pressure  $p_z = C_G C_t C_p q_z$

where  $C_t$  (for  $H/2L_u = 10/200 = 0.05$ ) = 1.19,  $C_p$  (for  $h/B = 42/30 = 1.4$ ,  $L/B = 40/30 = 1.33$ ) = 1.65

Also  $C_z$  and  $C_G$  change with height as shown below.

Since  $q_1 = 35$  psf at  $z = 12$  ft, the sustained pressures at other heights can be calculated from  $C_z$

Story	z (ft)	$C_z$	$q_z$ (psf)	$C_G$	$C_t$	$C_p$	$p_z$ (psf)	$h_{0z}$ (ft)	B (ft)	$F_z$ (kips)
1	12	0.801	35.00	1.321	1.19	1.65	90.78	11	30	29.96
2	22	0.866	37.84	1.300			96.59	10		28.98
3	32	0.958	41.86	1.270			104.38	10		31.32
4	42	1.051	45.92	1.239			111.72	5		16.76

4. If the windward pressure = 0, then  $0.07\alpha - 2.1 = 0 \Rightarrow \alpha = 30^\circ \Rightarrow$  Height of truss =  $60 \tan 30^\circ = 34.64'$

If the windward pressure = Leeward suction pressure = 0.7, then  $0.03\alpha - 0.9 = 0.7 \Rightarrow \alpha = 53.33^\circ$

$\Rightarrow$  Height of truss =  $60 \tan 53.33^\circ = 80.53'$

Using  $H/2L_u = 0.05 \Rightarrow C_t = 1.19$  and assuming  $C_G = 1.10$  (uniform) reasonably for both (i) and (ii).

(i)  $p_z$  (windward) = 0 and  $p_z$  (leeward) =  $1.10 \times 1.19 \times (-0.70) \times 40 = -33.32$  psf

(ii)  $p_z$  (windward) = 33.32 psf, and  $p_z$  (leeward) = -33.32 psf

5. For a flat terrain in Khulna,  $V_b = 150$  mph  $\Rightarrow q_z = 0.00256 (150)^2 = 57.6$  psf

$C_1 = 1.25$ ,  $C_t = 1.0$ , and assuming height =  $30/2 = 15'$  for Exposure B,  $C_z = 0.801$ ,  $C_G = 1.321$

$q_z = 1.25 \times 0.801 \times 57.6 = 57.67$  psf

(i) If wind blows from left,  $\alpha = \tan^{-1}(30/30) = 45^\circ$ ,

$C_p$  (windward) =  $0.03 \times 45 - 0.9 = 0.45$ , and  $C_p$  (leeward) = -0.7

$p_z$  (windward) =  $1.321 \times 1.0 \times (0.45) \times 57.67 = 34.28$  psf and

$p_z$  (leeward) =  $1.321 \times 1.0 \times (-0.7) \times 57.67 = -53.33$  psf

(ii) If wind blows from right,  $\alpha = \tan^{-1}(30/60) = 26.6^\circ$ ,

$C_p$  (windward) =  $0.07 \times 26.6 - 2.1 = -0.24$ , and  $C_p$  (leeward) = -0.7

$p_z$  (windward) =  $1.321 \times 1.0 \times (-0.24) \times 57.67 = -18.13$  psf and  $p_z$  (leeward) = -53.33 psf

6. For an essential structure in Dhaka,  $Z = 0.15$ ,  $I = 1.25$

Also natural frequency  $f_n = 3 \text{ Hz} \Rightarrow \text{Time period } T = 1/3 = 0.33 \text{ sec}$

For soil type  $S_3$ ,  $S = 1.5 \Rightarrow C = 1.25 S/T^{2/3} = 1.25 \times 1.5/0.33^{2/3} = 3.90 > 2.75$ ; i.e.,  $C = 2.75$

Also response modification factor  $R = 4.0$

Neglecting the weight of the column and tank and considering the weight of water only,

The total seismic weight  $W = 10 \times 10 \times 10 \times 62.5/1000 = 62.5 \text{ kips}$

$\therefore$  Seismic base shear force  $V = (ZIC/R) W = (0.15 \times 1.25 \times 2.75/4.0) \times 62.5 = 8.06 \text{ kips}$

7. The seismic force distribution is given by the equation,  $V = F_t + \sum F_i$

where,  $F_i$  = Lateral force applied at storey level  $i$ , and

$F_t$  = Additional concentrated lateral force considered at the top of the building

Here,  $T$  = Fundamental period of vibration =  $C_t (h_n)^{3/4} = 0.073 (42/3.28)^{3/4} = 0.49 \text{ sec} \leq 0.70 \text{ sec}$

$\therefore F_t = 0$

$\therefore F_j = V [w_j h_j / \sum w_i h_i] = V h_j / \sum h_i$ , if floor weights are assumed constant

$\therefore$  Seismic forces are assumed to be proportional to height from base

$\therefore F_1 = 5^k \Rightarrow F_2 = 5 \times 22/12 = 9.17^k$ ,  $F_3 = 5 \times 32/12 = 13.33^k$ ,  $F_4 = 5 \times 42/12 = 17.50^k$

$\therefore$  Base shear force  $V = F_1 + F_2 + F_3 + F_4 = 5 + 9.17 + 13.33 + 17.50 = 45.0 \text{ kips}$

## Dynamic Force, Dynamic System and Equation of Motion

### Dynamic Force and System

Time-varying loads are called dynamic loads. Structural dead loads and live loads have the same magnitude and direction throughout their application and are thus static loads. However there are several examples of forces that vary with time, i.e., those caused by wind, vortex, water wave, vehicle, blast or ground motion.

A dynamic system is a simple representation of physical systems and is modeled by mass, damping and stiffness. Stiffness is the resistance it provides to deformations, mass is the matter it contains and damping represents its ability to decrease its own motion with time. A dynamic system resists external forces by a combination of forces due to its stiffness (spring force), damping (viscous force) and mass (inertia force).

### Formulation of the Single-Degree-of-Freedom (SDOF) Equation

For the system shown in Fig. 1.1,  $k$  is the stiffness,  $c$  the viscous damping,  $m$  the mass and  $u(t)$  is the dynamic displacement due to the time-varying excitation force  $f(t)$ . Such systems are called Single-Degree-of-Freedom (SDOF) systems because they have only one dynamic displacement [ $u(t)$  here].

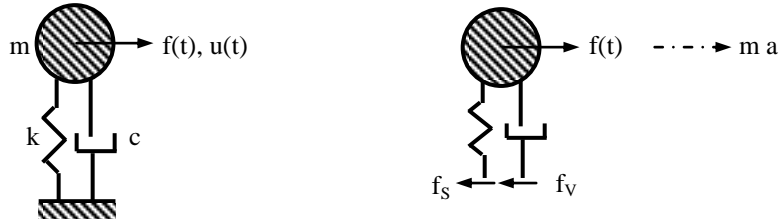


Fig. 1.1: Dynamic SDOF system subjected to dynamic force  $f(t)$

$$\text{Considering the free body diagram of the system, } f(t) - f_s - f_v = ma \Rightarrow ma + f_v + f_s = f(t) \quad \dots\dots\dots(1.1)$$

$$\text{where } f_s = \text{Spring force} = \text{Stiffness times the displacement} = k u \quad \dots\dots\dots(1.2)$$

$$f_v = \text{Viscous force} = \text{Viscous damping times the velocity} = c \, du/dt \quad \dots\dots\dots(1.3)$$

$$f_i = \text{Inertia force} = \text{Mass times the acceleration} = m \, d^2u/dt^2 \quad \dots\dots\dots(1.4)$$

$$\text{Combining the equations (1.2)-(1.4) with (1.1), the equation of motion for a SDOF system is derived as,} \\ m \, d^2u/dt^2 + c \, du/dt + ku = f(t) \quad \dots\dots\dots(1.5)$$

This is a 2<sup>nd</sup> order ordinary differential equation, which needs to be solved in order to obtain the dynamic displacement  $u(t)$ . Eq. (1.5) has several limitations; but it still has wide applications in Structural Dynamics. Several important derivations and conclusions in this field have been based on it.

### Governing Equation of Motion for Systems under Seismic Vibration

The loads induced by earthquake are not body-forces; rather it is a ground vibration that induces certain forces in the structure. For the SDOF system subjected to ground displacement  $u_g(t)$

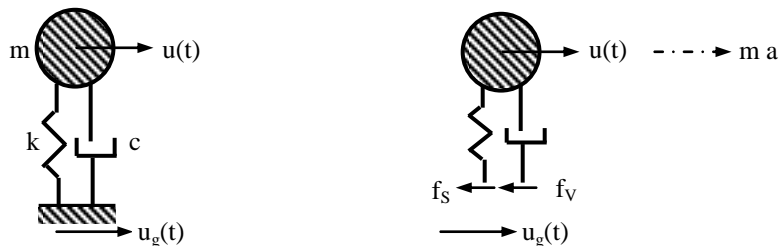


Fig. 1.2: Dynamic SDOF system subjected to ground displacement  $u_g(t)$

$$f_s = \text{Spring force} = k(u - u_g), \quad f_v = \text{Viscous force} = c(du/dt - du_g/dt), \quad f_i = \text{Inertia force} = m \, d^2u/dt^2$$

Combining the equations, the equation of motion for a SDOF system is derived as,

$$m \, d^2u/dt^2 + c(du/dt - du_g/dt) + k(u - u_g) = 0 \Rightarrow m \, d^2u/dt^2 + c \, du/dt + k u = c \, du_g/dt + k u_g \quad \dots\dots\dots(1.6)$$

$$\Rightarrow m \, d^2u_r/dt^2 + c \, du_r/dt + k u_r = -m \, d^2u_g/dt^2 \quad \dots\dots\dots(1.7)$$

where  $u_r = u - u_g$  is the relative displacement of the SDOF system with respect to the ground displacement. Eqs. (1.6) and (1.7) show that the ground motion appears on the right side of the equation of motion just like a time-dependent load. Therefore, although there is no body-force on the system, it is still subjected to dynamic excitation by the ground displacement.

## Free Vibration of Damped Systems

As mentioned in the previous section, the equation of motion of a dynamic system with mass ( $m$ ), linear viscous damping ( $c$ ) & stiffness ( $k$ ) undergoing free vibration is,

$$m \frac{d^2u}{dt^2} + c \frac{du}{dt} + ku = 0 \quad \dots\dots\dots(1.5)$$

$$\Rightarrow \frac{d^2u}{dt^2} + \frac{c}{m} \frac{du}{dt} + \frac{k}{m} u = 0 \Rightarrow \frac{d^2u}{dt^2} + 2\xi\omega_n \frac{du}{dt} + \omega_n^2 u = 0 \quad \dots\dots\dots(2.1)$$

where  $\omega_n = \sqrt{k/m}$ , is the *Natural Frequency* of the system .....(2.2)

and  $\xi = c/(2m\omega_n) = c\omega_n/(2k) = c/2\sqrt{km}$ , is the *Damping Ratio* of the system .....(2.3)

If  $\xi < 1$ , the system is called an *Underdamped System*. Practically, most structural systems are underdamped. The displacement  $u(t)$  for such a system is

$$u(t) = e^{-\xi\omega_n t} [C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] \quad \dots\dots\dots(2.4)$$

where  $\omega_d = \omega_n \sqrt{1-\xi^2}$  is called the *Damped Natural Frequency* of the system .....(2.5)

If  $u(0) = u_0$  and  $\dot{u}(0) = v_0$ , then the equation for free vibration of a damped system is given by

$$u(t) = e^{-\xi\omega_n t} [u_0 \cos(\omega_d t) + \{(v_0 + \xi\omega_n u_0)/\omega_d\} \sin(\omega_d t)] \quad \dots\dots\dots(2.6)$$

∴ Eq (2.6) ⇒ The system vibrates at its damped natural frequency (i.e., a frequency of  $\omega_d$  radian/sec).

Since  $\omega_d [= \omega_n \sqrt{1-\xi^2}]$  is less than  $\omega_n$ , the system vibrates more slowly than the undamped system.

Moreover, due to the exponential term  $e^{-\xi\omega_n t}$ , the amplitude of the motion of an underdamped system decreases steadily, and reaches zero after (a hypothetical) ‘infinite’ time of vibration.

### Example 2.1

A damped structural system with stiffness ( $k$ ) = 25 k/ft and mass ( $m$ ) = 1 k-sec<sup>2</sup>/ft is subjected to an initial displacement ( $u_0$ ) = 1 ft, and an initial velocity ( $v_0$ ) = 4 ft/sec. Plot the free vibration of the system vs. time if the Damping Ratio ( $\xi$ ) is (a) 0.00 (undamped system), (b) 0.05, (c) 0.50 (underdamped systems).

### Solution

The equations for  $u(t)$  are plotted against time for various damping ratios (DR) and shown below in Fig. 2.1. These figures show that the underdamped systems have sinusoidal variations of displacement with time. Their natural periods are lengthened (more apparent for  $\xi = 0.50$ ) and maximum amplitudes of vibration reduced due to damping.

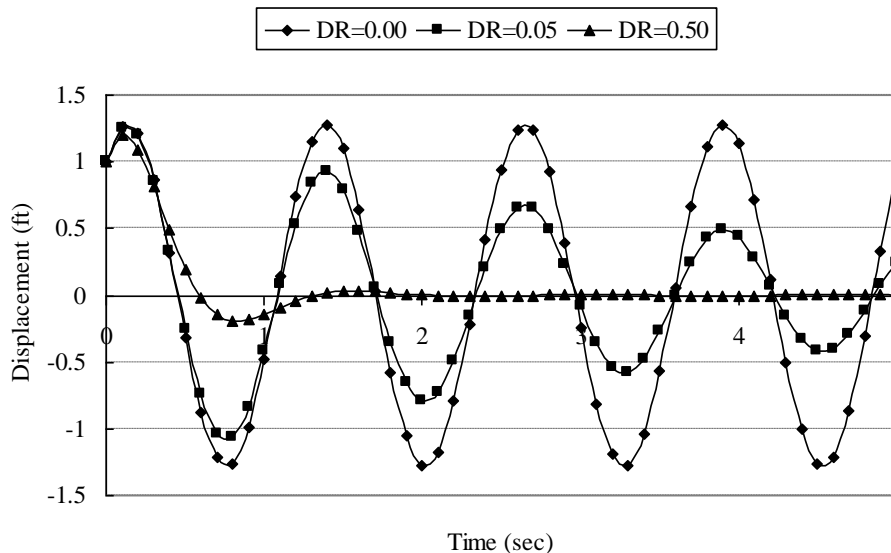


Fig. 2.1: Displacement vs. Time for free vibration of damped systems



Damping of Structures

Damping is the element that causes impedance of motion in a structural system. There are several sources of damping in a dynamic system. It can be due to internal resistance to motion between layers, friction between different materials or different parts of the structure (called frictional damping), drag between fluids or structures flowing past each other, etc. Sometimes, external forces themselves can contribute to (increase or decrease) the damping. Damping is also increased in structures artificially by external sources like dampers acting as control systems.

Viscous Damping of SDOF systems

Linear viscous damping is the most used damping and provides a force directly proportional to the structural velocity. This is a fair representation of structural damping in many cases and is often convenient to assume for the purpose of analysis. Viscous damping is usually an intrinsic property of the material and originates from internal resistance to motion between different layers within the material itself.

The free vibration response of SDOF system with linear viscous damping was found to be

$$u(t) = e^{-\xi\omega_n t} [u_0 \cos(\omega_d t) + \{(v_0 + \xi\omega_n u_0)/\omega_d\} \sin(\omega_d t)] \dots\dots\dots(2.6)$$

Therefore the displacement at N time periods ( $T_d = 2\pi/\omega_d$ ) later than u(t) is

$$u(t+NT_d) = e^{-\xi\omega_n(t+2\pi N/\omega_d)} [u_0 \cos(\omega_d t+2\pi N) + \{(v_0 + \xi\omega_n u_0)/\omega_d\} \sin(\omega_d t+2\pi N)] = e^{-\xi\omega_n(2\pi N/\omega_d)} u(t) \dots(2.7)$$

$$\text{From which, using } \omega_d = \omega_n \sqrt{1-\xi^2} \Rightarrow \xi/\sqrt{1-\xi^2} = \ln[u(t)/u(t+NT_d)]/2\pi N = \delta \Rightarrow \xi = \delta/\sqrt{1+\delta^2} \dots\dots\dots(2.8)$$

$$\text{For lightly damped structures (i.e., } \xi \ll 1), \xi \cong \delta = \ln[u(t)/u(t+NT_d)]/2\pi N \dots\dots\dots(2.9)$$

For example, if the free vibration amplitude of a SDOF system decays from 1.5" to 0.5" in 3 cycles

$$\delta = \ln(1.5/0.5)/(2\pi \times 3) = 0.0583 = 5.83\%$$

$$\therefore \text{The damping ratio, } \xi = \delta/\sqrt{1+\delta^2} = 0.0582 = 5.82\%$$

**Table 2.1: Recommended Damping Ratios for different Structural Elements**

Stress Level	Type and Condition of Structure	$\xi$ (%)
Working stress	Welded steel, pre-stressed concrete, RCC with slight cracking	2-3
	RCC with considerable cracking	3-5
	Bolted/riveted steel or timber	5-7
Yield stress	Welded steel, pre-stressed concrete	2-3
	RCC	7-10
	Bolted/riveted steel or timber	10-15

## Forced Vibration and Dynamic Magnification

Unlike free vibration, forced vibration is the dynamic motion caused by the application of external force (with or without initial displacement and velocity). Therefore,  $f(t) \neq 0$  in the equation of motion for forced vibration; rather, they have different equations for different variations of the applied force with time. The equations for displacement for various types of applied force are now studied analytically for undamped and underdamped vibration systems. The following cases are studied

### Case1: Step Loading

For a constant static load of  $p_0$ , the equation of motion becomes

$$m \frac{d^2u}{dt^2} + c \frac{du}{dt} + ku = p_0 \quad \dots\dots\dots(3.1)$$

If initial displacement  $u(0) = 0$  and initial velocity  $v(0) = 0$ , then

$$u(t) = (p_0/k)[1 - e^{-\zeta\omega_n t} \{ \cos(\omega_d t) + \zeta\omega_n/\omega_d \sin(\omega_d t) \}] \quad \dots\dots\dots(3.2)$$

For an undamped system,  $\zeta = 0$ ,  $\omega_d = \omega_n \therefore u(t) = (p_0/k)[1 - \cos(\omega_n t)] \quad \dots\dots\dots(3.3)$

### Example 3.1

For the system mentioned in Example 2.1 (i.e.,  $k = 25 \text{ k/ft}$ ,  $m = 1 \text{ k-sec}^2/\text{ft}$ ), plot the displacement vs. time if a static load  $p_0 = 25 \text{ k}$  is applied on the system if the Damping Ratio ( $\zeta$ ) is

- (a) 0.00 (undamped system), 0.05, (c) 0.50 (underdamped systems).

### Solution

In this case, the static displacement is  $= p_0/k = 25/25 = 1 \text{ ft}$ . The dynamic solutions are obtained from Eq. 3.2 and plotted below in Fig. 3.1. The main features of these results are

For Step Loading, the maximum dynamic response for an undamped system (i.e., 2 ft in this case) is twice the static response and continues indefinitely without converging to the static response.

Here, the maximum dynamic response for damped systems is between 1 and 2 ft, and eventually converges to the static solution (1 ft). The larger the damping ratio, the less the maximum response and the quicker it converges to the static solution.

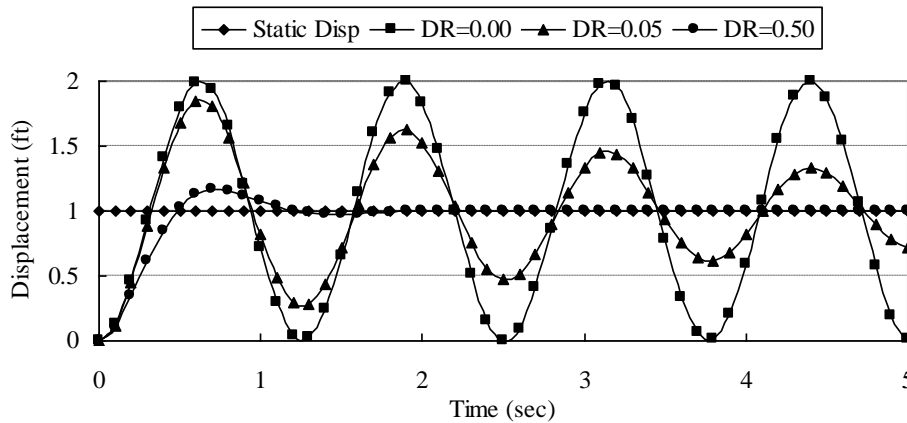


Fig. 3.1: Dynamic Response to Step Loading

### Case2: Harmonic Loading

For a harmonic load of amplitude  $p_0$  and frequency  $\omega$ , the equation of motion is

$$m \frac{d^2u}{dt^2} + c \frac{du}{dt} + ku = p_0 \cos(\omega t), \text{ for } t > 0 \quad \dots\dots\dots(3.4)$$

If initial displacement  $u(0) = 0$  and initial velocity  $v(0) = 0$ , then

$$u(t) = (p_0/k_d)[\cos(\omega t - \phi) - e^{-\zeta\omega_n t} \{ \cos\phi \cos(\omega_d t) + (\omega/\omega_d \sin\phi + \zeta\omega_n/\omega_d \cos\phi) \sin(\omega_d t) \}] \quad \dots\dots\dots(3.5)$$

$\therefore$  For an undamped system,  $u(t) = [p_0/(k - \omega^2 m)] [\cos(\omega t) - \cos(\omega_n t)] \quad \dots\dots\dots(3.6)$

### Dynamic Magnification

If the motion of a SDOF system subjected to harmonic loading is allowed to continue for long (theoretically infinite) durations, the total response converges to the steady state solution given by the particular solution of the equation of motion.

$$\therefore u_{\text{steady}}(t) = u_p(t) = (p_0/k_d) \cos(\omega t - \phi) \quad \dots\dots\dots(3.5)$$

[where  $k_d = \sqrt{\{(k - \omega^2 m)^2 + (\omega c)^2\}}$ ,  $\phi = \tan^{-1}\{(\omega c) / (k - \omega^2 m)\}$ ]

$\therefore$  Putting the value of  $k_d$  in Eq. (3.5), the amplitude of steady vibration is found to be

$$\therefore u_{\text{amplitude}} = p_0/k_d = p_0/\sqrt{\{(k - \omega^2 m)^2 + (\omega c)^2\}} \quad \dots\dots\dots(3.8)$$

Using  $u_{\text{static}} = p_0/k \Rightarrow u_{\text{amplitude}}/u_{\text{static}} = 1/\sqrt{\{(1 - \omega^2/\omega_n^2)^2 + (2\xi\omega/\omega_n)^2\}} \quad \dots\dots\dots(3.9)$

Eq. (3.9) gives the ratio of the dynamic and static amplitude of motion as a function of frequency  $\omega$  (as well as structural properties like  $\omega_n$  and  $\xi$ ). This ratio is called the steady state dynamic magnification factor (DMF) for harmonic motion. Putting the frequency ratio  $\omega/\omega_n = r$ , Eq. (3.9) can be rewritten as

$$\text{DMF} = 1/\sqrt{\{(1 - r^2)^2 + (2\xi r)^2\}} \quad \dots\dots\dots(3.10)$$

From which the maximum value of DMF is found =  $(1/2\xi)/\sqrt{(1 - \xi^2)}$ , when  $r = \sqrt{(1 - 2\xi^2)}$   $\dots\dots\dots(3.11)$

The variation of the steady state dynamic magnification factor (DMF) with frequency ratio ( $r = \omega/\omega_n$ ) is shown in Fig. 3.2 for different values of  $\xi$  (= DR). The main features of this graph are

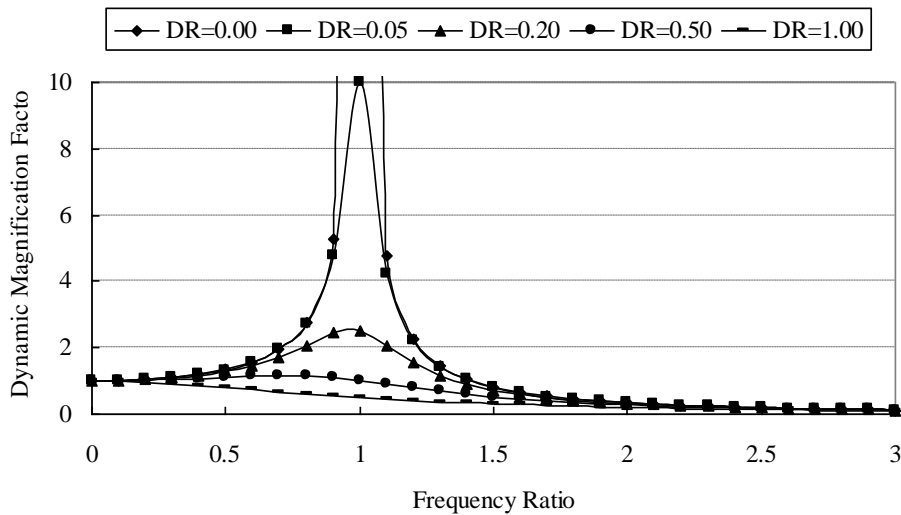
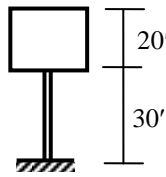


Fig. 3.2: Steady State Dynamic Magnification Factor

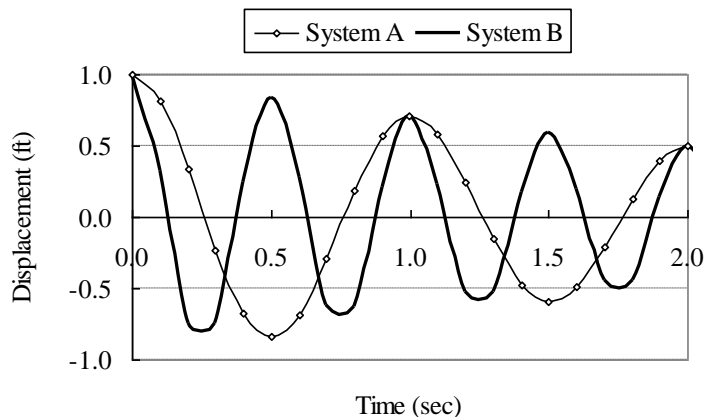
1. The curves for the smaller values of  $\xi$  show pronounced peaks  $[\cong 1/(2\xi)]$  at  $\omega/\omega_n \cong 1$ . This situation is called Resonance and is characterized by large dynamic amplifications of motion. This situation can be derived from Eq. (3.10), where  $\xi \ll 1 \Rightarrow \text{DMF}_{\text{max}} \cong 1/\sqrt{(4\xi^2)} = (1/2\xi)$ , when  $r \cong \sqrt{(1 - 2\xi^2)} = 1$ .
2. For undamped system, the resonant peak is infinity.
3. Since resonance is such a critical condition from structural point of view, it should be avoided in practical structures by making it either very stiff (i.e.,  $r \ll 1$ ) or very flexible (i.e.,  $r \gg 1$ ) with respect to the frequency of the expected harmonic load.
4. The resonant condition mentioned in (1) is not applicable for large values of  $\xi$ , because the condition of maxima at  $r = \sqrt{(1 - 2\xi^2)}$  is meaningless if  $r$  is imaginary; i.e.,  $\xi > (1/\sqrt{2}) = 0.707$ . Therefore, another way of avoiding the critical effects of resonance is by increasing the damping of the system.

## Problems on the Dynamic Analysis of SDOF Systems

1. For the (20' × 20' × 20') overhead water tank shown below supported by a 25" × 25" square column, calculate the undamped natural frequency for (i) horizontal vibration ( $k = 3EI/L^3$ ), (ii) vertical vibration ( $k = EA/L$ ). Assume the total weight of the system to be concentrated in the tank  
 [Given: Modulus of elasticity of concrete =  $400 \times 10^3$  k/ft<sup>2</sup>, Unit weight of water = 62.5 lb/ft<sup>3</sup>].



2. The free vibration responses of two underdamped systems (A and B) are shown below.  
 (i) Calculate the undamped natural frequency and damping ratio of system B.  
 (ii) Explain (qualitatively) which one is stiffer and which one is more damped of the two systems.



3. For the undamped water tank described in Question 1, calculate the  
 (i) maximum displacement when subjected to a sustained wind pressure of 40 psf,  
 (ii) maximum steady-state displacement when subjected to a harmonic wind pressure of  $[40 \cos(2t)]$  psf.
4. A SDOF system with  $k = 10$  k/ft,  $m = 1$  k-sec<sup>2</sup>/ft, with  $c = 0$ , and  $0.5$  k-sec/ft is subjected to a force (lbs)  
 (i)  $p(t) = 500$ , (ii)  $p(t) = 500 \cos(3t)$ .  
 Calculate maximum displacement of the system for (i) and maximum steady-state displacement for (ii).
5. An undamped SDOF system suffers resonant vibration when subjected to a harmonic load (i.e., of frequency  $\omega = \omega_n$ ). Of the control measures suggested below, explain which one will minimize the steady-state vibration amplitude.  
 (i) Doubling the structural stiffness, (ii) Doubling the structural stiffness and the mass,  
 (iii) Adding a damper to make the structural damping ratio = 10%.

## Solution of Problems on the Dynamic Analysis of SDOF Systems

1. Mass of the tank (filled with water),  $m = 20 \times 20 \times 20 \times 62.5/32.2 = 15528 \text{ lb-ft/sec}^2$   
 Modulus of elasticity  $E = 400 \times 10^3 \text{ k/ft}^2 = 400 \times 10^6 \text{ lb/ft}^2$ , Length of column  $L = 30 \text{ ft}$   
 (i) Moment of inertia,  $I = (25/12)^4/12 = 1.570 \text{ ft}^4$   
 Stiffness for horizontal vibration,  $k_h = 3EI/L^3 = 3 \times 400 \times 10^6 \times 1.570/(30)^3 = 69770 \text{ lb/ft}$   
 Natural frequency,  $\omega_{nh} = \sqrt{(k_h/m)} = \sqrt{(69770/15528)} = 2.120 \text{ rad/sec}$   
 (ii) Area,  $A = (25/12)^2 = 4.340 \text{ ft}^2$   
 Stiffness for vertical vibration,  $k_v = EA/L = 400 \times 10^6 \times 4.340/30 = 5787 \times 10^4 \text{ lb/ft}$   
 Natural frequency,  $\omega_{nv} = \sqrt{(k_v/m)} = \sqrt{(5787 \times 10^4/15528)} = 61.05 \text{ rad/sec}$
  
2. (i) System B takes 1.0 second to complete two cycles of vibration.  
 $\therefore$  Damped natural period  $T_d$  for system B =  $1.0/2 = 0.50 \text{ sec}$   
 $\therefore$  Damped natural frequency,  $\omega_d = 2\pi/T_d = 12.566 \text{ rad/sec}$   
 Using as reference the displacements at  $t = 0$  (1.0 ft) and  $t = 2.0 \text{ sec}$  (0.5 ft); i.e., for  $N = 4$   
 $\delta = \ln[u(0.0)/u(2.0)]/(2\pi \times 4) = \ln[1.0/0.5]/8\pi = 0.0276 \Rightarrow \xi = \delta/\sqrt{(1+\delta^2)} = 0.0276$   
 Undamped natural frequency,  $\omega_n = \omega_d/\sqrt{(1-\xi^2)} = 12.566/\sqrt{(1-0.0276^2)} = 12.571 \text{ rad/sec}$   
 (ii) System A completes only two vibrations while (in 2.0 sec) system B completes four vibrations.  
 $\therefore$  System B is stiffer.  
 However, system A decays by the same ratio (i.e., 0.50 or 50%) in two vibrations system B decays in four vibrations.  
 $\therefore$  System A is more damped.
  
3. For the water tank filled with water,  
 Mass,  $m = 15528 \text{ lb-ft/sec}^2$ , Stiffness for horizontal (i.e., due to wind) vibration,  $k_h = 69770 \text{ lb/ft}$   
 Natural frequency,  $\omega_{nh} = 2.120 \text{ rad/sec}$   
 (i)  $p(t) = p_0 = 40 \times 20 \times 20 = 16000 \text{ lb}$   
 For the undamped system,  $u_{\max} = 2(p_0/k_h) = 2(16000/69770) = 0.459 \text{ ft}$   
 (ii)  $p(t) = p_0 \cos(\omega t) = 16000 \cos(2t)$   
 For the undamped system,  $u(\text{steady})_{\max} = p_0/(k_h - \omega^2 m) = 16000/(69770 - 2^2 \times 15528) = 2.09 \text{ ft}$
  
4. For the SDOF system,  $k = 10 \text{ k/ft}$ ,  $m = 1 \text{ k-sec}^2/\text{ft}$ ,  $c = 0$ , and  $c = 0.5 \text{ k-sec/ft}$   
 $\therefore$  Natural frequency,  $\omega_n = \sqrt{(k/m)} = \sqrt{(10/1)} = 3.162 \text{ rad/sec}$ , Damping Ratio  $\xi = c/(2\omega_n) = 0$ , and  $\xi = 0.079$   
 (i)  $p(t) = p_0 = 500 \text{ lb} = 0.5 \text{ kip}$   
 For undamped system,  $u_{\max} = 2(p_0/k) = 2(0.5/10) = 0.10 \text{ ft}$   
 For damped system,  $u_{\max} = (p_0/k) [1 + e^{\{-\pi\xi/(1-\xi^2)\}}] = (0.5/10) [1 + \exp\{-0.079\pi/(1-0.079^2)\}] = 0.089 \text{ ft}$   
 (ii)  $p(t) = 0.5 \cos(3t)$   
 For undamped system,  $u(\text{steady})_{\max} = p_0/(k - \omega^2 m) = (0.50)/(10 - 3^2 \times 1) = 0.50 \text{ ft}$   
 For damped system,  $u(\text{steady})_{\max} = p_0/\sqrt{\{(k - \omega^2 m)^2 + (\omega c)^2\}} = (0.50)/\sqrt{\{(10 - 3^2 \times 1)^2 + (3 \times 0.5)^2\}} = 0.277 \text{ ft}$
  
5. Maximum dynamic response amplitude,  $u_{\max} = p_0/(k - \omega^2 m)$   
 If  $\omega = \omega_n$ ,  $u_{\max} = p_0/(k - \omega_n^2 m) = p_0/(k - k) = \infty$   
 (i) Doubling the structural stiffness  $\Rightarrow u_{\max} = p_0/(2k - k) = p_0/k$   
 (ii) Doubling the structural stiffness and the mass  $\Rightarrow u_{\max} = p_0/(2k - \omega_n^2 2m) = p_0/(2k - 2k) = \infty$   
 (iii) Adding a damper to make the structural damping ratio,  $\xi = 10\% = 0.10$   
 $\Rightarrow u_{\max} = p_0/\sqrt{\{(k - \omega_n^2 m)^2 + (\omega_n c)^2\}} = (p_0/k)/(2\xi) = 5(p_0/k)$   
 $\therefore$  Option (i) is the most effective [since it minimizes  $u_{\max}$ ].

## Vortex Induced Vibration (VIV)

The pressure difference around a bluff body in flowing fluid may result in separated flow and shear layers over a large portion of its surface. The outermost shear layers (in contact with the fluid) move faster than the innermost layers, which are in contact with the structure. If the fluid velocity is large enough, this causes the shear layers to roll into the near wake and form periodic vortices. The interaction of the structure with these vortices causes it to vibrate transverse to the flow direction, and this vibration is called VIV.

The frequency of vortex-shedding is called Strouhal frequency (after Strouhal 1878), given by

$$f_s = SU/D \quad \dots\dots\dots(4.1)$$

where U = Fluid Velocity, D = Transverse dimension (diameter) of the structure, and S = Strouhal number, which is a function of Reynolds number and the geometry of the structure. For circular cylinders,  $S \cong 0.20$ .

Experimental results suggest that VIV usually causes structural vibrations at the frequency  $f_s$ . However, if  $f_s$  is near (about 0.9-1.2 times) the natural frequency ( $f_n$ ) of the structure, the structure vibrates at its natural frequency. Obviously, this is the most significant feature of VIV and is known as synchronization or lock-in.

For a structure, VIV can be important enough to cause fatigue or instability failure; and over the years it has been observed in both wind-driven as well as marine structures. The research on the fundamental nature of VIV has developed in two different directions. One is the rigorous treatment by computational fluid dynamics using Navier-Stokes equations. Although it has enjoyed success, this approach is still not developed enough for practical design purposes. The other direction has been primarily aimed at predicting structural response and modeling the associated fluid-dynamic phenomena by coupled/uncoupled dynamic equations of motion. Since we primarily aim to predict structural responses due to VIV, and therefore focus on a simple dynamic model, developed by Iwan and Blevins (1974).

Incorporating experimental results into the equation of motion is equivalent to adding a harmonic load and an additional damping into the system. The harmonic load has a frequency  $\omega_e \cong \omega_s = 2\pi f_s$ , and is similar to the form

$$0.5 \rho U^2 D C_L \sin(\omega_e t), \text{ where } C_L \cong 0.50, \text{ for several experimental data.}$$

$$\text{Also, the added damping/length is } \cong 0.4 \rho U D \quad \dots\dots\dots(4.2)$$

Therefore, for uniform structural dimensions throughout its length L, the dynamic SDOF system is modeled as

$$m \, d^2u/dt^2 + (c + 0.4 \rho U D L) \, du/dt + ku = 0.25 \rho U^2 D L \sin(\omega_s t) \quad \dots\dots\dots(4.3)$$

### Example 4.1

Calculate the dynamic force and added damping due to wind-driven VIV for a cylindrical water tank of 20'-diameter, 20'-height and supported by two 30'-high concrete columns of 25"-diameter each, considering (i) Design wind speed in Dhaka, (ii) Lock-in phenomenon.

Mass of the tank (filled with water),  $m = (\pi/4) \times 20^2 \times 20 \times 62.5/32.2 = 12196 \text{ lb-ft/sec}^2$   
 Modulus of elasticity  $E = 400 \times 10^6 \text{ lb/ft}^2$ , Length of column  $L = 30 \text{ ft}$   
 Moment of inertia,  $I = \pi(25/12)^4/64 = 0.925 \text{ ft}^4$   
 Stiffness for horizontal vibration,  $k_n = 2 \times 3EI/L^3 = 82196 \text{ lb/ft}$ , Natural frequency,  $\omega_n = 2.60 \text{ rad/sec}$

If  $U = 130 \text{ mph} = 190.67 \text{ ft/sec}$ ,  $\omega_s = 2\pi \times (0.2 \times 190.67/20) = 11.98 \text{ rad/sec}$   
 $\therefore \rho = \text{Density of air} = 0.0765/32.2 = 23.76 \times 10^{-4} \text{ slug/ft}^3$   
 $\Rightarrow \text{Dynamic force amplitude} = 0.25 \rho U^2 D L = 0.25 \times 23.76 \times 10^{-4} \times (190.67)^2 \times 20 \times 20 = 8638 \text{ lb}$   
 Added damping  $= 0.4 \rho U D L = 0.4 \times 23.76 \times 10^{-4} \times 190.67 \times 20 \times 20 = 72.48 \text{ lb-sec/ft}$   
 Since  $\omega_s/\omega_n = 11.98/2.60 = 4.61 \gg 1$ , VIV is not very important at this velocity.

Lock-in occurs when  $\omega_s = \omega_n = 2.60 \text{ rad/sec} \Rightarrow 2\pi \times (0.2 \times U/20) = 2.60 \Rightarrow U = 41.32 \text{ ft/sec} = 28.2 \text{ mph}$   
 $\Rightarrow \text{Dynamic force amplitude} = 0.25 \rho U^2 D L = 0.25 \times 23.76 \times 10^{-4} \times (41.32)^2 \times 20 \times 20 = 405 \text{ lb}$   
 Added damping  $c_{add} = 0.4 \rho U D L = 0.4 \times 23.76 \times 10^{-4} \times 41.32 \times 20 \times 20 = 15.71 \text{ lb-sec/ft}$   
 $\therefore \text{Added damping ratio } \xi_{add} = c_{add}/[2\sqrt{(km)}] = 15.71/[2\sqrt{(82196 \times 12196)}] = 0.00025$ , which is very small  
 Therefore, unless the system itself is sufficiently damped, the static deflection of  $(405/82196 =) 0.0049 \text{ ft}$  will be hugely amplified.

## Different Types of Dynamic Loads

### Hydrodynamic Force

Morison's equation is easily the most used equation for calculation of hydrodynamic force on marine/offshore structures. According to the Morison's equation, the horizontal wave-force on a structure of height  $L$  for a water-depth of  $d$  is given by

$$f(t) = \text{Inertia Force} + \text{Drag Force} = f_I + f_D \quad \dots\dots\dots(4.4)$$

For regular waves, the 'linearized' force can be calculated (after adjustments for structural acceleration) as

$$f_I^{(1)} = K_I(a\omega^2/k)[1 - \sinh(k(d-L))/\sinh(kd)] \sin \theta = f_{I0} \sin(\theta) \quad \dots\dots\dots(4.5)$$

$$\text{and } f_D^{(1)} \cong (4/3\pi) K_D(a\omega)^2 [\{2kL + \sinh(2kd) - \sinh(2k(d-L))\} / \sinh^2 kd] |\cos \theta| \cos \theta = f_{D0} |\cos \theta| \cos \theta \quad \dots\dots\dots(4.6)$$

$$\text{with additional damping } c_a \cong (4/3\pi) K_D(a\omega/k) [1 - \sinh(k(d-L))/\sinh kd] \quad \dots\dots\dots(4.7)$$

where  $K_I = \rho C_I A$ ,  $K_D = \rho C_D R$ ,  $\theta = kx - \omega t + \beta$

$\rho$  = Water density,  $C_I$  = Inertia coefficient,  $C_D$  = Drag coefficient

$A$  = Cross-sectional area,  $R$  = Radius = Half-width of projected surface

$a_x$  = Horizontal wave-acceleration,  $u_r$  = Horizontal (wave velocity – structural velocity)

### Vehicular Load

As shown in Fig. 4.1, a wheel load of  $p_0$  is traveling with a velocity  $v$  over a simply supported beam of length  $L$ . The wheel traveling with a velocity  $v$  takes time  $t_d = L/v$  to cross the bridge. Its position at any time  $t$  is as shown in Fig 4.1.

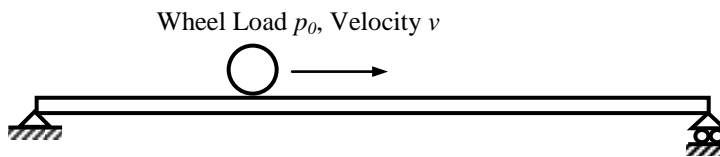


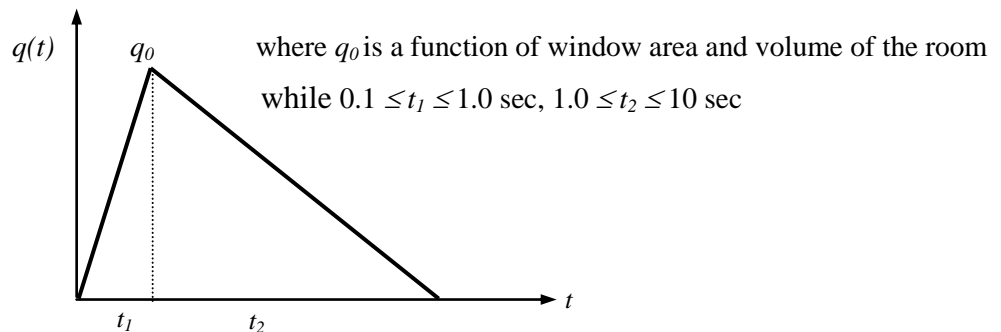
Fig. 4.1: Single wheel on Simply Supported Beam

The moving load can be described mathematically as

$$\begin{aligned} f(t) &= p_0 \sin(\pi vt/L), \text{ when } 0 \leq t \leq t_d \\ &= 0, \text{ when } t > t_d \end{aligned} \quad \dots\dots\dots(4.8)$$

### Blast Loading

According to BNBC, the internal overpressure ( $q$ ) developed from an internal explosion such as due to leaks in gas pipes, evaporation of volatile liquids, etc. in typical room sizes may be calculated from the following time-dependent function



### Machine Vibrations

The centrifugal force due to vibration of a mass  $m$  rotating at an angular velocity  $\omega$  and radius  $r_0$  that transfers to a machine foundation can be calculated from

$$f(t) = m\omega^2 r_0 \sin(\omega t) \quad \dots\dots\dots(4.9)$$

## Wave Force using Morison's Equation

Morison's equation is easily the most used equation for calculation of hydrodynamic force on marine/offshore structures. According to the (modified) Morison's equation, the horizontal wave-force on a differential vertical segment  $dz$  is given (after adjustments for structural acceleration) by

$$F_I^{(1)} = K_I (a\omega^2/k) [1 - \sinh(k(d-L))/\sinh(kd)] \sin \theta = F_{D0} \sin \theta \quad \dots\dots\dots(4.10)$$

$$dF_x = [K_I a_x + K_D |u_r| u] dz \quad \dots\dots\dots(4.4)$$

where  $K_I = \rho C_I A$ ,  $K_D = \rho C_D R$

$\rho$  = Water density,  $C_I$  = Inertia coefficient,  $C_D$  = Drag coefficient

$A$  = Cross-sectional area,  $R$  = Radius = Half-width of projected surface

$a_x$  = Horizontal wave-acceleration,  $u_r$  = Horizontal (wave velocity – structural velocity)

In equation (4.4), the first term gives the inertia force, and the second is the drag force.

$\therefore$  Total horizontal force  $F_x = \int dF_x$ , where  $\int$  implies integration between  $z = -L$  (bottom of the structure) and  $z = \eta$  (instantaneous wave-elevation).

Once the forces are calculated, the structural dynamic problem can be modeled as

$$M d^2u/dt^2 + C du/dt + k u = F_x \quad \dots\dots\dots(4.5)$$

However, the mass  $M$  is now a combination of the structural mass  $m$  and the 'added' mass  $m_a$ . The 'added' mass is obtained from the mathematical adjustment of the equation of motion and is proportional to mass  $m$  added by the mass of water occupying the same volume as the structure. For example,  $m_a$  for a uniform cylinder is  $= \rho \pi R^2 L$

$$\therefore \text{The effective mass, } M = m + m_a \quad \dots\dots\dots(4.6)$$

The damping  $C$  is a combination of the structural damping  $c$  and the damping  $c_a$  added from the relative velocity (drag) term of the load vector, which is obtained from the mathematical adjustment of the equation of motion.

$$\therefore \text{The mass-matrix, } C = c + c_a \quad \dots\dots\dots(4.7)$$

Common values of  $C_I$  are 2.0 for a cylinder and 1.5 for sphere. However, the drag coefficient  $C_D$  strongly depends on the Reynolds number ( $= uD/v$ ); whereas it converges to nearly 0.6-0.7 for high Reynolds numbers, it can be as high as 3.0 for small Reynolds numbers (i.e., for thinner structural elements like truss members).

### First Order Inertia Force

Integration of the 1<sup>st</sup> order inertia force over the depth of the structure gives the 1<sup>st</sup> order potential force with

$$F_I^{(1)} = K_I \Sigma (a_i \omega_i^2 / k_i) [1 - \sinh(k_i(d-L))/\sinh(k_i d)] \sin \theta_i \quad \dots\dots\dots(4.8)$$

where,  $\theta_i = k_i x - \omega_i t + \beta_i$ ; and 'wave number'  $k_i$  can be obtained from  $\omega_i^2 = g k_i \tanh(k_i d) \dots\dots\dots(4.9)$

For regular waves, it simplifies to  $F_I^{(1)} = K_I (a\omega^2/k) [1 - \sinh(k(d-L))/\sinh(kd)] \sin \theta = F_{D0} \sin \theta \quad \dots\dots\dots(4.10)$

### First Order (Linearized) Drag

The drag force is given by  $K_D \int |u_r^{(1)}| u^{(1)} dz$ , which can be approximately 'linearized'. In addition to the drag force involved, this turns out to be a major contributor to the damping of the system.

For irregular Gaussian waves, the total 'linearized' drag force  $F_D^{(1)} \cong \sqrt{(8/\pi)} K_D \int \sigma_{ur(1)} u dz \quad \dots\dots\dots(4.11)$

However for regular waves, the following equation can be used conveniently

$$F_D^{(1)} \cong (4/3\pi) K_D (a\omega)^2 [\{2kL + \sinh(2kd) - \sinh(2k(d-L))\} / \sinh^2 kd] |\cos \theta| \cos \theta = F_{D0} |\cos \theta| \cos \theta \dots\dots(4.12)$$

and the additional damping

$$c_a \cong (4/3\pi) K_D (a\omega/k) [1 - \sinh(k(d-L))/\sinh kd] \quad \dots\dots\dots(4.13)$$

$$\therefore \text{The maximum hydrodynamic force from Eq. (4.4) is } F_{\max} = \sqrt{(F_{I0}^2 + F_{D0}^2)} \quad \dots\dots\dots(4.14)$$

In addition to the inertia and drag forces, a mean drift force acts on the structure that cannot be predicted by Morison's equation. However the drag term in Morison's equation can account for the force ( $F_c$ ) and damping ( $c_c$ ) due to uniform current of velocity  $U_c$ , the force being given by

$$F_c \cong K_D U_c^2 L \quad \dots\dots\dots(4.15)$$

$$c_c \cong K_D U_c L \quad \dots\dots\dots(4.16)$$