Structural Dynamics, Dynamic Force and Dynamic System

Structural Dynamics

Conventional structural analysis is based on the concept of statics, which can be derived from Newton's 1st law of motion. This law states that it is necessary for some force to act in order to initiate motion of a body at rest or to change the velocity of a moving body. Conventional structural analysis considers the external forces or joint displacements to be static and resisted only by the stiffness of the structure. Therefore, the resulting displacements and forces resulting from structural analysis do not vary with time.

Structural Dynamics is an extension of the conventional static structural analysis. It is the study of structural analysis that considers the external loads or displacements to vary with time and the structure to respond to them by its stiffness as well as inertia and damping. Newton's 2nd law of motion forms the basic principle of Structural Dynamics. This law states that the resultant force on a body is equal to its mass times the acceleration induced. Therefore, just as the 1st law of motion is a special case of the 2nd law, static structural analysis is also a special case of Structural Dynamics.

Although much less used by practicing engineers than conventional structural analysis, the use of Structural Dynamics has gradually increased with worldwide acceptance of its importance. At present, it is being used for the analysis of tall buildings, bridges, towers due to wind, earthquake, and for marine/offshore structures subjected wave, current, wind forces, vortex etc.

Dynamic Force

The time-varying loads are called dynamic loads. Structural dead loads and live loads have the same magnitude and direction throughout their application and are thus static loads. However there are several examples of forces that vary with time, such as those caused by wind, vortex, water wave, vehicle, impact, blast or ground motion like earthquake.

Dynamic System

A dynamic system is a simple representation of physical systems and is modeled by mass, damping and stiffness. Stiffness is the resistance it provides to deformations, mass is the matter it contains and damping represents its ability to decrease its own motion with time.

Mass is a fundamental property of matter and is present in all physical systems. This is simply the weight of the structure divided by the acceleration due to gravity. Mass contributes an inertia force (equal to mass times acceleration) in the dynamic equation of motion.

Stiffness makes the structure more rigid, lessens the dynamic effects and makes it more dependent on static forces and displacements. Usually, structural systems are made stiffer by increasing the cross-sectional dimension, making the structures shorter or using stiffer materials.

Damping is often the least known of all the elements of a structural system. Whereas the mass and the stiffness are well-known properties and measured easily, damping is usually determined from experimental results or values assumed from experience. There are several sources of damping in a dynamic system. Viscous damping is the most used damping system and provides a force directly proportional to the structural velocity. This is a fair representation of structural damping in many cases and for the purpose of analysis, it is convenient to assume viscous damping (also known as linear viscous damping). Viscous damping is usually an intrinsic property of the material and originates from internal resistance to motion between different layers within the material itself. However, damping can also be due to friction between different materials or different parts of the structure (called frictional damping), drag between fluids or structures flowing past each other, etc. Sometimes, external forces themselves can contribute to (increase or decrease) the damping. Damping is also increased in structures artificially by external sources.

Free Vibration of Undamped Single-Degree-of-Freedom (SDOF) System

Formulation of the Single-Degree-of-Freedom (SDOF) Equation

A dynamic system resists external forces by a combination of forces due to its stiffness (spring force), damping (viscous force) and mass (inertia force). For the system shown in Fig. 2.1, k is the stiffness, c the viscous damping, m the mass and u(t) is the dynamic displacement due to the time-varying excitation force f(t). Such systems are called *Single-Degree-of-Freedom (SDOF) systems* because they have only one dynamic displacement [u(t) here].

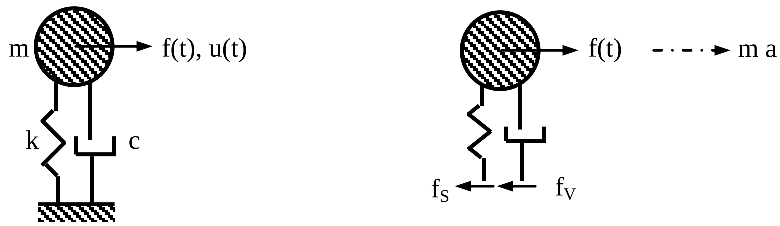


Fig. 2.1: Dynamic SDOF system subjected to dynamic force f(t)

Considering the free body diagram of the system, $f(t) - f_S - f_V = ma$ …………..(2.1)

where f_S = Spring force = Stiffness times the displacement = k u ………….(2.2)

f_V = Viscous force = Viscous damping times the velocity = c du/dt ………….(2.3)

f_I = Inertia force = Mass times the acceleration = $m\, d^2u/dt^2$ ………….(2.4)

Combining the equations (2.2)-(2.4) with (2.1), the equation of motion for a SDOF system is derived as,

$m\, d^2u/dt^2 + c\, du/dt + ku = f(t)$ ………….(2.5)

This is a 2nd order ordinary differential equation (ODE), which needs to be solved in order to obtain the dynamic displacement u(t). As will be shown subsequently, this can be done analytically or numerically.

Eq. (2.5) has several limitations; e.g., it is assumed on linear input-output relationship [constant spring (k) and dashpot (c)]. It is only a special case of the more general equation (2.1), which is an equilibrium equation and is valid for linear or nonlinear systems. Despite these, Eq. (2.5) has wide applications in Structural Dynamics. Several important derivations and conclusions in this field have been based on it.

Free Vibration of Undamped Systems

Free Vibration is the dynamic motion of a system without the application of external force; i.e., due to initial excitement causing displacement and velocity.

The equation of motion of a general dynamic system with m, c and k is,

$m\, d^2u/dt^2 + c\, du/dt + ku = f(t)$ ………….(2.5)

For free vibration, f(t) = 0; i.e., $m\, d^2u/dt^2 + c\, du/dt + ku = 0$

For undamped free vibration, $c = 0 \Rightarrow m\, d^2u/dt^2 + ku = 0 \Rightarrow d^2u/dt^2 + \omega_n^2 u = 0$ ………….(2.6)

where $\omega_n = \sqrt{(k/m)}$, is called the *natural frequency* of the system ………….(2.7)

Assume $u = e^{st}$, $d^2u/dt^2 = s^2e^{st} \Rightarrow s^2 e^{st} + \omega_n^2 e^{st} = 0 \Rightarrow s = \pm i\omega_n$

$\Rightarrow u(t) = A\, e^{i\omega n\, t} + B\, e^{-i\omega n\, t} = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$ ………….(2.8)

$\therefore v(t) = du/dt = -C_1\, \omega_n \sin(\omega_n t) + C_2\, \omega_n \cos(\omega_n t)$ ………….(2.9)

If $u(0) = u_0$ and $v(0) = v_0$, then $C_1 = u_0$ and $C_2\omega_n = v_0 \Rightarrow C_2 = v_0/\omega_n$ ………….(2.10)

$\therefore u(t) = u_0 \cos(\omega_n t) + (v_0/\omega_n) \sin(\omega_n t)$ ………….(2.11)

Natural Frequency and Natural Period of Vibration

Eq (2.11) implies that the system vibrates indefinitely with the same amplitude at a frequency of ω_n radian/sec. Here, ω_n is the angular rotation (radians) traversed by a dynamic system in unit time (one second). It is called the *natural frequency of the system (in radians/sec)*.

Alternatively, the number of cycles completed by a dynamic system in one second is also called its *natural frequency (in cycles/sec or Hertz)*. It is often denoted by f_n. $\therefore$ $f_n = \omega_n/2\pi$ …………(2.12)

The time taken by a dynamic system to complete one cycle of revolution is called its natural period (T_n). It is the inverse of natural frequency.

$$\therefore T_n = 1/f_n = 2\pi/\omega_n \qquad \text{…………..(2.13)}$$

Example 2.1

An undamped structural system with stiffness (k) = 25 k/ft and mass (m) = 1 k-sec^2/ft is subjected to an initial displacement (u_0) = 1 ft and an initial velocity (v_0) = 4 ft/sec.

(i) Calculate the natural frequency and natural period of the system.

(ii) Plot the free vibration of the system vs. time.

Solution

(i) For the system, natural frequency, $\omega_n = \sqrt{(k/m)} = \sqrt{(25/1)} = 5$ radian/sec

$\therefore$ $f_n = \omega_n/2\pi = 5/2\pi = 0.796$ cycle/sec

$\therefore$ Natural period, $T_n = 1/f_n = 1.257$ sec

(ii) The free vibration of the system is given by Eq (2.11) as

$u(t) = u_0 \cos(\omega_n t) + (v_0/\omega_n) \sin(\omega_n t) = (1) \cos(5t) + (4/5) \sin(5t) = (1) \cos(5t) + (0.8) \sin(5t)$

The maximum value of u(t) is = $\sqrt{(1^2 + 0.8^2)} = 1.281$ ft.

The plot of u(t) vs. t is shown below in Fig. 2.2.

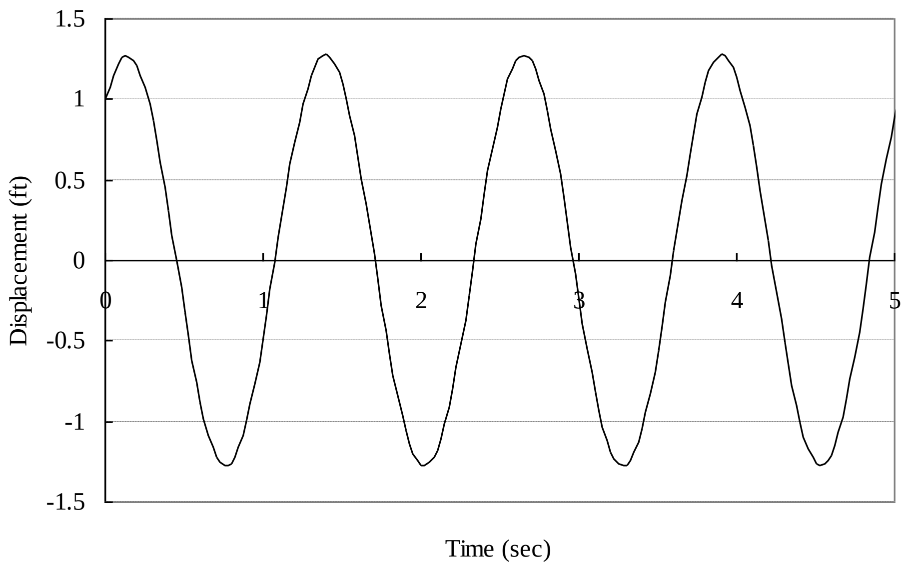


Fig. 2.2: Displacement vs. Time for Free Vibration of an Undamped System

Free Vibration of Damped Systems

As mentioned in the previous section, the equation of motion of a dynamic system with mass (m), linear viscous damping (c) & stiffness (k) undergoing free vibration is,

$m\, d^2u/dt^2 + c\, du/dt + ku = 0$ …………………(2.5)

$\Rightarrow d^2u/dt^2 + (c/m)\, du/dt + (k/m)\, u = 0 \Rightarrow d^2u/dt^2 + 2\xi\omega_n\, du/dt + \omega_n^2\, u = 0$ …………………(3.1)

where $\omega_n = \sqrt{(k/m)}$, is the *natural frequency* of the system …………………(2.7)

and $\xi = c/(2m\omega_n) = c\omega_n/(2k) = c/2\sqrt{(km)}$, is the *damping ratio* of the system …………………(3.2)

Assume $u = e^{st}$, $d^2u/dt^2 = s^2e^{st} \Rightarrow s^2\, e^{st} + 2\xi\omega_n\, s\, e^{st} + \omega_n^2 e^{st} = 0 \Rightarrow s = \omega_n\, (-\xi \pm \sqrt{(\xi^2-1)})$ …………………(3.3)

1. If $\xi>1$, the system is called an *overdamped system*. Here, the solution for s is a pair of different real numbers $[\omega_n(-\xi+\sqrt{(\xi^2-1)}),\ \omega_n(-\xi-\sqrt{(\xi^2-1)})]$. Such systems, however, are not very common. The displacement u(t) for such a system is

$u(t) = e^{-\xi\omega n\, t}\, (A\, e^{\omega 1\, t} + B\, e^{-\omega 1\, t})$ …………………(3.4)

where $\omega_1 = \omega_n\sqrt{(\xi^2-1)}$

2. If $\xi = 1$, the system is called a *critically damped system*. Here, the solution for s is a pair of identical real numbers $[-\omega_n, -\omega_n]$. Critically damped systems are rare and mainly of academic interest only.
The displacement u(t) for such a system is

$u(t) = e^{-\omega n\, t}\, (A + Bt)$ …………………(3.5)

3. If $\xi<1$, the system is called an *underdamped system*. Here, the solution for s is a pair of different complex numbers $[\omega_n(-\xi+i\sqrt{(1-\xi^2)}),\ \omega_n(-\xi-i\sqrt{(1-\xi^2)})]$.
Practically, most structural systems are underdamped.
The displacement u(t) for such a system is

$u(t) = e^{-\xi\omega nt}\, (A\, e^{i\omega d\, t} + B\, e^{-i\omega d\, t}) = e^{-\xi\omega nt}\, [C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)]$ …………………(3.6)

where $\omega_d = \omega_n\sqrt{(1-\xi^2)}$ is called the *damped natural frequency* of the system.

Since underdamped systems are the most common of all structural systems, the subsequent discussion will focus mainly on those. Differentiating Eq (3.6), the velocity of an underdamped system is obtained as

$v(t) = du/dt$

$= e^{-\xi\omega nt}\, [\omega_d\{-C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)\} - \xi\omega_n\{C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\}]$ …………………(3.7)

If $u(0) = u_0$ and $v(0) = v_0$, then

$C_1 = u_0$ and $\omega_d C_2 - \xi\omega_n C_1 = v_0 \Rightarrow C_2 = (v_0 + \xi\omega_n u_0)/\omega_d$ …………………(3.8)

$\therefore u(t) = e^{-\xi\omega nt}\, [u_0 \cos(\omega_d t) + \{(v_0 + \xi\omega_n u_0)/\omega_d\} \sin(\omega_d t)]$ …………………(3.9)

$\therefore$ Eq (3.9) $\Rightarrow$ The system vibrates at its damped natural frequency (i.e., a frequency of ω_d radian/sec). Since the damped natural frequency $\omega_d\ [= \omega_n\sqrt{(1-\xi^2)}]$ is less than ω_n, the system vibrates more slowly than the undamped system.

Moreover, due to the exponential term $e^{-\xi\omega nt}$, the amplitude of the motion of an underdamped system decreases steadily, and reaches zero after (a hypothetical) 'infinite' time of vibration.

Similar equations can be derived for critically damped and overdamped dynamic systems in terms of their initial displacement, velocity and damping ratio.

Example 3.1

A damped structural system with stiffness (k) = 25 k/ft and mass (m) = 1 k-sec^2/ft is subjected to an initial displacement (u_0) = 1ft and an initial velocity (v_0) = 4 ft/sec. Plot the free vibration of the system vs. time if the Damping Ratio (ξ) is

(a) 0.00 (undamped system),
(b) 0.05, (c) 0.50 (underdamped systems),
(d) 1.00 (critically damped system),
(e) 1.50 (overdamped system).

Solution

The equations for u(t) are plotted against time for various damping ratios (DR) and shown below in Fig. 3.1. The main features of these figures are

(1) The underdamped systems have sinusoidal variations of displacement with time. Their natural periods are lengthened (more apparent for ξ = 0.50) and maximum amplitudes of vibration reduced due to damping.

(2) The critically damped and overdamped systems have monotonic rather than harmonic (sinusoidal) variations of displacement with time. Their maximum amplitudes of vibration are less than the amplitudes of underdamped systems.

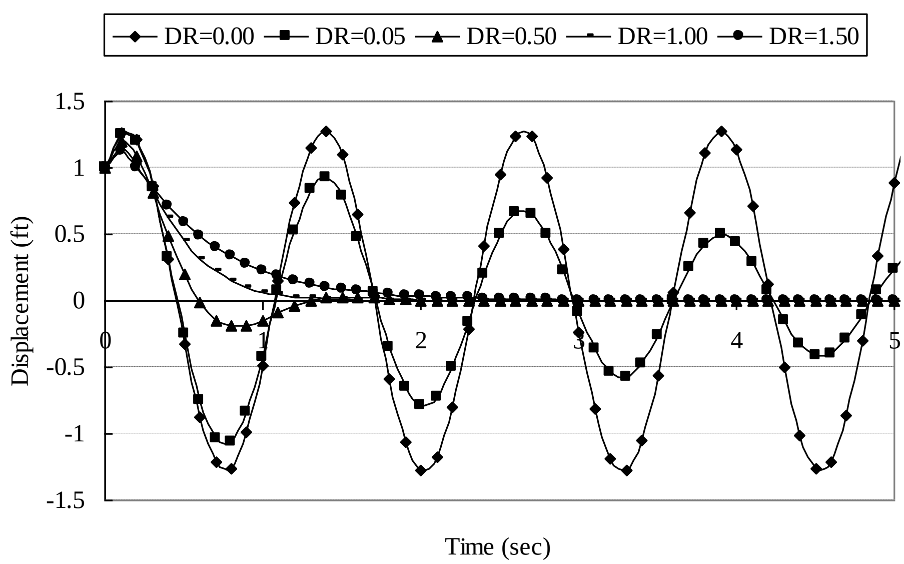


Fig. 3.1: Displacement vs. Time for free Vibration of Damped Systems

Damping of Structures

Damping is the element that causes impedance of motion in a structural system. There are several sources of damping in a dynamic system. Damping can be due to internal resistance to motion between layers, friction between different materials or different parts of the structure (called frictional damping), drag between fluids or structures flowing past each other, etc. Sometimes, external forces themselves can contribute to (increase or decrease) the damping. Damping is also increased in structures artificially by external sources like dampers acting as control systems.

Viscous Damping of SDOF systems

Viscous damping is the most used damping and provides a force directly proportional to the structural velocity. This is a fair representation of structural damping in many cases and for the purpose of analysis it is convenient to assume viscous damping (also known as linear viscous damping). Viscous damping is usually an intrinsic property of the material and originates from internal resistance to motion between different layers within the material itself.

While discussing different types of viscous damping, it was mentioned that underdamped systems are the most common of all structural systems. This discussion focuses mainly on underdamped SDOF systems, for which the free vibration response was found to be

$$u(t) = e^{-\xi\omega nt} [u_0 \cos(\omega_d t) + \{(v_0 + \xi\omega_n u_0)/\omega_d\} \sin(\omega_d t)] \qquad \text{………………..(3.9)}$$

Eq (3.9) $\Rightarrow$ The system vibrates at its damped natural frequency (i.e., a frequency of ω_d radian/sec). Since ω_d [$= \omega_n\sqrt{(1-\xi^2)}$] is less than ω_n, the system vibrates more slowly than the undamped system. Due to the exponential term $e^{-\xi\omega nt}$ the amplitude of motion decreases steadily and reaches zero after (a hypothetical) 'infinite' time of vibration.

However, the displacement at N time periods ($T_d = 2\pi/\omega_d$) later than u(t) is

$$u(t + NT_d) = e^{-\xi\omega n(t+2\pi N/\omega d)} [u_0 \cos(\omega_d t + 2\pi N) + \{(v_0 + \xi\omega_n u_0)/\omega_d\} \sin(\omega_d t + 2\pi N)]$$
$$= e^{-\xi\omega n(2\pi N/\omega d)}\, u(t) \qquad \text{....………….…(3.10)}$$

From which, using $\omega_d = \omega_n\sqrt{(1-\xi^2)} \Rightarrow \xi/\sqrt{(1-\xi^2)} = \ln[u(t)/u(t + NT_d)]/2\pi N = \delta$

$$\Rightarrow \xi = \delta/\sqrt{(1+\delta^2)} \qquad \text{……………...…(3.11)}$$

For lightly damped structures (i.e., $\xi<<1$), $\xi \cong \delta = \ln[u(t)/u(t + NT_d)]/2\pi N$ ……..…..……….(3.12)

For example, if the free vibration amplitude of a SDOF system decays from 1.5″ to 0.5″ in 3 cycles, the damping ratio, $\xi = \ln(1.5/0.5)/(2\pi\times3) = 0.0583 = 5.83\%$.

Table 3.1: Recommended Damping Ratios for different Structural Elements

Stress Level	Type and Condition of Structure	ξ (%)
Working stress	Welded steel, pre-stressed concrete, RCC with slight cracking	2-3
	RCC with considerable cracking	3-5
	Bolted/riveted steel or timber	5-7
Yield stress	Welded steel, pre-stressed concrete	2-3
	RCC	7-10
	Bolted/riveted steel or timber	10-15

Forced Vibration

The discussion has so far concentrated on free vibration, which is caused by initialization of displacement and/or velocity and without application of external force after the motion has been initiated. Therefore, free vibration is represented by putting $f(t) = 0$ in the dynamic equation of motion.

Forced vibration, on the other hand, is the dynamic motion caused by the application of external force (with or without initial displacement and velocity). Therefore, $f(t) \neq 0$ in the equation of motion for forced vibration. Rather, they have different equations for different variations of the applied force with time.

The equations for displacement for various types of applied force are now derived analytically for undamped and underdamped vibration systems. The following cases are studied

1. Step Loading; i.e., constant static load of p_0; i.e., $f(t) = p_0$, for $t > 0$

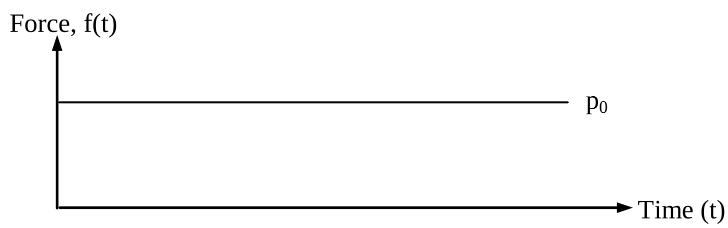


Fig. 4.1: Step Loading Function

2. Ramped Step Loading; i.e., load increasing linearly with time up to p_0 in time t_0 and remaining constant thereafter; i.e., $f(t) = p_0(t/t_0)$, for $0 < t < t_0$

$= p_0$, for $t > t_0$

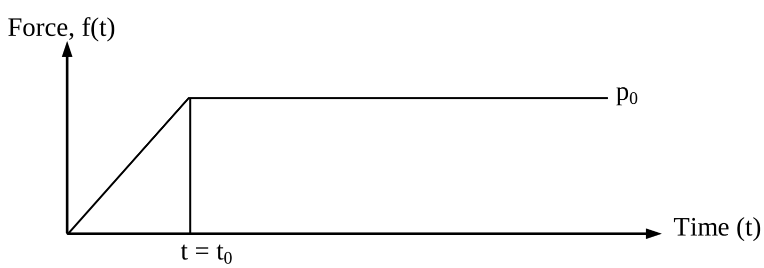


Fig. 4.2: The Ramped Step Loading Function

3. Harmonic Load; i.e., a sinusoidal load of amplitude p_0 and frequency ω; i.e., $f(t) = p_0 \cos(\omega t)$, for $t>0$

In all these cases, the dynamic system will be assumed to start from rest; i.e., initial displacement u(0) and velocity v(0) will both be assumed zero.

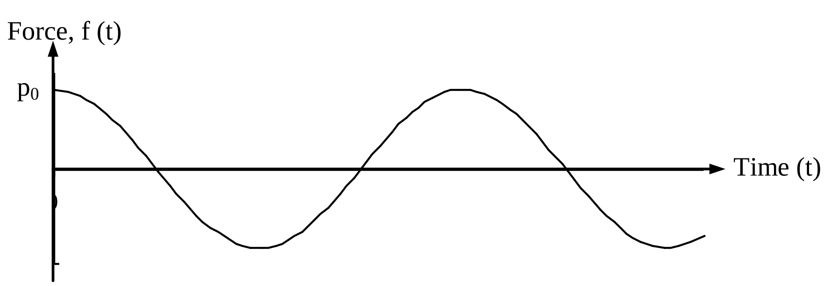


Fig. 4.3: The Harmonic Load Function

Case 1 - Step Loading:

For a constant static load of p_0, the equation of motion becomes

$$m\, d^2u/dt^2 + c\, du/dt + ku = p_0 \qquad \text{………………..(4.1)}$$

The solution of this differential equation consists of two parts; i.e., the general solution and the particular solution. The general solution assumes the excitation force to be zero and thus it will be the same as the free vibration solution (with two arbitrary constants). The particular solution of u(t) will satisfy Eq. 4.1. The total solution will be the summation of these two solutions.

The general solution for an underdamped system is (using Eq. 3.6)

$$u_g(t) = e^{-\xi\omega nt}\,[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] \qquad \text{…………………..(4.2)}$$

and the particular solution of Eq. 4.1 is $u_p(t) = p_0/k$ …………………..(4.3)

Combining the two, the total solution for u(t) is

$$u(t) = u_g(t) + u_p(t) = e^{-\xi\omega nt}\,[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] + p_0/k \qquad \text{…..……………...(4.4)}$$

$$\therefore v(t) = du/dt = e^{-\xi\omega nt}\,[\omega_d\{-C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)\} - \xi\omega_n\{C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\}] \qquad \text{……....(4.5)}$$

If initial displacement u(0) = 0 and initial velocity v(0) = 0, then

$$C_1 + p_0/k = 0 \Rightarrow C_1 = -\,p_0/k, \text{ and } \omega_d C_2 - \xi\omega_n C_1 = 0 \Rightarrow C_2 = -\,\xi\omega_n\,(p_0/k)/\omega_d \qquad \text{……...………...(4.6)}$$

$$\therefore \text{Eq. (4.4)} \Rightarrow u(t) = (p_0/k)[1 - e^{-\xi\omega nt}\{\cos(\omega_d t) + \xi\omega_n/\omega_d \sin(\omega_d t)\}] \qquad \text{...………...…...(4.7)}$$

For an undamped system, $\xi = 0$, $\omega_d = \omega_n \therefore u(t) = (p_0/k)[1 - \cos(\omega_n t)]$ …........………...(4.8)

Example 4.1

For the system mentioned in Examples 2.1 and 3.1 (i.e., k = 25 k/ft, m = 1 k-sec^2/ft), plot the displacement vs. time if a static load p_0 = 25 k is applied on the system if the Damping Ratio (ξ) is

(a) 0.00 (undamped system), 0.05, (c) 0.50 (underdamped systems).

Solution

In this case, the static displacement is = p_0/k = 25/25 = 1 ft. The dynamic solutions are obtained from Eq. 4.7 and plotted below in Fig. 4.4. The main features of these results are

(1) For Step Loading, the maximum dynamic response for an undamped system (i.e., 2 ft in this case) is twice the static response and continues indefinitely without converging to the static response.

(2) The maximum dynamic response for damped systems is between 1 and 2, and eventually converges to the static solution. The larger the damping ratio, the less the maximum response and the quicker it converges to the static solution. In general, the dynamic response converges to the particular solution of the dynamic equation of motion, and is therefore called the *steady state response*.

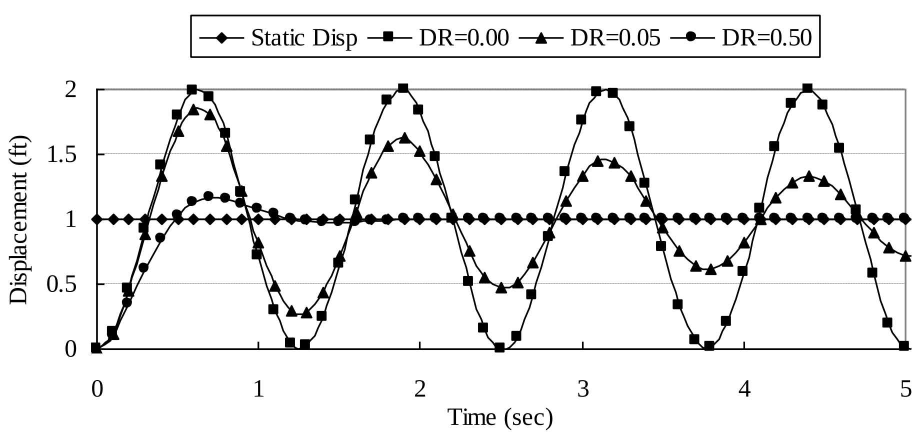


Fig. 4.4: Dynamic Response to Step Loading

Case 2 - Ramped Step Loading:
For a ramped step loading up to p_0 in time t_0, the equation of motion is

$$m\, d^2u/dt^2 + c\, du/dt + ku = p_0(t/t_0),\ \text{for } 0< t< t_0$$
$$= p_0,\ \text{for } t>t_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4.9)$$

The solution will be different (u1 and u2) for the two stages of loading. The loading in the first stage is a linearly varying function of time, while that of the second stage is a constant.

The general solution for an underdamped system has been shown in Eq. 3.6 and 4.2, while the particular solution is $u1_p(t) = (p_0/kt_0)\,(t - c/k)$ ……………………(4.10)

$$\therefore u1(t) = e^{-\xi\omega nt}\,[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] + (p_0/kt_0)(t - c/k) \qquad \ldots\ldots(4.11)$$
$$v1(t) = e^{-\xi\omega nt}[\omega_d\{-C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)\} - \xi\omega_n\{C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\}] + p_0/kt_0 \qquad \ldots\ldots(4.12)$$

If initial displacement $u1(0) = 0$, initial velocity $v1(0) = 0$, then $C_1 - (p_0/k)(c/kt_0) = 0 \Rightarrow C_1 = (p_0/k)(c/kt_0)$ and $\omega_d C_2 - \xi\omega_n C_1 + p_0/kt_0 = 0 \Rightarrow C_2 = (p_0/k)(\xi\omega_n c/k - 1)/(\omega_d\, t_0)$ ……………(4.13)

$$\therefore u1(t) = (p_0/kt_0)[(t - c/k) + e^{-\xi\omega nt}\{(c/k)\cos(\omega_d t) + (\xi\omega_n c/k - 1)/(\omega_d)\sin(\omega_d t)\}] \qquad \ldots\ldots(4.14)$$

$\therefore$ For an undamped system, $u1(t) = (p_0/k)\,[t/t_0 - \sin(\omega_n t)/(\omega_n t_0)]$ ……………(4.15)

The second stage of the solution may be considered as the difference between two ramped step functions, one beginning at $t = 0$ and another at $t = t_0$.

$$\therefore u2(t) = u1(t) - u1(t - t_0) = u1(t) - u1(t');\ \text{where } t' = t - t_0 \qquad \ldots\ldots(4.16)$$

$\therefore$ For undamped system, $u2(t) = (p_0/k)\,[1 - \{\sin(\omega_n t) - \sin(\omega_n t')\}/(\omega_n t_0)]$ ……………(4.17)

Example 4.2
For the system mentioned in Example 4.1, plot the displacement vs. time if a ramped step load with $p_0 =$ 25 k is applied on the system with $\xi = 0.00$ if t_0 is (a) 0.5 second, (b) 2 seconds.

Solution
In this case, the static displacement is $= p_0/k = 25/25 = 1$ ft. The dynamic solutions are obtained from Eqs. 4.14~4.17 and plotted in Fig. 4.5. The main features of these results are

(1) For Ramped Step Loading, the maximum dynamic response for an undamped system is less than the response due to Step Loading, which is twice as much as the static response (i.e., 2 ft in this case).

(2) The larger the ramp duration, the smaller the maximum dynamic response. Eventually the dynamic response takes the form of an oscillating sinusoid about the steady state (static in this case) response.

(3) The response for damped system is not shown here. However, the response for a damped system would be qualitatively similar for an undamped system and would eventually converge to the steady state solution.

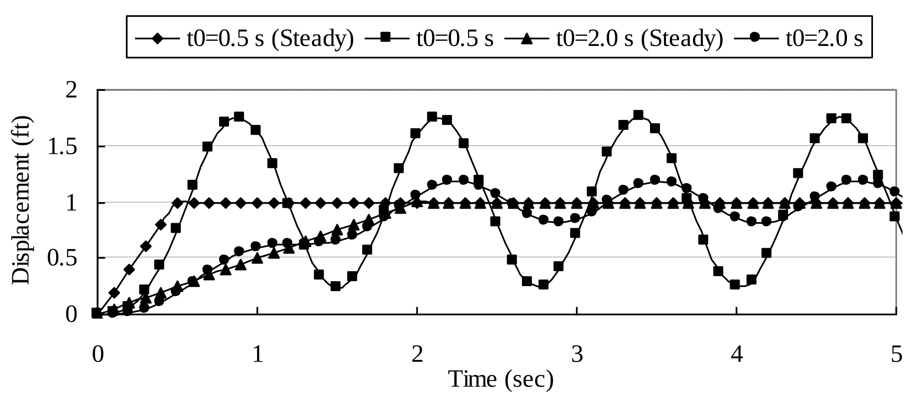


Fig. 4.5: Dynamic Response to Ramped Step Loading

Case 3 - Harmonic Loading:

For a harmonic load of amplitude p_0 and frequency ω, the equation of motion is

$$m\, d^2u/dt^2 + c\, du/dt + ku = p_0 \cos(\omega t), \text{ for } t>0 \qquad \text{.....………………(4.18)}$$

The general solution for an underdamped system has been shown before, and the particular solution

$$u_p(t) = [p_0/\sqrt{\{(k-\omega^2 m)^2+(\omega c)^2\}}] \cos(\omega t-\phi) = (p_0/k_d) \cos(\omega t-\phi) \qquad \text{………………...…..(4.19)}$$

$$[\text{where } k_d = \sqrt{\{(k-\omega^2 m)^2 + (\omega c)^2\}},\ \phi = \tan^{-1}\{(\omega c)/(k - \omega^2 m)\}]$$

$$\therefore u(t) = e^{-\xi\omega nt} [C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] + (p_0/k_d) \cos(\omega t-\phi) \qquad \text{…...………......…(4.20)}$$

$$v(t) = e^{-\xi\omega nt}[\omega_d\{-C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)\}-\xi\omega_n\{C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\}] - (p_0\omega/k_d) \sin(\omega t-\phi)$$
$$\text{…...………......….(4.21)}$$

If initial displacement u(0) = 0 and initial velocity v(0) = 0, then

$$C_1 + (p_0/k_d) \cos(\phi) = 0 \Rightarrow C_1 = -(p_0/k_d) \cos\phi$$

$$\text{and } \omega_d C_2 - \xi\omega_n C_1 + (p_0\omega/k_d) \sin\phi = 0 \Rightarrow C_2 = -(p_0/k_d) (\omega \sin\phi + \xi\omega_n \cos\phi)/\omega_d \qquad \text{…………(4.22)}$$

$$\therefore u(t) = (p_0/k_d)[\cos(\omega t-\phi) - e^{-\xi\omega nt}\{\cos\phi \cos(\omega_d t) + (\omega/\omega_d \sin\phi + \xi\omega_n/\omega_d \cos\phi) \sin(\omega_d t)\}] \text{ ...…(4.23)}$$

$$\therefore \text{For an undamped system, } u(t) = [p_0/(k - \omega^2 m)] [\cos(\omega t) - \cos(\omega_n t)] \qquad \text{……....………..(4.24)}$$

Example 4.3

For the system mentioned in previous examples, plot the displacement vs. time if a harmonic load with p_0 = 25 k is applied on the system with ξ = 0.05 and 0.00, if ω is (a) 2.0, (d) 5.0, (e) 10.0 radian/sec.

Solution

The variation of load f(t) with time in shown in Fig. 4.3. The solutions for u(t) are obtained from Eqs. 4.23 and 4.24 and are plotted in Figs. 4.6~4.8. The main features of these results are

(1) The responses for undamped systems are larger than the damped responses. This is true in general for all three loading cases.

(2) The responses for the second loading case are larger than the other two, because the frequency of the load is equal to the natural frequency of the system. As will be explained later, this is the ***resonant condition***. At resonance, the damped response reaches a maximum amplitude (the steady state amplitude) and oscillates with that amplitude subsequently. This amplitude is 10 ft, which the damped system would eventually reach if it were allowed to vibrate 'long enough'. The amplitude of the undamped system, on the other hand, increases steadily with time and would eventually reach infinity.

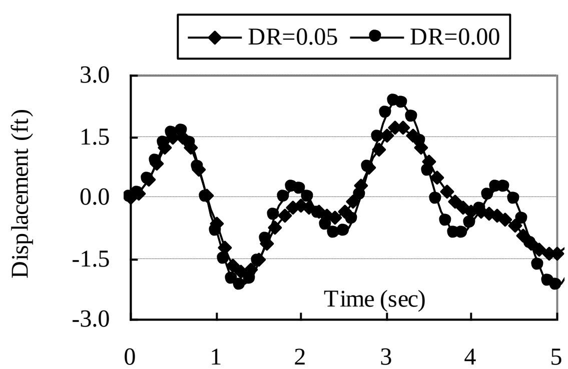


Fig. 4.6: Dynamic Response to 25 Cos(2t)

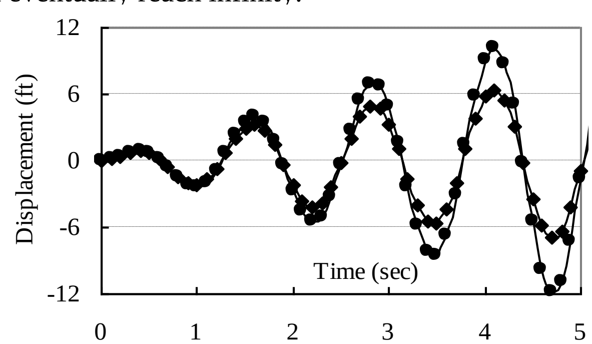


Fig. 4.7: Dynamic Response to 25 Cos(5t)

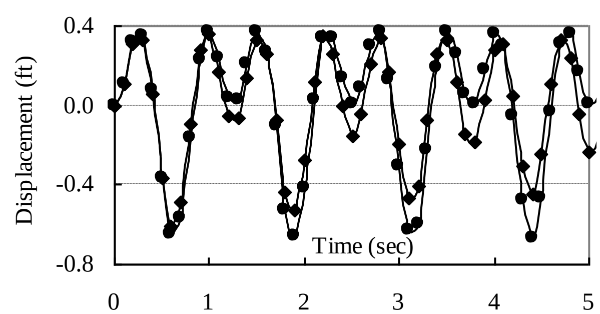


Fig. 4.8: Dynamic Response to 25 Cos(10t)

Dynamic Magnification

In Section 4, it was observed that the maximum dynamic displacements are different from their static counterparts. However, the effect of this magnification (increase or decrease) was more apparent in the harmonic loading case. There, for sinusoidal loads (cosine functions of time) of the same amplitude (25 k) the maximum vibrations varied between 0.5 ft to 12 ft depending on the frequency of the harmonic load.

If the motion is allowed to continue for long (theoretically infinite) durations, the total response converges to the steady state solution given by the particular solution of the equation of motion.

$$\therefore u_{steady}(t) = u_p(t) = (p_0/k_d)\cos(\omega t-\phi) \qquad \text{……………..….(4.19)}$$

[where $k_d = \sqrt{\{(k-\omega^2 m)^2 + (\omega c)^2\}}$, $\phi = \tan^{-1}\{(\omega c)/(k-\omega^2 m)\}$]

$\therefore$ Putting the value of k_d in Eq. (4.19), the amplitude of steady vibration is found to be

$$\therefore u_{amplitude} = p_0/k_d = p_0/\sqrt{\{(k-\omega^2 m)^2 + (\omega c)^2\}} \qquad \text{……………..….(5.1)}$$

This can be written as, $u_{amplitude} = (p_0/k)/\sqrt{\{(1-\omega^2 m/k)^2 + (\omega c/k)^2\}}$

Using $u_{static} = p_0/k$, $m/k = 1/\omega_n^2$, $c/k = 2\xi/\omega_n \Rightarrow u_{amplitude}/u_{static} = 1/\sqrt{\{(1-\omega^2/\omega_n^2)^2 + (2\xi\omega/\omega_n)^2\}}$ ………(5.2)

Eq. (5.2) gives the ratio of the dynamic and static amplitude of motion as a function of frequency ω (as well as structural properties like ω_n and ξ). This ratio is called the steady state dynamic magnification factor (DMF) for harmonic motion. Putting the frequency ratio $\omega/\omega_n = r$, Eq. (6.2) can be rewritten as

$$DMF = 1/\sqrt{\{(1-r^2)^2 + (2\xi r)^2\}} \qquad \text{..…………………(5.3)}$$

From which the maximum value of DMF is found to be $= (1/2\xi)/\sqrt{(1-\xi^2)}$, when $r = \sqrt{(1-2\xi^2)}$ ……....(5.4)

The variation of the steady state dynamic magnification factor (DMF) with frequency ratio ($r = \omega/\omega_n$) is shown in Fig. 5.1 for different values of ξ (= DR). The main features of this graph are

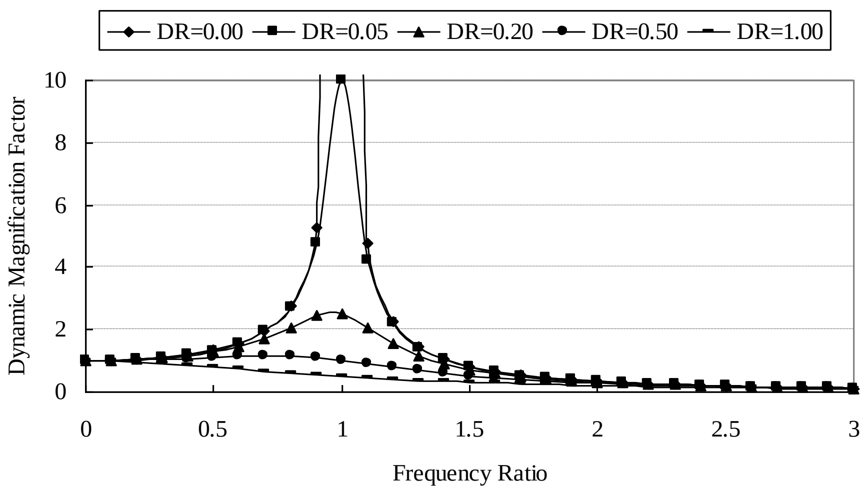


Fig. 5.1: Steady State Dynamic Magnification Factor

1. The curves for the smaller values of ξ show pronounced peaks [$\cong 1/(2\xi)$] at $\omega/\omega_n \cong 1$. This situation is called *resonance* and is characterized by large dynamic amplifications of motion. This situation can be derived from Eq. (5.4), where $\xi<<1 \Rightarrow DMF_{max} \cong 1/\sqrt{(4\xi^2)} = (1/2\xi)$, when $r \cong \sqrt{(1-2\xi^2)} = 1$.
2. For undamped system, the resonant peak is infinity, which is consistent with the earlier conclusion from Section 4 that and vibration amplitude of an undamped system tends steadily to infinity.
3. Since resonance is such a critical condition from structural point of view, it should be avoided in practical structures by making it either very stiff (i.e., r<<1) or very flexible (i.e., r>>1) with respect to the frequency of the expected harmonic load.
4. The resonant condition mentioned in (1) is not applicable for large values of ξ, because the condition of maxima at $r = \sqrt{(1-2\xi^2)}$ is meaningless if r is imaginary; i.e., $\xi>(1/\sqrt{2} =)\ 0.707$. Therefore, another way of avoiding the critical effects of resonance is by increasing the damping of the system.

Numerical Solution of SDOF Equation

So far the equation of motion for a SDOF system has been solved analytically for different loading functions. For mathematical convenience, the dynamic loads have been limited to simple functions of time and the initial conditions had been set equal to zero. Even if the assumptions of linear structural properties and initial 'at rest' conditions are satisfied; the practical loading situations can be more complicated and not convenient to solve analytically. Numerical methods must be used in such situations.

The most widely used numerical approach for solving dynamic problems is the ***Newmark-β method***. Actually, it is a set of solution methods with different physical interpretations for different values of β. The total simulation time is divided into a number of intervals (usually of equal duration Δt) and the unknown displacement (as well as velocity and acceleration) is solved at each instant of time. The method solves the dynamic equation of motion in the $(i + 1)^{th}$ time step based on the results of the i^{th} step.

The equation of motion for the $(i +1)^{th}$ time step is

$$m\,(d^2u/dt^2)_{i+1} + c\,(du/dt)_{i+1} + k\,(u)_{i+1} = f_{i+1} \Rightarrow m\,a_{i+1} + c\,v_{i+1} + k\,u_{i+1} = f_{i+1} \qquad \text{.........(6.1)}$$

where 'a' stands for the acceleration, 'v' for velocity and 'u' for displacement.

To solve for the displacement or acceleration at the $(i + 1)^{th}$ time step, the following equations are assumed for the velocity and displacement at the $(i + 1)^{th}$ step in terms of the values at the i^{th} step.

$$v_{i+1} = v_i + \{(1-\alpha)\,a_i + \alpha\,a_{i+1}\}\Delta t \qquad \text{.............................(6.2)}$$

$$u_{i+1} = u_i + v_i\,\Delta t + \{(0.5-\beta)\,a_i + \beta\,a_{i+1}\}\Delta t^2 \qquad \text{.............................(6.3)}$$

By putting the value of v_{i+1} from Eq. (6.2) and u_{i+1} from Eq. (6.3) in Eq. (6.1), the only unknown variable a_{i+1} can be solved from Eq. (6.1).

In the solution set suggested by the Newmark-β method, the ***Constant Average Acceleration (CAA) method*** is the most popular because of the stability of its solutions and the simple physical interpretations it provides. This method assumes the acceleration to remain constant during each small time interval Δt, and this constant is assumed to be the average of the accelerations at the two instants of time t_i and t_{i+1}. The CAA is a special case of Newmark-β method where $\alpha = 0.50$ and $\beta = 0.25$. Thus in the CAA method, the equations for velocity and displacement [Eqs. (6.2) and (6.3)] become

$$v_{i+1} = v_i + (a_i + a_{i+1})\Delta t/2 \qquad \text{...............(6.4)}$$

$$u_{i+1} = u_i + v_i\,\Delta t + (a_i + a_{i+1})\Delta t^2/4 \qquad \text{...............(6.5)}$$

Inserting these values in Eq. (6.1) and rearranging the coefficients, the following equation is obtained,

$$(m + c\Delta t/2 + k\Delta t^2/4)a_{i+1} = f_{i+1} - ku_i - (c + k\Delta t)v_i - (c\Delta t/2 + k\Delta t^2/4)a_i \qquad \text{............(6.6)}$$

To obtain the acceleration a_{i+1} at an instant of time t_{i+1} using Eq. (6.6), the values of u_i, v_i and a_i at the previous instant t_i have to be known (or calculated) before. Once a_{i+1} is obtained, Eqs. (6.4) and (6.5) can be used to calculate the velocity v_{i+1} and displacement u_{i+1} at time t_{i+1}. All these values can be used to obtain the results at time t_{i+2}. The method can be used for subsequent time-steps also.

The simulation should start with two initial conditions, like the displacement u_0 and velocity v_0 at time $t_0 = 0$. The initial acceleration can be obtained from the equation of motion at time $t_0 = 0$ as

$$a_0 = (f_0 - cv_0 - ku_0)/m \qquad \text{...............(6.7)}$$

Example 6.1

For the undamped SDOF system described before (m = 1 k-sec^2/ft, k = 25 k/ft, c = 0 k-sec/ft), calculate the dynamic response for a Ramped Step Loading with p0 = 25 k and t0 = 0.5 sec.

Results using the CAA Method (for time interval Δt = 0.05 sec) as well as the exact analytical equation are shown below in tabular form.

Table 6.1: Acceleration, Velocity and Displacement for Δt = 0.05 sec

m (k-sec^2/ft)	c (k-sec/ft)	k (k/ft)	t_0 (sec)	dt (sec)	m_{eff} (k-sec^2/ft)	c_{eff} (k-sec/ft)	m_1 (k-sec^2/ft)
1.00	0.00	25.00	0.50	0.05	1.0156	1.2500	0.0156

i	t (sec)	f_i (kips)	a_i (ft/sec^2)	v_i (ft/sec)	u_i (ft)	u_{ex} (ft)
0	0.00	0.0	0.0000	0.0000	0.0000	0.0000
1	0.05	2.5	2.4615	0.0615	0.0015	0.0010
2	0.10	5.0	4.7716	0.2424	0.0091	0.0082
3	0.15	7.5	6.7880	0.5314	0.0285	0.0273
4	0.20	10.0	8.3867	0.9107	0.0645	0.0634
5	0.25	12.5	9.4693	1.3571	0.1212	0.1204
6	0.30	15.0	9.9692	1.8431	0.2012	0.2010
7	0.35	17.5	9.8556	2.3387	0.3058	0.3064
8	0.40	20.0	9.1354	2.8135	0.4346	0.4363
9	0.45	22.5	7.8531	3.2382	0.5859	0.5888
10	0.50	25.0	6.0876	3.5867	0.7565	0.7606
11	0.55	25.0	1.4858	3.7760	0.9406	0.9463
12	0.60	25.0	-3.2073	3.7330	1.1283	1.1353
13	0.65	25.0	-7.7031	3.4603	1.3081	1.3159
14	0.70	25.0	-11.7249	2.9746	1.4690	1.4769
15	0.75	25.0	-15.0251	2.3058	1.6010	1.6082
16	0.80	25.0	-17.4007	1.4952	1.6960	1.7017
17	0.85	25.0	-18.7055	0.5925	1.7482	1.7516
18	0.90	25.0	-18.8592	-0.3466	1.7544	1.7547
19	0.95	25.0	-17.8523	-1.2644	1.7141	1.7109
20	1.00	25.0	-15.7468	-2.1044	1.6299	1.6230
21	1.05	25.0	-12.6723	-2.8149	1.5069	1.4962
22	1.10	25.0	-8.8179	-3.3521	1.3527	1.3387
23	1.15	25.0	-4.4209	-3.6831	1.1768	1.1600
24	1.20	25.0	0.2481	-3.7874	0.9901	0.9715
25	1.25	25.0	4.9019	-3.6586	0.8039	0.7846
26	1.30	25.0	9.2540	-3.3048	0.6298	0.6112
27	1.35	25.0	13.0367	-2.7475	0.4785	0.4620
28	1.40	25.0	16.0171	-2.0211	0.3593	0.3462
29	1.45	25.0	18.0118	-1.1704	0.2795	0.2711
30	1.50	25.0	18.8981	-0.2477	0.2441	0.2412
31	1.55	25.0	18.6214	0.6903	0.2551	0.2586
32	1.60	25.0	17.1989	1.5858	0.3120	0.3220
33	1.65	25.0	14.7179	2.3837	0.4113	0.4276
34	1.70	25.0	11.3312	3.0350	0.5468	0.5688
35	1.75	25.0	7.2472	3.4994	0.7101	0.7368
36	1.80	25.0	2.7172	3.7485	0.8913	0.9212
37	1.85	25.0	-1.9800	3.7670	1.0792	1.1105
38	1.90	25.0	-6.5553	3.5536	1.2622	1.2929
39	1.95	25.0	-10.7273	3.1215	1.4291	1.4570
40	2.00	25.0	-14.2391	2.4974	1.5696	1.5928

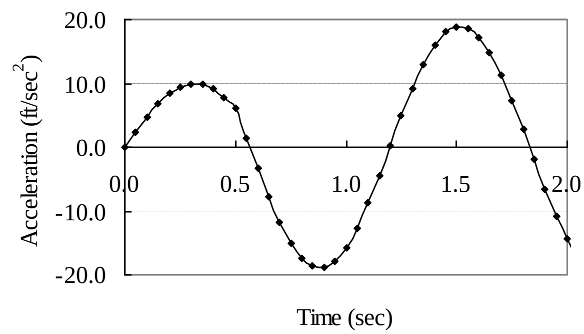


Fig. 6.1: Acceleration vs. Time

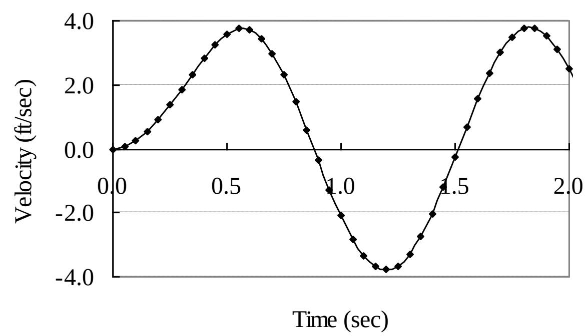


Fig. 6.2: Velocity vs. Time

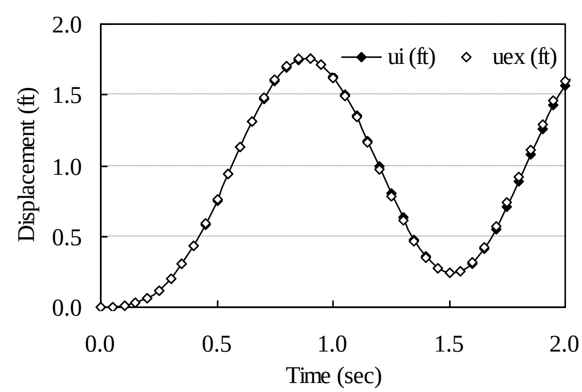
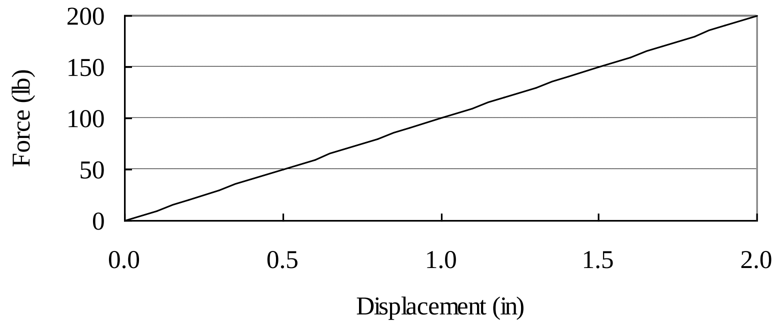


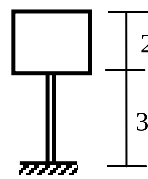
Fig. 6.3: Displacement vs. Time

Problems on the Dynamic Analysis of SDOF Systems

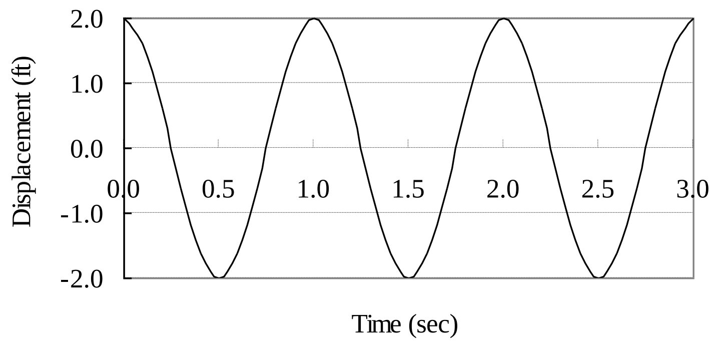
1. The force vs. displacement relationship of a spring is shown below. If the spring weighs 10 lb, calculate its natural frequency and natural period of vibration. If the damping ratio of the spring is 5%, calculate its damping (c, in lb-sec/in).



2. For the (20′ × 20′ × 20′) overhead water tank shown below supported by a 25″ × 25″ square column, calculate the undamped natural frequency for (i) horizontal vibration ($k = 3EI/L^3$), (ii) vertical vibration ($k = EA/L$). Assume the total weight of the system to be concentrated in the tank [Given: Modulus of elasticity of concrete = 400×10^3 k/ft^2, Unit weight of water = 62.5 lb/ft^3].

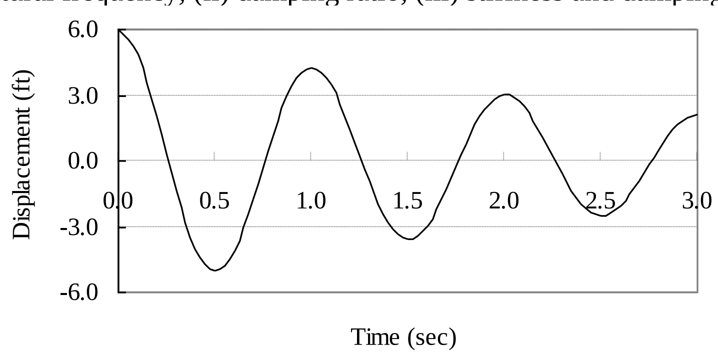


3. The free vibration of an undamped system is shown below. Calculate its (i) undamped natural period, (ii) undamped natural frequency in Hz and radian/second, (iii) stiffness if its mass is 2 lb-sec^2/ft.

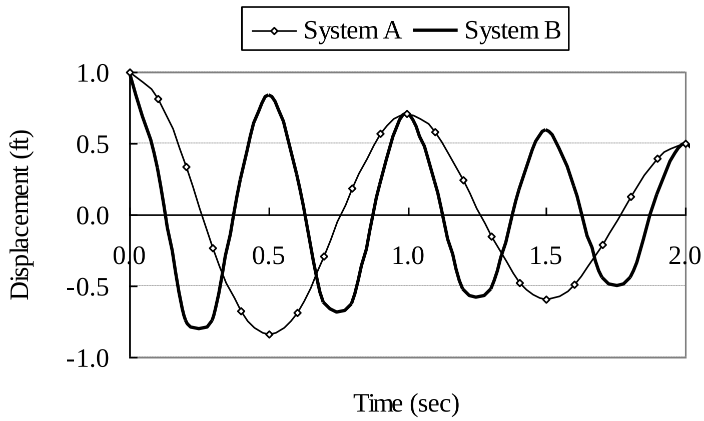


4. If a linear viscous damper 1.5 lb-sec/ft is added to the system described in Question 3, calculate its (i) damping ratio, (ii) damped natural period, (ii) free vibration at t = 2 seconds [Initial velocity = 0].

5. The free vibration response of a SDOF system is shown in the figure below. Calculate its (i) damped natural frequency, (ii) damping ratio, (iii) stiffness and damping if its weight is 10 lb.



6. The free vibration responses of two underdamped systems (A and B) are shown below.
   (i) Calculate the undamped natural frequency and damping ratio of system B.
   (ii) Explain (qualitatively) which one is stiffer and which one is more damped of the two systems.



7. A SDOF system with k = 10 k/ft, m = 1 k-sec^2/ft, c = 0 is subjected to a force (in kips) given by
   (i) p(t) = 50, (ii) p(t) = 100 t, (iii) p(t) = 50 cos(3t).
   In each case, calculate the displacement (u) of the system at time t = 0.1 seconds, if the initial displacement and velocity are both zero.

8. Calculate the maximum displacement of the water tank described in Problem 2 when subjected to
   (i) a sustained wind pressure of 40 psf, (ii) a harmonic wind pressure of 40 cos(2t) psf.

9. An undamped SDOF system suffers resonant vibration when subjected to a harmonic load (i.e., of frequency $\omega = \omega_n$). Of the control measures suggested below, explain which one will minimize the steady-state vibration amplitude.
   (i) Doubling the structural stiffness, (ii) Doubling the structural stiffness and the mass,
   (iii) Adding a damper to make the structural damping ratio = 10%.

10. For the system defined in Question 7, calculate u(0.1) in each case using the CAA method.

Solution of Problems on the Dynamic Analysis of SDOF Systems

1. From the force vs. displacement relationship, spring stiffness $k = 200/2.0 = 100$ lb/in
   Weight of the spring is $W = 10$ lb $\Rightarrow$ Mass $m = 10/(32.2 \times 12) = 0.0259$ lb-sec^2/in
   $\therefore$ Natural frequency, $\omega_n = \sqrt{(k/m)} = \sqrt{(100/0.0259)} = 62.16$ rad/sec $\Rightarrow f_n = \omega_n/2\pi = 9.89$ Hz
   $\therefore$ Natural period, $T_n = 1/f_n = 0.101$ sec
   Damping ratio, $\xi = 5\% = 0.05$
   $\Rightarrow$ Damping, $c = 2\xi\sqrt{(km)} = 2 \times 0.05\ \sqrt{(100 \times 0.0259)} = 0.161$ lb-sec/in

2. Mass of the tank (filled with water), $m = 20 \times 20 \times 20 \times 62.5/32.2 = 15528$ lb-ft/sec^2
   Modulus of elasticity $E = 400 \times 10^3$ k/ft^2 $= 400 \times 10^6$ lb/ft^2, Length of column $L = 30$ ft
   (i) Moment of inertia, $I = (25/12)^4/12 = 1.570$ ft^4
   Stiffness for horizontal vibration, $k_h = 3EI/L^3 = 3 \times 400 \times 10^6 \times 1.570/(30)^3 = 69770$ lb/ft
   Natural frequency, $\omega_{nh} = \sqrt{(k_h/m)} = \sqrt{(69770/15528)} = 2.120$ rad/sec
   (ii) Area, $A = (25/12)^2 = 4.340$ ft^2
   Stiffness for vertical vibration, $k_v = EA/L = 400 \times 10^6 \times 4.340/30 = 5787 \times 10^4$ lb/ft
   Natural frequency, $\omega_{nv} = \sqrt{(k_v/m)} = \sqrt{(5787 \times 10^4/15528)} = 61.05$ rad/sec

3. (i) The same displacement (2 ft) is reached after 1.0 second intervals.
   $\therefore$ Undamped natural period, $T_n = 1.0$ sec
   (ii) Undamped natural frequency, $f_n = 1/T_n = 1.0$ Hz $\Rightarrow \omega_n = 2\pi f_n = 6.283$ radian/second
   (iii) Mass, $m = 2$ lb-sec^2/ft.
   $\therefore$ Stiffness, $k = m\omega_n^2 = 78.96$ rad/sec

4. If a linear viscous damper, $c = 1.5$ lb-sec/ft, (i) damping ratio, (ii) damped natural period, (iii) free vibration at $t = 2$ seconds [Initial velocity = 0].
   (i) Damping ratio, $\xi = c/[2\sqrt{(km)}] = 1.5/[2\sqrt{(78.96 \times 2)}] = 0.0597 = 5.97\%$
   (ii) Damped natural period, $T_d = T_n/\sqrt{(1-\xi^2)} = 1.0/\sqrt{(1-0.0597^2)} = 1.002$ sec
   (iii) Damped natural frequency, $\omega_d = 2\pi/T_d = 6.272$ rad/sec
   $u(t) = e^{-\xi\omega nt}\,[u_0 \cos(\omega_d t) + \{(v_0 + \xi\omega_n u_0)/\omega_d\} \sin(\omega_d t)]$
   $= e^{-0.0597 \times 6.283 \times 2}\,[2\ \cos(6.272 \times 2) + \{(0 + 0.0597 \times 6.283 \times 2)/6.272\} \sin(6.272 \times 2)]$
   $= 0.943$ ft

5. (i) The figure shows that the peak displacement is repeated in every 1.0 second
   $\therefore$ Damped natural period, $T_d = 1.0$ sec
   $\therefore$ Damped natural frequency, $\omega_d = 2\pi/T_d = 6.283$ rad/sec
   (ii) Damping ratio, $\xi = \delta/\sqrt{(1+\delta^2)}$; where $\delta = \ln[u(t)/u(t + NT_d)]/2\pi N$
   Using as reference the displacements at $t = 0$ (6.0 ft) and $t = 2.0$ sec (3.0 ft); i.e., for $N = 2$
   $\delta = \ln[u(0.0)/u(2.0)]/(2\pi\times2) = \ln[6.0/3.0]/4\pi = 0.0552 \Rightarrow \xi = \delta/\sqrt{(1+\delta^2)} = 0.0551$
   (iii) Weight, $W = 10$ lb $\Rightarrow$ Mass, $m = 10/32.2 = 0.311$ lb-sec^2/ft
   Undamped natural frequency, $\omega_n = \omega_d/\sqrt{(1-\xi^2)} = 6.283/\sqrt{(1-0.0551^2)} = 6.293$ rad/sec
   $\therefore$ Stiffness, $k = m\omega_n^2 = 0.311 \times 6.293^2 = 12.30$ k/ft
   and Damping, $c = 2\xi\sqrt{(km)} = 2 \times 0.0515\ \sqrt{(12.30 \times 0.311)} = 0.215$ lb-sec/ft

6. (i) System B takes 1.0 second to complete two cycles of vibration.
   $\therefore$ Damped natural period T_d for system B $= 1.0/2 = 0.50$ sec
   $\therefore$ Damped natural frequency, $\omega_d = 2\pi/T_d = 12.566$ rad/sec
   Using as reference the displacements at $t = 0$ (1.0 ft) and $t = 2.0$ sec (0.5 ft); i.e., for $N = 4$
   $\delta = \ln[u(0.0)/u(2.0)]/(2\pi\times4) = \ln[1.0/0.5]/8\pi = 0.0276 \Rightarrow \xi = \delta/\sqrt{(1+\delta^2)} = 0.0276$
   Undamped natural frequency, $\omega_n = \omega_d/\sqrt{(1-\xi^2)} = 12.566/\sqrt{(1-0.0276^2)} = 12.571$ rad/sec

(ii) System A completes only two vibrations while (in 2.0 sec) system B completes four vibrations.

$\therefore$ System B is stiffer.

However, system A decays by the same ratio (i.e., 0.50 or 50%) in two vibrations system B decays in four vibrations.

$\therefore$ System A is more damped.

7. For the SDOF system, k = 10 k/ft, m = 1 k-sec^2/ft, c = 0

$\therefore$ Natural frequency, $\omega_n = \sqrt{(k/m)} = \sqrt{(10/1)} = 3.162$ rad/sec

(i) $p(t) = p_0 = 50$

For an undamped system, $u(t) = (p_0/k)[1-\cos(\omega_n t)]$

$\Rightarrow u(0.1) = (50/10)[1-\cos(3.162 \times 0.1)] = 0.248$ ft

(ii) p(t) = 100 t

For an undamped system, $u(t) = (p_0/k)\,[t/t_0 - \sin(\omega_n t)/(\omega_n t_0)] = (p_0/t_0)/k\,[t - \sin(\omega_n t)/\omega_n]$

$\Rightarrow u(0.1) = (100)/10\,[0.1 - \sin(3.162 \times 0.1)/3.162] = 0.0166$ ft

(iii) p(t) = 50 cos(3t)

For an undamped system, $u(t) = [p_0/(k-\omega^2 m)]\,[\cos(\omega t) - \cos(\omega_n t)]$

$\Rightarrow u(0.1) = (50)/(10 - 3^2 \times 1)\,[\cos(3 \times 0.1) - \cos(3.162 \times 0.1)] = 0.246$ ft

8. For the water tank filled with water,

Mass, m = 15528 lb-ft/sec^2, Stiffness for horizontal (i.e., due to wind) vibration, k_h = 69770 lb/ft

Natural frequency, ω_{nh} = 2.120 rad/sec

(i) $p(t) = p_0 = 40 \times 20 \times 20 = 16000$ lb

For an undamped system, $u_{max} = 2(p_0/k_h) = 2\,(16000/69770) = 0.459$ ft

(ii) $p(t) = p_0 \cos(\omega t) = 16000 \cos(2t)$

For an undamped system, $u(t) = [p_0/(k-\omega^2 m)]\,[\cos(\omega t)-\cos(\omega_n t)]$

$= [p_0/(k-\omega^2 m)]\,[\cos(2t)-\cos(2.120t)]$

$\therefore$ Possible $u_{max} \cong p_0/(k_h-\omega^2 m)\,[1-(-1)] \cong 2p_0/(k_h-\omega^2 m)$ [when $t = \pi$]

$= 2 \times 16000/(69770-2^2 \times 15528) = 4.178$ ft

9. Maximum dynamic response amplitude, $u_{max} = p_0/(k-\omega^2 m)$

If $\omega = \omega_n$, $u_{max} = p_0/(k-\omega_n^2 m) = p_0/(k-k) = \propto$

(i) Doubling the structural stiffness $\Rightarrow u_{max} = p_0/(2k-k) = p_0/k$

(ii) Doubling the structural stiffness and the mass $\Rightarrow u_{max} = p_0/(2k-\omega_n^2\, 2m) = p_0/(2k-2k) = \propto$

(iii) Adding a damper to make the structural damping ratio, ξ = 10% = 0.10

$\Rightarrow u_{max} = p_0/\sqrt{\{(k-\omega_n^2 m)^2 + (\omega_n c)^2\}} = (p_0/k)/(2\xi) = 5\,(p_0/k)$

$\therefore$ Option (i) is the most effective [since it minimizes u_{max}].

10. Using k = 10 k/ft, m = 1 k-sec^2/ft, c = 0, Δt = 0.1 sec, $u_0 = u(0) = 0$, $v_0 = v(0) = 0$, also $f_0 = f(0)$, $f_1 = f(0.1)$, $a_0 = a(0)$, $u_1 = u(0.1)$, $v_1 = v(0.1)$, $a_1 = a(0.1)$, the basic equations of the CAA method become

$a_0 = f_0$ …………… using (6.7)

$(1.025)\, a_1 = f_1 - 0.025\, a_0$ …………….. using (6.6)

$u_1 = 0.0025\,(a_0 + a_1)$ …………… using (6.5)

(i) $a_0 = f_0 = 50$ ft/sec^2

$(1.025)\, a_1 = 50 - 0.025 \times 50 \Rightarrow a_1 = 47.561$ ft/sec^2

$u_1 = 0.0025\,(50 + 47.561) = 0.244$ ft

(ii) $a_0 = f_0 = 0$ ft/sec^2

$(1.025)\, a_1 = 100 \times 0.1 - 0.025 \times 0 \Rightarrow a_1 = 9.756$ ft/sec^2

$u_1 = 0.0025\,(0 + 9.756) = 0.0244$ ft

(iii) $a_0 = f_0 = 50$ ft/sec^2

$(1.025)\, a_1 = 50 \cos(3 \times 0.1) - 0.025 \times 50 \Rightarrow a_1 = 45.382$ ft/sec^2

$u_1 = 0.0025\,(50 + 45.382) = 0.238$ ft

Computer Implementation of Numerical Solution of SDOF Equation

The numerical time-step integration method of solving the SDOF dynamic equation of motion using the Newmark-β method or its special case CAA (Constant Average Acceleration) method can be used for any dynamic system with satisfactory agreement with analytical solutions. Numerical solution is the only option for problems that cannot be solved analytically. They are particularly useful for computer implementation, and are used in the computer solution of various problems of structural dynamics. These are implemented in standard softwares for solving structural dynamics problems.

A computer program written in FORTRAN77 for the Newmark-β method is listed below for a general linear system and dynamic loading. Although the forcing function is defined here (as the Ramped Step Function mentioned before) the algorithm can be used in any version of FORTRAN to solve dynamic SDOF problems, with slight modification for the forcing function. Also, the resulting acceleration, velocity and displacement are printed out only once in every twenty steps solved numerically. This can also be modified easily depending on the required output. The program listing is shown below.

```
      OPEN(1,FILE='SDOF.IN',STATUS='OLD')
      OPEN(2,FILE='OUT',STATUS='NEW')

      READ(1,*)RM0,RK0,DRATIO
      READ(1,*)DT,NSTEP

      C0=2.*DRATIO*SQRT(RK0*RM0)

      TIME=0.
      DIS0=0.
      VEL0=0.
      FRC=0.

      ACC0=(FRC–RK0*DIS0–C0*VEL0)/RM0
      WRITE(2,4)TIME,ACC0,VEL0,DIS0

      RKEFF=RK0
      CEFF=C0+RK0*DT
      RMEFF=C0*DT/2.+RK0*DT**2/4.
      DO 10 I=1,NSTEP
       TIME=DT*I

      FRC=25.
      IF(TIME.LE.0.5)FRC=50.*TIME

      ACC=(FRC-RKEFF*DIS0–CEFF*VEL0-RMEFF*ACC0)/(RM0+RMEFF)
      VEL=VEL0+(ACC0+ACC)*DT/2.
      DIS=DIS0+VEL0*DT+(ACC0+ACC)*DT**2/4.

      IF(I/20.EQ.I/20.)WRITE(2,4)TIME,ACC,VEL,DIS

      DIS0=DIS
      VEL0=VEL
      ACC0=ACC
10    CONTINUE

4     FORMAT(10(2X,F8.4))
      END
```

Example 7.1
For the SDOF system described before (m = 1 k-sec^2/ft, k = 25 k/ft) with damping ratio = 0 (c = 0 k-sec/ft), calculate the dynamic displacements for a Ramped Step Loading with p0 = 25 k and t0 = 0.5 sec.

The output file for the FORTRAN77 program listed in the previous section is shown below in tabular form (in Table 7.1). The numerical integrations are carried out for time intervals of Δt = 0.01 sec and results are printed in every 0.20 second up to 5.0 seconds.

Table 7.1: Acceleration, Velocity and Displacement for Δt = 0.01 sec (Results shown in 0.20 second intervals)

Time (sec)	Acceleration (ft/sec^2)	Velocity (ft/sec)	Displacement (ft)
0.0000	0.0000	0.0000	0.0000
0.2000	8.4136	0.9190	0.0635
0.4000	9.0947	2.8315	0.4362
0.6000	-3.3760	3.7351	1.1350
0.8000	-17.5373	1.4506	1.7015
1.0000	-15.5811	-2.1670	1.6232
1.2000	0.6949	-3.7931	0.9722
1.4000	16.3322	-1.9331	0.3467
1.6000	16.9595	1.7034	0.3216
1.8000	2.0003	3.7745	0.9200
2.0000	-14.7973	2.3766	1.5919
2.2000	-17.9955	-1.2055	1.7198
2.4000	-4.6550	-3.6797	1.1862
2.6000	12.9636	-2.7721	0.4815
2.8000	18.6681	0.6832	0.2533
3.0000	7.2158	3.5105	0.7114
3.2000	-10.8682	3.1116	1.4347
3.4000	-18.9638	-0.1471	1.7586
3.6000	-9.6308	-3.2706	1.3852
3.8000	8.5533	-3.3883	0.6579
4.0000	18.8766	-0.3920	0.2449
4.2000	11.8514	2.9645	0.5259
4.4000	-6.0657	3.5965	1.2426
4.6000	-18.4082	0.9231	1.7363
4.8000	-13.8327	-2.5986	1.5533
5.0000	3.4557	-3.7322	0.8618

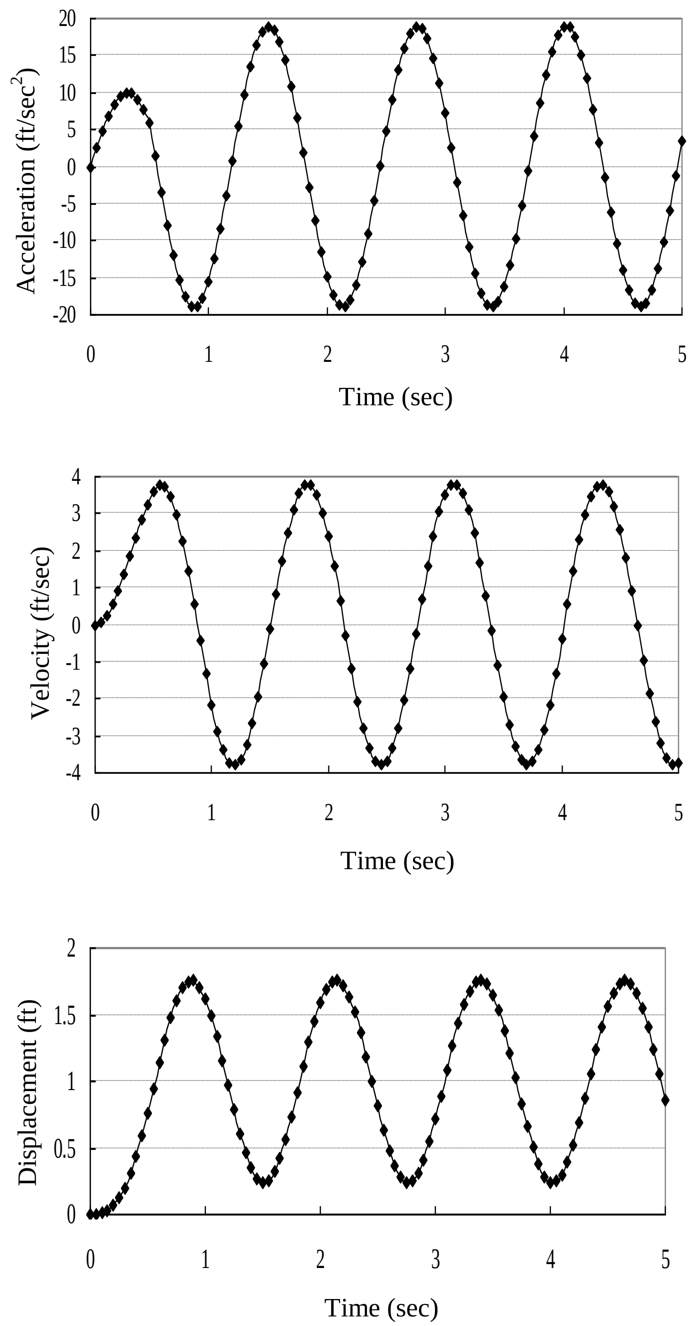


Fig. 7.1: Acceleration, Velocity and Displacement vs. Time

The numerical results (i.e., displacements only) obtained for Δt = 0.01 are presented in Table 7.2 along with exact analytical results and results for Δt = 0.025 and 0.05 sec. In the table, it is convenient to notice the deterioration of accuracy with increasing Δt, although those results are also very accurate and the deterioration of accuracy cannot be detected in Fig. 7.2, where they are also plotted.

Numerical predictions are worse for larger Δt but the CAA guarantees convergence for any value of Δt, even if the results are not very accurate. Table 7.2 also shows the results for Δt = 0.10 and 0.20 sec. These results are clearly unsatisfactory compared to the corresponding exact results, but overall there is only a shift in the dynamic responses and there if no tendency to diverge towards infinity.

Table 7.2: Exact Displacement and Displacement for Δt = 0.01, 0.025, 0.05, 0.10, 0.20 sec

Time (sec)	Displacement (ft)					
	Exact	Δt = 0.01 sec	Δt = 0.025 sec	Δt = 0.05 sec	Δt = 0.10 sec	Δt = 0.20 sec
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2000	0.0634	0.0635	0.0637	0.0645	0.0678	0.0800
0.4000	0.4363	0.4362	0.4358	0.4346	0.4299	0.4160
0.6000	1.1353	1.1350	1.1336	1.1283	1.1080	1.0192
0.8000	1.7017	1.7015	1.7003	1.6960	1.6787	1.5670
1.0000	1.6230	1.6232	1.6247	1.6299	1.6482	1.6612
1.2000	0.9715	0.9722	0.9761	0.9901	1.0435	1.2265
1.4000	0.3462	0.3467	0.3494	0.3593	0.4003	0.6105
1.6000	0.3220	0.3216	0.3194	0.3120	0.2883	0.3061
1.8000	0.9212	0.9200	0.9136	0.8913	0.8068	0.5569
2.0000	1.5928	1.5919	1.5871	1.5696	1.4964	1.1621
2.2000	1.7194	1.7198	1.7220	1.7291	1.7463	1.6377
2.4000	1.1846	1.1862	1.1947	1.2246	1.3351	1.6031
2.6000	0.4801	0.4815	0.4888	0.5156	0.6271	1.0860
2.8000	0.2536	0.2533	0.2518	0.2477	0.2494	0.5002
3.0000	0.7134	0.7114	0.7010	0.6650	0.5366	0.3142

The accuracy of the results depends on the choice of Δt with respect to the natural period of the system or the period of the forcing function itself. For example, whereas Δt = 0.1 and 0.2 sec do not give accurate results for the given system (with natural frequency = 5.0 rad/sec, natural period = 1.257 seconds) it gives much more accurate predictions for 'System2' where m = 1 k-sec^2/ft and k = 4 k/ft (i.e., natural frequency = 2 rad/sec, natural period = 3.142 seconds). The results are shown in Table 7.3.

Table 7.3: Exact Displacement and Displacement for Δt = 0.1 and 0.2 sec for System2

Time (sec)	[Exact]	[Δt = 0.1 sec]	[Δt = 0.2 sec]
0.0000	0.0000	0.0000	0.0000
0.2000	0.0661	0.0738	0.0962
0.4000	0.5165	0.5281	0.5621
0.6000	1.6664	1.6714	1.6628
0.8000	3.5317	3.5205	3.4211
1.0000	5.8261	5.7978	5.6146
1.2000	8.1874	8.1459	7.9059
1.4000	10.2429	10.1967	9.9424
1.6000	11.6679	11.6284	11.4109
1.8000	12.2376	12.2166	12.0853
2.0000	11.8620	11.8689	11.8621
2.2000	10.6004	10.6399	10.7754
2.4000	8.6519	8.7224	8.9926
2.6000	6.3242	6.4170	6.7877
2.8000	3.9848	4.0855	4.5002
3.0000	2.0031	2.0935	2.4819

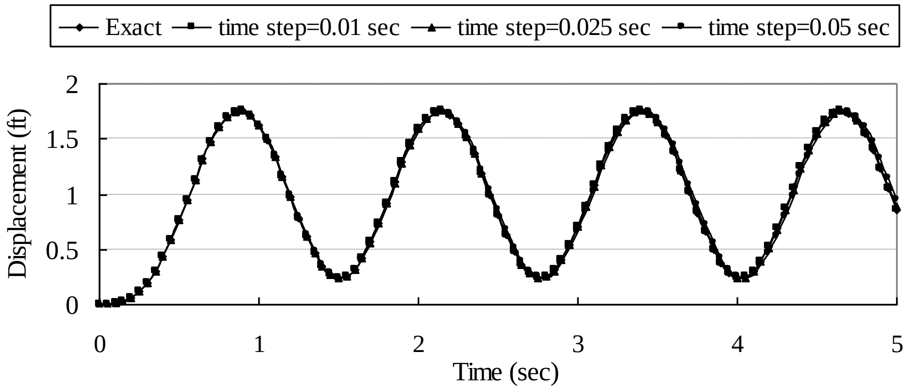


Fig. 7.2: Displacement vs. Time for ‘small’ time steps

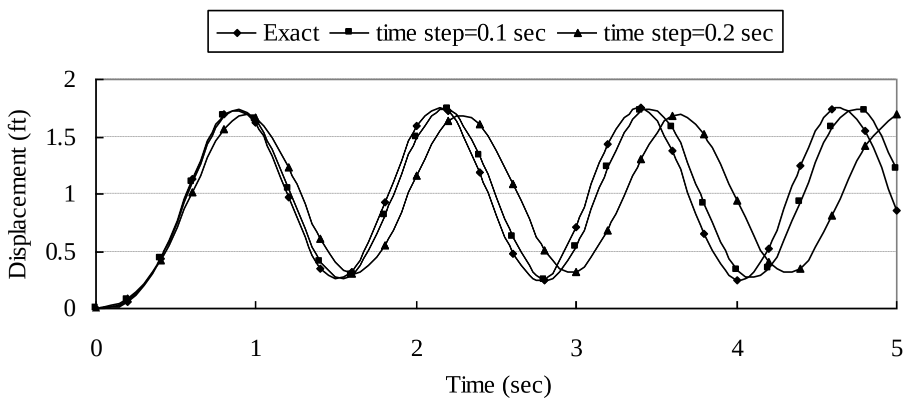


Fig. 7.3: Displacement vs. Time for ‘large’ time steps

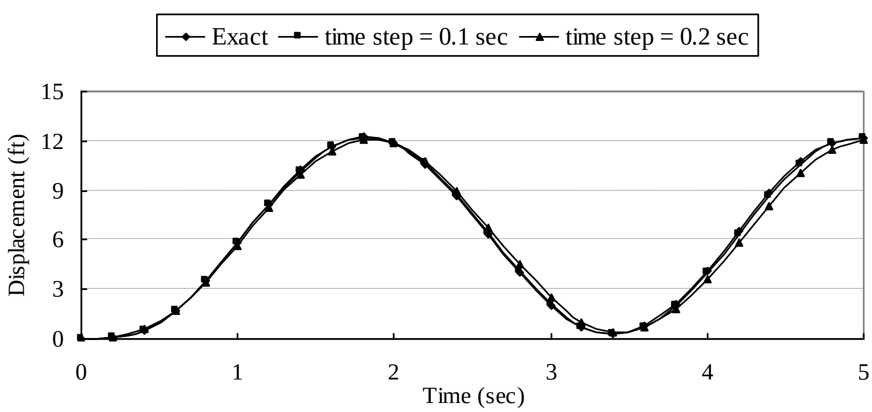


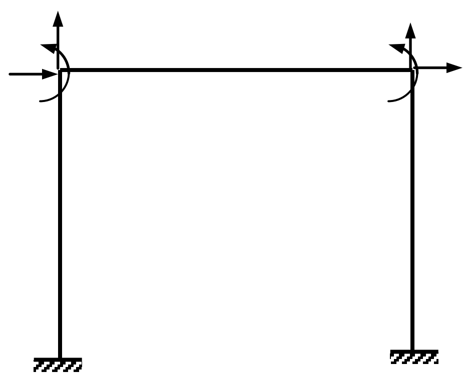
Fig. 7.4: Displacement vs. Time for System2

Introduction to Multi-Degree-of-Freedom (MDOF) System

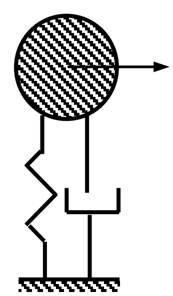
The lectures so far had dealt with Single-Degree-of-Freedom (SDOF) systems, i.e., systems with only one displacement. Although important concepts like free vibration, natural frequency, forced vibration, dynamic magnification and resonance were explained, the conclusions based on such a simplified model have limitations while applying to real structures. Real systems can be modeled as SDOF systems only if it is possible to express the physical properties of the system by a single motion. However, in most cases the SDOF system is only a simplification of real systems modeled by assuming simplified deflected shapes that satisfy the essential boundary conditions.

Real structural systems often consist of an infinite number of independent displacements/rotations and need to be modeled by several degrees of freedom for an accurate representation of their structural response. Therefore, real structural systems are called Multi-Degree-of-Freedom (MDOF) systems in contrast to the SDOF systems discussed before.

A commonly used dynamic model of a 1-storied building is as shown in Fig. 8.1(b), represented by the story sidesway only. Since weight carried by the building is mainly concentrated at the slab and beams while the columns provide the resistance to lateral deformations, the SDOF model assumes a spring and a dashpot for the columns and a mass for the slabs. However the SDOF model may not be an adequate model for real building structures, which calls for modeling as MDOF systems. The infinite number of deflections and rotations of the 1-storied frame shown in Fig. 8.1(a) (subjected to the vertical and horizontal loads as shown) can also be represented by the joint displacements and rotations. A detailed formulation of the 1-storied building frame would require at least three degrees of freedom per joint; i.e., twelve degrees of freedom overall for the four joints (reduced to six after applying boundary conditions). The models become even more complicated for larger structures.



(a)



(b)

Fig. 8.1: One-storied building frame (a) with infinite degrees of freedom, (b) modeled as a SDOF system

Some of the comparative features of the SDOF and MDOF systems are

1. Several basic concepts used for the analysis of SDOF systems like free and forced vibration, dynamic magnification can also be used for MDOF systems.
2. However, some differences between the analyses of SDOF and MDOF systems are mainly due to the more elaborate nature of the MDOF systems. For example, the basic SDOF concepts are valid for each degree of freedom in a MDOF system. Therefore, the MDOF system has several natural frequencies, modes of vibration, damping ratios, modal masses.
3. The basic method of numerical analysis of SDOF systems can be applied for MDOF systems after replacing the displacement, velocity and acceleration by the corresponding vectors and the stiffness, mass and damping by corresponding matrices.

Formulation of the 2-DOF Equations for Lumped Systems

The simplest extension of the SDOF system is a two-degrees-of-freedom (2-DOF) system, i.e., a system with two unknown displacements for two masses. The two masses may be connected to each other by several spring-dashpot systems, which will lead to two differential equations of motion, the solution of which gives the displacements and internal forces in the system.

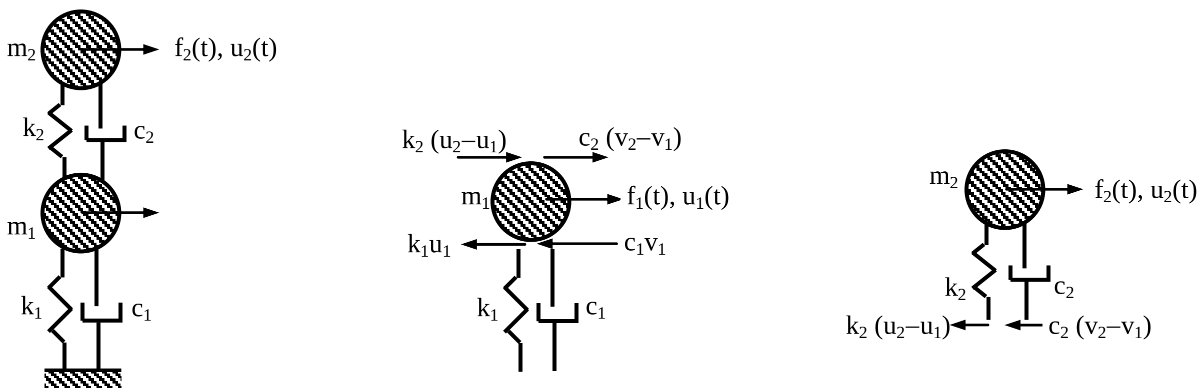


Fig. 8.2: Dynamic 2-DOF system and free body diagrams of m_1 and m_2

Fig. 8.2 shows a 2-DOF dynamic system and the free body diagrams of the two masses m_1 and m_2. In the figure, 'u' stands for displacement (i.e., u_1 and u_2) while 'v' stands for velocity (v_1 and v_2). Denoting accelerations by a_1 and a_2, the differential equations of motion can be applied by applying Newton's 2nd law of motion to m_1 and m_2; i.e.,

$$m_1a_1 = f_1(t) + k_2(u_2–u_1) + c_2(v_2–v_1) – k_1u_1 – c_1v_1$$

$$\Rightarrow m_1a_1 + (c_1+c_2)\, v_1 + (k_1+k_2)u_1 – c_2v_2 – k_2u_2 = f_1(t) \qquad \text{……..…(8.1)}$$

and $m_2a_2 = f_2(t) – k_2\,(u_2–u_1) – c_2(v_2–v_1) \Rightarrow m_2a_2 – c_2v_1 + c_2v_2 – k_2u_1 + k_2u_2 = f_2(t)$ ……..…(8.2)

Putting $v = du/dt$ (i.e., $v_1 = du_1/dt$, $v_2 = du_2/dt$) and $a = d^2u/dt^2$ (i.e., $a_1 = d^2u_1/dt^2$, $a_2 = d^2u_2/dt^2$) in Eqs. (8.1) and (8.2), the following equations are obtained

$$m_1\, d^2u_1/dt^2 + (c_1+c_2)\, du_1/dt – c_2\, du_2/dt + (k_1+k_2)\, u_1 – k_2\, u_2 = f_1(t) \qquad \text{………....(8.3)}$$

$$m_2\, d^2u_2/dt^2 – c_2\, du_1/dt + c_2\, du_2/dt – k_2\, u_1 + k_2\, u_2 = f_2\,(t) \qquad \text{…………(8.4)}$$

Eqs. (8.3) and (8.4) can be arranged in matrix form as

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{Bmatrix} d^2u_1/dt^2 \\ d^2u_2/dt^2 \end{Bmatrix} + \begin{pmatrix} c_1 + c_2 & –c_2 \\ –c_2 & c_2 \end{pmatrix} \begin{Bmatrix} du_1/dt \\ du_2/dt \end{Bmatrix} + \begin{pmatrix} k_1+k_2 & –k_2 \\ –k_2 & k_2 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \qquad \text{………....(8.5)}$$

Eqs. (8.5) represent in matrix form the set of equations [i.e. (8.3) and (8.4)] to evaluate the displacements $u_1(t)$ and $u_2(t)$. In this set, the matrix consisting of the masses (m_1 and m_2) is called the *mass matrix*, the one consisting of the dampings (c_1 and c_2) is called the *damping matrix* and the one consisting of the stiffnesses (k_1 and k_2) is called the *stiffness matrix* of this particular system. These matrices are different for various 2-DOF systems, so that Eq. (8.5) cannot be taken as a general form of governing equations of motion for any 2-DOF system.

For a MDOF system, the mass, damping and stiffness matrices can be generalized by their coefficients, so that Eq. (8.5) can be written in the general form of the dynamic equations of motion,

$$\mathbf{M}\, d^2\mathbf{u}/dt^2 + \mathbf{C}\, d\mathbf{u}/dt + \mathbf{K}\,\mathbf{u} = \mathbf{f}(t) \qquad \text{…………..(8.6)}$$

where the bold capital letters (**M**, **C** and **K**) represent matrices, while the bold small letters ($d^2\mathbf{u}/dt^2$, $d\mathbf{u}/dt$ and $\mathbf{u}$) represent vectors.

For an undamped system, $\mathbf{C} = \mathbf{0} \Rightarrow \mathbf{M}\, d^2\mathbf{u}/dt^2 + \mathbf{K}\,\mathbf{u} = \mathbf{f}(t)$ ………….(8.7)

Eigenvalue Problem and Calculation of Natural Frequencies of a MDOF System

In the previous section, the general equations of motion of a general MDOF system was mentioned to be

$$\mathbf{M}\, d^2\mathbf{u}/dt^2 + \mathbf{C}\, d\mathbf{u}/dt + \mathbf{K}\,\mathbf{u} = \mathbf{f}(t) \qquad \text{……………………….….(8.6)}$$

The free vibration condition for the dynamic motion of MDOF system is obtained by setting $\mathbf{f}(t) = \mathbf{0}$; i.e.,

$$\mathbf{M}\, d^2\mathbf{u}/dt^2 + \mathbf{C}\, d\mathbf{u}/dt + \mathbf{K}\,\mathbf{u} = \mathbf{0} \qquad \text{……………………….….(9.1)}$$

In order to obtain the natural frequency of the undamped system, if $\mathbf{C}$ is also set equal to zero, the equations of motion reduce to

$$\mathbf{M}\, d^2\mathbf{u}/dt^2 + \mathbf{K}\,\mathbf{u} = \mathbf{0} \qquad \text{……………………….….(9.2)}$$

If the displacement vector can be chosen as the summation of a number (equal to the DOF) of variable separable vectors $\mathbf{u}(t) = \sum q_r(t)\, \boldsymbol{\phi}_\mathbf{r}$ ………………………..(9.3)

where $q_r(t)$ is a time-dependent scalar and $\boldsymbol{\phi}_\mathbf{r}$ is a space-dependent vector.

With $q(t) = A_r\, e^{\,i\omega nrt}$, or $q_r(t) = C_{1r} \cos(\omega_{nr}t) + C_{2r} \sin(\omega_{nr}t)$ ……………...………..(9.4)

Eq. (9.2) can be written as $[-\omega_{nr}^2\, \mathbf{M} + \mathbf{K}]\, q_r(t)\, \boldsymbol{\phi}_\mathbf{r} = \mathbf{0}$

$$\Rightarrow [\mathbf{K} - \omega_{nr}^2\, \mathbf{M}]\, \boldsymbol{\phi}_\mathbf{r} = \mathbf{0} \qquad \text{……………………….….(9.5)}$$

Since the vector $\mathbf{u}$ is not zero, Eq. (9.5) turns into the following eigenvalue problem

$$\left| \mathbf{K} - \omega_{nr}^2\, \mathbf{M} \right| = 0 \qquad \text{……………………….….(9.6)}$$

i.e., the determinant of the matrix $(\mathbf{K} - \omega_{nr}^2\, \mathbf{M})$ is zero.

Eq. (9.6) is satisfied for different values of the 'natural frequency'ω_{nr}, which implies that there can be several natural frequencies of a MDOF system. In fact, the number of natural frequencies of the system is equal to the degrees of freedom of the system, i.e., size of the displacement vector. However, consideration of only the first few can adequately model the structural behavior of a dynamic system.

There are several ways to solve the eigenvalue problem of Eq. (9.6), the suitability of which depends on the size of the matrices and the number of eigenvalues required to represent the system accurately.

For each value of ω_{nr}, the vector $\boldsymbol{\phi}_\mathbf{r}$ is called a modal vector for the r^{th} mode of vibration. Once a natural frequency is known, Eq. (9.5) can be solved for the corresponding vector $\boldsymbol{\phi}_\mathbf{r}$ to within a multiplicative constant. The eigenvalue problem does not fix the absolute amplitude of the vectors $\boldsymbol{\phi}_\mathbf{r}$, only the shape of the vector is given by the relative values of the displacements.

Thus the vector $\boldsymbol{\phi}_\mathbf{r}$ (i.e., the eigenvectors, also called the *natural mode of vibration*, *normal mode*, *characteristic vector*, etc.) physically represents the *modal shape* of the system corresponding to the natural frequency. The relative values of the displacements in the vector $\boldsymbol{\phi}_\mathbf{r}$ indicate the shape that the structure would assume while undergoing free vibration at the relevant natural frequency.

The undamped natural frequencies and modal shapes calculated from the above procedure usually prove to be adequate in the subsequent dynamic analyses, since the damped natural frequencies are often quite similar to the damped natural frequencies for typical (undamped) systems, as mentioned in the discussion on SDOF systems.

However, the damped natural frequencies and modal shapes can also be calculated by the methods mentioned before. For that, $q_r(t) = A_r\, e^{\,i\omega nrt}$ will lead to the following equation

$$[\mathbf{K} + i\omega_{nr}\, \mathbf{C} - \omega_{nr}^2\, \mathbf{M}]\, \boldsymbol{\phi}_\mathbf{r} = \mathbf{0} \Rightarrow \left| \mathbf{K} + i\omega_{nr}\, \mathbf{C} - \omega_{nr}^2\, \mathbf{M} \right| = 0 \qquad \text{…………………..…….(9.7)}$$

The solution of Eq. (9.7) provides the natural frequencies of the system, from which the natural modes can also be obtained.

Example 9.1

Calculate the natural frequencies and determine the natural modes of vibration of the 2-storied building system shown in Figs. 8.1 and 8.2, whose governing equations of motion are given by Eq. (8.5). Assume, $k_1 = k_2 = 25$ k/ft, $m_1 = m_2 = 1$ k-sec^2/ft, $c_1 = c_2 = 0$ (i.e., the same stiffnesses and masses as used for the SDOF system before are used here for an undamped 2-DOF system).

Solution

The mass and stiffness matrices of the system are given by **M** and **K** as follows

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} = \begin{pmatrix} 50 & -25 \\ -25 & 25 \end{pmatrix}$$

while the damping matrix $\mathbf{C} = \mathbf{0}$

Thus the eigenvalue problem is given by

$$\begin{bmatrix} 50-\omega_{nr}^2(1) & -25-\omega_{nr}^2(0) \\ -25-\omega_{nr}^2(0) & 25-\omega_{nr}^2(1) \end{bmatrix} \begin{Bmatrix} \phi_{1,r} \\ \phi_{2,r} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

So that the natural frequencies can be obtained from the equation

$(50-\omega_{nr}^{\ 2})\,(25-\omega_{nr}^{\ 2}) - (-25)\,(-25) = 0 \Rightarrow 1250 - 75\,\omega_{nr}^{\ 2} + \omega_{nr}^{\ 4} - 625 = 0 \Rightarrow \omega_{nr}^{\ 2} = 9.55,\ 65.45$

$\Rightarrow \omega_{nr} = 3.09,\ 8.09$ rad/sec $\Rightarrow f_{nr} = \omega_{nr}/2\pi = 0.492,\ 1.288$ cycle/sec; $\therefore T_{nr} = 1/f_{nr} = 2.033,\ 0.777$ sec

The two values of the natural frequency indicate the first and second natural frequency of the system.

$\therefore \omega_{n1} = 3.09$ and $\omega_{n2} = 8.09$ rad/sec for this system.

[Recall that the natural frequency ω_n was equal to 5 rad/sec (i.e., $f_n = 0.796$ cycle/sec) for the SDOF system in Example 2.1, which is greater than ω_{n1} but less than ω_{n2}]

Once the natural frequencies are known, modal shapes can be determined from the eigenvalue equation. For the first natural frequency, the eigenvalue equations are

$$\begin{bmatrix} 40.45 & -25 \\ -25 & 15.45 \end{bmatrix} \begin{Bmatrix} \phi_{1,1} \\ \phi_{2,1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from both these equation, $1.618\,\phi_{1,1} - \phi_{2,1} = 0 \Rightarrow \phi_{1,1} : \phi_{2,1} = 1 : 1.618$

For the second natural frequency, the equations are

$$\begin{bmatrix} -15.45 & -25 \\ -25 & -40.45 \end{bmatrix} \begin{Bmatrix} \phi_{1,2} \\ \phi_{2,2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

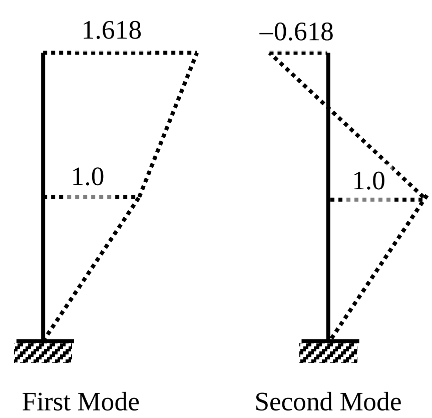


Fig. 9.1: Modal Shapes of the system

from both these equations, $-\,\phi_{1,2} - 1.618\,\phi_{2,2} = 0 \Rightarrow \phi_{1,2} : \phi_{2,2} = 1 : -0.618$

Thus, the first two modal shapes are as shown in Fig. 9.1.

Modal Analysis of MDOF Systems

Calculation of the natural frequencies and the corresponding natural modes of vibration are important in developing a general method of dynamic analysis called the Modal Analysis. This method decomposes the dynamic system into different SDOF systems after solving the eigenvalue problem for natural frequencies and natural modes and considers the individual modes separately to obtain the total solution.

The Modal Analysis uses a very important characteristic of the modal vectors, i.e., the orthogonality conditions. The derivation of the orthogonality conditions is avoided here, but they are available in any standard text on Structural Dynamics. If ω_{ni} and ω_{nj} are the i^{th} and j^{th} natural frequencies of an undamped system and ϕ_i and ϕ_j are the i^{th} and j^{th} modes of vibration, then if $j \neq i$, the mass and stiffness matrices satisfy the following orthogonality conditions

$$\boldsymbol{\phi}_j^T \mathbf{K} \boldsymbol{\phi}_i = 0 \qquad \text{……………….(10.1)}$$
$$\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_i = 0 \qquad \text{……………….(10.2)}$$

where the superscripts T indicate the transpose of the matrices. If j = i, the ratio of the products $\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i$ give the square of the i^{th} natural frequency of the system; i.e.,

$$\omega_{ni}^2 = (\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i)/(\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i) \qquad \text{……………….(10.3)}$$

Choosing the displacement vector as the summation of a number (equal to the DOF) of variable separable vectors [using Eq. (9.3)]

$$\mathbf{u}(t) = \sum q_i(t)\, \boldsymbol{\phi}_i \qquad \text{………………………..(9.3)}$$

so that the governing equations of motion $\mathbf{M}\, d^2\mathbf{u}/dt^2 + \mathbf{K}\,\mathbf{u} = \mathbf{f}(t)$ ……………………….….(8.7)

can be written as, $\sum(\mathbf{M}\, d^2q_i/dt^2\, \boldsymbol{\phi}_i + \mathbf{K}\, q_i\, \boldsymbol{\phi}_i) = \mathbf{f}(t)$ …..……………….….(10.4)

Pre-multiplying (10.4) by $\boldsymbol{\phi}_j^T \Rightarrow \sum\boldsymbol{\phi}_j^T (\mathbf{M}\, d^2q_i/dt^2\, \boldsymbol{\phi}_i + \mathbf{K}\, q_i\, \boldsymbol{\phi}_i) = \boldsymbol{\phi}_j^T\, \mathbf{f}(t)$ ….……………….….(10.5)

Using the orthogonality equations $\Rightarrow (\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i)\, d^2q_i/dt^2 + (\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i)\, q_i = \boldsymbol{\phi}_i^T\, \mathbf{f}(t)$ ……...…………….….(10.6)

where $\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i$ is called the 'modal mass' M_i, $\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i$ the 'modal stiffness' K_i and $\boldsymbol{\phi}_i^T\, \mathbf{f}(t)$ the 'modal load' f_i for the i^{th} mode of the system. Eq. (10.6) is an uncoupled differential equation that can be solved to get $q_i(t)$ as a function of time.

Since $\boldsymbol{\phi}_i$ is already known by solving the eigenvalue problem, $q_i(t)$ can be inserted in Eq. (9.3) and summing up similar components gives u(t). Therefore, the main advantage of the orthogonality conditions is to uncouple the equations of motion so that they can be solved as separate SDOF systems.

For a damped system, the damping matrix **C** can also be formed to satisfy orthogonality condition; i.e.,

$$\boldsymbol{\phi}_j^T \mathbf{C} \boldsymbol{\phi}_i = 0 \qquad \text{……………….(10.7)}$$

This can be possible if the matrix **C** is proportional to the mass matrix **M** or the stiffness matrix **K**, or more rationally a combination of the two; i.e.,

$$\mathbf{C} = a_0\, \mathbf{M} + a_1\, \mathbf{K} \qquad \text{……………….(10.8)}$$

Thus formulated, the equation of motion for the i^{th} mode can be written as

$$(\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i)\, d^2q_i/dt^2 + (\boldsymbol{\phi}_i^T \mathbf{C} \boldsymbol{\phi}_i)\, dq_i/dt + (\boldsymbol{\phi}_i^T \mathbf{K} \boldsymbol{\phi}_i)\, q_i = \boldsymbol{\phi}_i^T\, \mathbf{f}(t) \qquad \text{……………...(10.9)}$$

$\therefore$ The ratio $(\boldsymbol{\phi}_i^T \mathbf{C} \boldsymbol{\phi}_i)/(\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i) = 2\xi_i\omega_{ni}$

$\Rightarrow$ The modal damping ratio, $\xi_i = (\boldsymbol{\phi}_i^T \mathbf{C} \boldsymbol{\phi}_i)/(\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i)/(2\omega_{ni})$ ..……………(10.10)

Modal Analysis is helpful in illustrating the basic features of MDOF system, and is preferred in analytical works. However for practical purposes, it has many drawbacks. The solution of the eigenvalue problem can be cumbersome for large dynamic systems, and the method cannot be applied for nonlinear systems.

Example 10.1

For the 2-storied building system described in Example 9.1, calculate the dynamic displacement vector if step loads of 25 kips are applied at both stories when the system is at rest; i.e., $f_1(t) = f_2(t) = 25$ kips.

Solution

The mass and stiffness matrices of the system are given by

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} = \begin{pmatrix} 50 & -25 \\ -25 & 25 \end{pmatrix}$$

while the damping matrix $\mathbf{C} = \mathbf{0}$

From Example 9.1, the natural frequencies of the system are found to be

$\omega_{n1} = 3.09$ rad/sec, and $\omega_{n2} = 8.09$ rad/sec, while the modal vectors are given by

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix} \quad \text{and} \quad \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$$

$\therefore$ The modal masses are, $M_1 = \boldsymbol{\phi}_1^{\mathbf{T}}\, \mathbf{M}\, \boldsymbol{\phi}_1 = 3.618$ k-sec^2/ft, $M_2 = \boldsymbol{\phi}_2^{\mathbf{T}}\, \mathbf{M}\, \boldsymbol{\phi}_2 = 1.382$ k-sec^2/ft

The modal stiffnesses are, $K_1 = \boldsymbol{\phi}_1^{\mathbf{T}}\, \mathbf{K}\, \boldsymbol{\phi}_1 = 34.55$ k/ft, $K_2 = \boldsymbol{\phi}_2^{\mathbf{T}}\, \mathbf{K}\, \boldsymbol{\phi}_2 = 90.45$ k/ft

The modal loads are, $f_1(t) = \boldsymbol{\phi}_1^{\mathbf{T}}\, \mathbf{f} = 65.45$ k, $f_2(t) = \boldsymbol{\phi}_2^{\mathbf{T}}\, \mathbf{f} = 9.55$ k

$\therefore$ The uncoupled modal equations of motion are

$$3.618\, d^2q_1/dt^2 + 34.55\, q_1 = 65.45$$
$$1.382\, d^2q_2/dt^2 + 90.45\, q_2 = 9.55$$

The solution of these equations starting 'at rest' is

$$q_1(t) = (65.45/34.55)\,[1 - \cos(3.09t)] = 1.894\,[1 - \cos(3.09t)]$$
$$\text{and } q_2(t) = (9.55/90.45)\,[1 - \cos(8.09t)] = 0.1056\,[1 - \cos(8.09t)]$$

$\therefore \mathbf{u}(t) = \sum q_i(t)\, \boldsymbol{\phi}_i = q_1(t)\, \boldsymbol{\phi}_1 + q_2(t)\, \boldsymbol{\phi}_2$

$$\therefore \mathbf{u}(t) = 1.894\,[1 - \cos(3.09t)] \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix} + 0.1056\,[1 - \cos(8.09t)] \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$$

$\Rightarrow u_1(t) = 1.894\,[1 - \cos(3.09t)] + 0.1056\,[1 - \cos(8.09t)]$

$\quad u_2(t) = 3.065\,[1 - \cos(3.09t)] - 0.065\,[1 - \cos(8.09t)]$

The displacements are plotted with time in Fig. 10.1 and 10.2. Fig. 10.1 shows the contribution of the two modes to the total displacements, which are shown in Fig. 10.2. The figures indicate that u_2 is larger in this case, and by far the bigger contributions to the displacements come from the 1st mode of vibration.

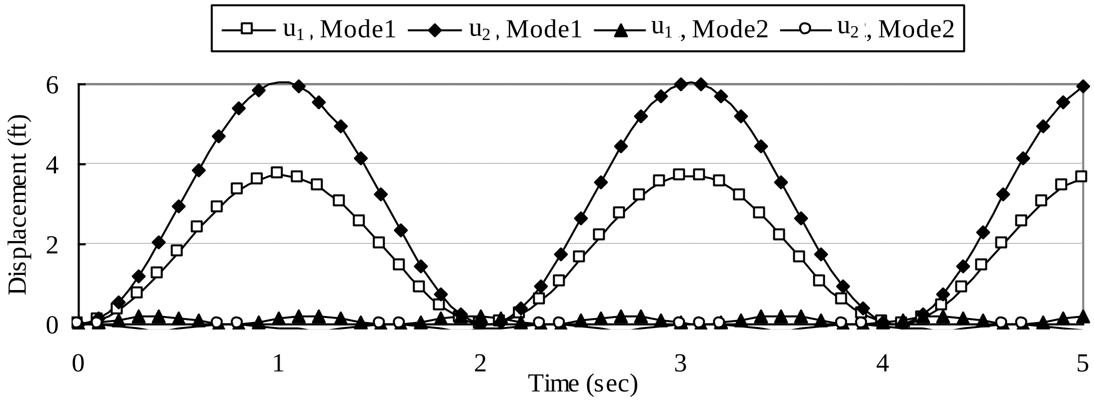


Fig. 12.1: Contribution of various Modes

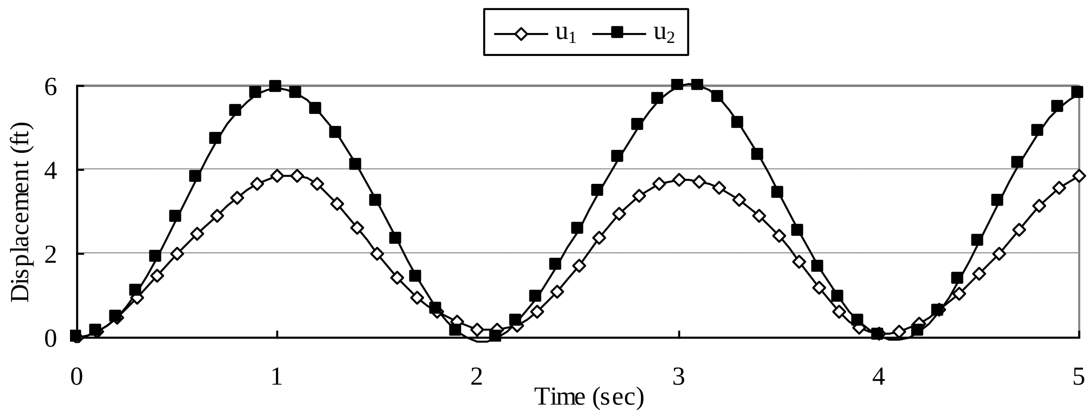


Fig. 12.2: Total Responses

Example 10.2

Calculate the modal damping ratios for the 2-storied system described in Example 10.1 if dampers equivalent to ones for a 5% damped SDOF system are included in each story; i.e., $c_1 = c_2 = 0.5$ k-sec/ft.

Solution

$$\mathbf{C} = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix} = \begin{pmatrix} 1.0 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}$$

The modal dampings are, $C_1 = \boldsymbol{\phi}_1^{\mathbf{T}}\,\mathbf{C}\,\boldsymbol{\phi}_1 = 0.691$ k-sec/ft, $C_2 = \boldsymbol{\phi}_2^{\mathbf{T}}\,\mathbf{C}\,\boldsymbol{\phi}_2 = 1.809$ k-sec/ft
In Example 10.1, the modal masses were calculated to be $M_1 = 3.618$ k-sec^2/ft, $M_2 = 1.382$ k-sec^2/ft, and in Example 9.1, the natural frequencies of the system were found to be
$\omega_{n1} = 3.09$ rad/sec, and $\omega_{n2} = 8.09$ rad/sec

$\therefore$ Using Eq. (10.10), the modal damping ratios are

$$\xi_1 = C_1/(2M_1\omega_{n1}) = 0.691/(2\times3.618\times3.09) = 0.0309$$
$$\xi_2 = C_2/(2M_2\omega_{n2}) = 1.809/(2\times1.382\times8.09) = 0.0809$$

ξ_1 is lower while ξ_2 is greater than 0.05. Particularly the damping ratio of the second mode is much higher, which helps to suppress it even further.

Numerical Solution of MDOF Equations

The equations of motion for a MDOF system have been solved analytically using the Modal Analysis. Although Modal Analysis is helpful in formulating and understanding some basic concepts of dynamic analysis, it has several limitations of convenience and applicability. In fact, it has even more limitations than the analytical methods used to solve SDOF systems.

In addition to the considerable mathematical effort needed to solve eigenvalue problems and uncouple the simultaneous equations (i.e., make the system matrices diagonal), its formulation requires several assumptions. For example, the method is valid for linear systems only. The orthogonality condition that makes the Modal Analysis convenient, is not guaranteed to be valid for the damping matrix. The practical loading situations can be more complicated and not convenient to solve analytically. Numerical methods must be used in such situations.

As mentioned for SDOF systems, the most widely used numerical approach for solving dynamic problems is the *Newmark-β method*. The method solves the dynamic equation of motion in the $(i+1)^{th}$ time step based on the results of the i^{th} step.

The dynamic equations of motion for the $(i+1)^{th}$ time step is

$$\mathbf{M\,a_{i+1} + C\,v_{i+1} + K\,u_{i+1} = f_{i+1}} \qquad \text{..………………(11.1)}$$

where the bold small letter '**a**' stands for the acceleration vector, '**v**' for velocity vector and '**u**' for displacement vector. In the *Constant Average Acceleration (CAA) method* (a special case of Newmark-β method where $\alpha = 0.50$ and $\beta = 0.25$), the velocity and displacement vectors are given by

$$\mathbf{v_{i+1} = v_i} + (\mathbf{a_i + a_{i+1}})\Delta t/2 \qquad \text{…………………(11.2)}$$

$$\mathbf{u_{i+1} = u_i} + \mathbf{v_i}\,\Delta t + (\mathbf{a_i + a_{i+1}})\Delta t^2/4 \qquad \text{…………………(11.3)}$$

Inserting these values in Eq. (13.1) and rearranging the coefficients, the following equation is obtained,

$$(\mathbf{M + C}\Delta t/2 + \mathbf{K}\Delta t^2/4)\mathbf{a_{i+1} = f_{i+1} - Ku_i} - (\mathbf{C + K}\Delta t)\mathbf{v_i} - (\mathbf{C}\Delta t/2 + \mathbf{K}\Delta t^2/4)\mathbf{a_i} \qquad \text{…………......…..(11.4)}$$

Therefore, if the forcing function $\mathbf{f_{i+1}}$ is known, the only unknown in Eq. (11.4) is the acceleration vector $\mathbf{a_{i+1}}$, which can be obtained by matrix inversion (by Gauss Elimination or some other method). Once $\mathbf{a_{i+1}}$ is obtained, Eqs. (11.2) and (11.3) can be used to calculate the velocity vector $\mathbf{v_{i+1}}$ and the displacement vector $\mathbf{u_{i+1}}$ at time t_{i+1}. These values are used to obtain the results at time t_{i+2} and subsequent time-steps.

The simulation needs two initial conditions, e.g., the displacement vector $\mathbf{u_0}$ and velocity vector $\mathbf{v_0}$ at time $t_0 = 0$. Then the initial acceleration vector can be obtained as

$$\mathbf{a_0 = M^{-1}(f_0 - Cv_0 - Ku_0)} \qquad \text{…………………(11.5)}$$

Again, any standard method of matrix inversion can be used to solve Eq. (11.5).

Among other methods of numerical solution of the MDOF equations of motion, the *Linear Acceleration method* and *Central Difference method* are quite popular. The Linear Acceleration Method is a special case of the Newmark-β method with $\alpha = 0.50$ and $\beta = 1/6$. Instead of assuming constant average acceleration between two time intervals, it assumes the acceleration to vary linearly in between two intervals. Unlike the CAA method, the Linear Acceleration method is not unconditionally stable. However, the time increment needed for its stability is much greater than the interval needed for accurate results, therefore stability is usually not a problem for this method.

Incremental solutions of the equations of motion are also popular, particularly for nonlinear systems. Instead of solving for the total displacement or acceleration at any time, this method solves for the increment (change) in displacement or acceleration. There again, the CAA is widely used.

Computer Implementation of Numerical Solution of MDOF Equations

The numerical time-step integration method of solving the MDOF dynamic equation of motion has been described in the previous section. Just as the computer implementation in SDOF system, the Constant Average Acceleration (CAA) method can be used for any dynamic system with satisfactory agreement with analytical solutions.

A computer program written in FORTRAN77 for the CAA method is listed below for a general linear system and dynamic loading. The forcing function is defined here as the Step Function, but the algorithm can be used in any version of FORTRAN to solve dynamic MDOF problems with slight modification for the forcing function, which can be used as input also. Besides, the stiffness, damping and mass matrices are input for the discrete systems, but in practice they can be assembled from structural properties. The resulting displacements are printed only once in every ten steps solved numerically. This can also be modified easily depending on the required output. The program listing is shown below.

```
      PROGRAM MDOF
      DIMENSION DIS(100),VEL(100),ACC(100),DISV(100),VELV(100)
      DIMENSION DIS0(100),VEL0(100),ACC0(100),FORCE(100)
      DIMENSION SK(100,100),SC(100,100),SM(100,100)
      COMMON/SOLVER/SMEFF(100,100),PEFF(100),NDF

      OPEN(1,FILE='MDOF.IN',STATUS='OLD')
      OPEN(2,FILE='MDOF.OUT',STATUS='NEW')

      PI=4.*ATAN(1.)
      READ(1,*)NDF
      READ(1,*)((SK(I,J),J=1,NDF),I=1,NDF)
      READ(1,*)((SC(I,J),J=1,NDF),I=1,NDF)
      READ(1,*)((SM(I,J),J=1,NDF),I=1,NDF)

C*******TIME-STEP INTEGRATION USING CAA METHOD****************
      READ(1,*)DSTEP,NSTEP
      A1=DSTEP/2.
      A2=DSTEP**2/4.

      READ(1,*)(DIS0(I),VEL0(I),FORCE(I),I=1,NDF)

C*******INITIAL ACCELERATION**********************************
      DO 16 I=1,NDF
       DO 16 J=1,NDF
        SMEFF(I,J)=SM(I,J)
  16  CONTINUE

      DO 18 I=1,NDF
       PEFF(I)=FORCE(I)
       DO 18 J=1,NDF
        PEFF(I)=PEFF(I)–SC(I,J)*VEL0(J)–SK(I,J)*DIS0(J)
  18  CONTINUE

      CALL GAUSS

      DO 19 I=1,NDF
  19  ACC0(I)=PEFF(I)

      WRITE(2,6)TIME,(DIS0(I),I=1,NDF)
  6   FORMAT(10(1X,F8.4))
      DO 26 IT=1,NSTEP
       TIME=IT*DSTEP

       DO 27 I=1,NDF
        FORCE(I)=25.
```

```
       VELV(I)=VEL0(I)+ACC0(I)*A1
27    DISV(I)=DIS0(I)+VEL0(I)*DSTEP+ACC0(I)*A2

      DO 28 I=1,NDF
       PEFF(I)=FORCE(I)
       DO 28 J=1,NDF
        PEFF(I)=PEFF(I)–SC(I,J)*VELV(J)–SK(I,J)*DISV(J)
28    CONTINUE

      DO 15 I=1,NDF
       DO 15 J=1,NDF
        SMEFF(I,J)=SM(I,J)+SC(I,J)*A1+SK(I,J)*A2
15    CONTINUE

      CALL GAUSS

      DO 29 I=1,NDF
       ACC(I)=PEFF(I)
       VEL(I)=VEL0(I)+(ACC0(I)+ACC(I))*A1
       DIS(I)=DIS0(I)+VEL0(I)*DSTEP+(ACC0(I)+ACC(I))*A2

       DIS0(I)=DIS(I)
       VEL0(I)=VEL(I)
29     ACC0(I)=ACC(I)

C********DISPLACEMENTS*****************************************
      IF(IT/10.EQ.IT/10.)WRITE(2,6)TIME,(DIS(I),I=1,NDF)
26    CONTINUE

20    END

C*************************************************************
C*******GAUSS ELIMINATION**************************************
      SUBROUTINE GAUSS
      COMMON/SOLVER/AG(100,100),BG(100),N

      N1=N–1
      DO 10 I=1,N1
       I1=I+1
       CG=1./AG(I,I)
       DO 11 KS=I1,N
        DG=AG(KS,I)*CG
        DO 12 J=I1,N
12     AG(KS,J)=AG(KS,J)–DG*AG(I,J)
11    BG(KS)=BG(KS)–DG*BG(I)
10    CONTINUE

      BG(N)=BG(N)/AG(N,N)

      DO 13 II=1,N1
       I=N–II
       I1=I+1
       SUM=0.
       DO 14 J=I1,N
14    SUM=SUM+AG(I,J)*BG(J)
13    BG(I)=(BG(I)–SUM)/AG(I,I)

      END
```

Example 11.1

For the 2-DOF system described before ($m_1 = m_2 = 1$ k-sec^2/ft, $k_1 = k_2 = 25$ k/ft) with damping ratio = 0.0 or similar damping as the SDOF system with 5% damping ($c_1 = c_2 = 0.5$ k-sec/ft), calculate the dynamic displacements for a Step Loading with $f_1 = f_2 = 25$ k.

The results from the FORTRAN77 program listed in the previous section are plotted in Fig. 11.1 and Fig. 11.2. The numerical integrations are carried out for time intervals of $\Delta t = 0.01$ sec and results are printed in every 0.10 second up to 5.0 seconds.

Fig. 11.1 shows that the results from the numerical method can hardly be distinguished from the theoretical results obtained from Modal Analysis. The maximum values of the displacements (u_1 and u_2) are 3.92 ft and 6.00 ft respectively. Since the static displacements in this case are $u_1 = 2$ ft, $u_2 = 3$ ft, the dynamic magnifications are nearly 2.

Fig. 11.2 shows the response of a damped 2-DOF system where the damping of a SDOF system with 5% damping is included in both stories of the 2-storied structure. Everything else remaining the same, the displacements still oscillate about the static displacements after 5 seconds, but the convergence to the static solutions can be noticed.

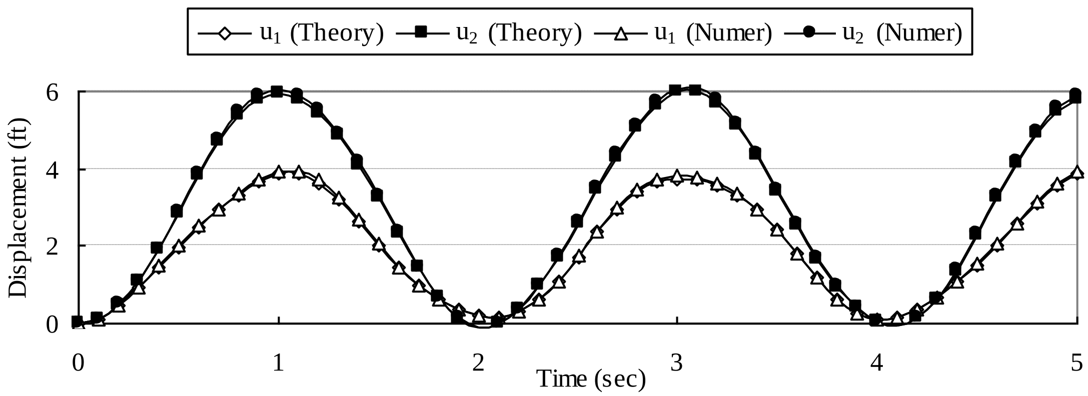


Fig.11.1 : Response of Undamped 2-DOF System

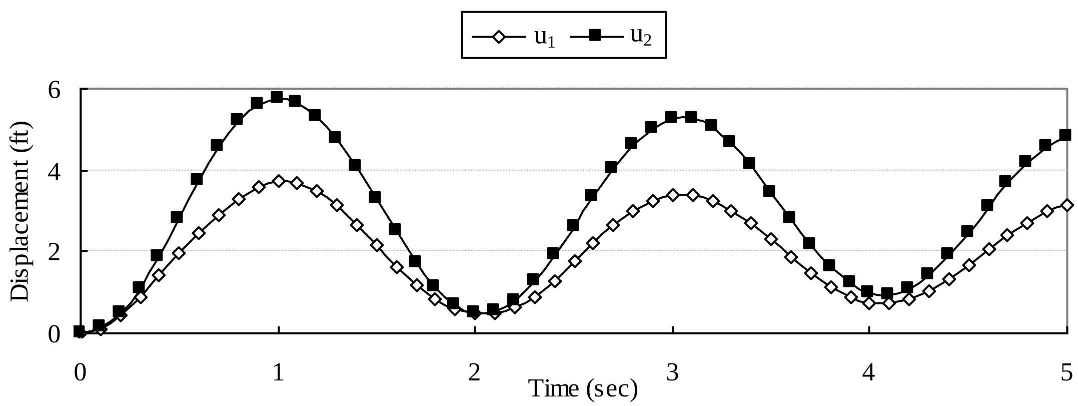


Fig.11.2 : Response of Damped 2-DOF System

Example 11.2

For a 4-DOF system with structural properties similar to the 2-DOF system described before ($m_1 = m_2 = m_3 = m_4 = 1$ k-sec^2/ft, $k_1 = k_2 = k_3 = k_4 = 25$ k/ft) with similar damping as the SDOF system with 5% damping ($c_1 = c_2 = c_3 = c_4 = 0.5$ k-sec/ft), calculate the dynamic displacements for a Step Loading with $f_1 = f_2 = f_3 = f_4 = 25$ k.

This problem is difficult to solve analytically because it involves solution of a 4×4 matrix. However, the numerical method is used here easily using the computer program listed before. The resulting dynamic displacements are shown in Fig. 11.3. The maximum values of u_1, u_2, u_3 and u_4 are 7.56, 13.55, 17.58 and 19.5 ft respectively, reached at the first peak at nearly 1.8 seconds. This shows that the fundamental time period of this system is about 3.6 seconds and since the static solutions of u_1, u_2, u_3 and u_4 are 4, 7, 9 and 10 ft respectively, the dynamic magnifications are nearly 2 again for all the displacements.

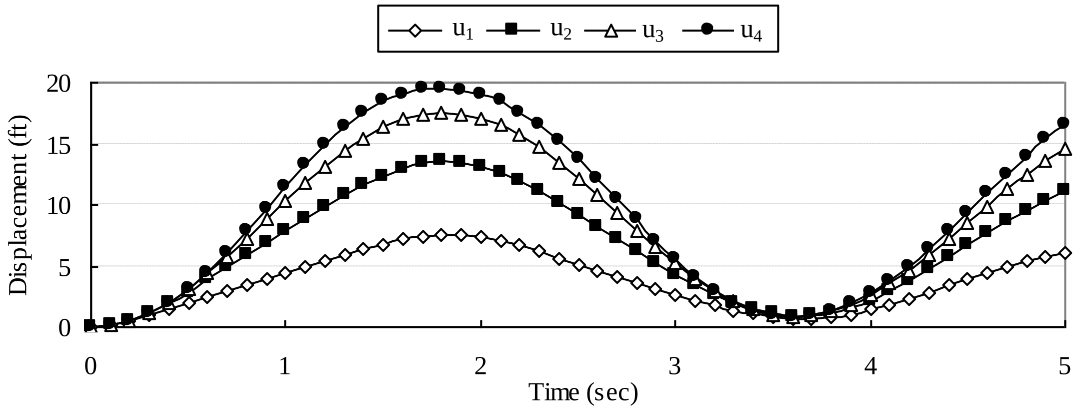


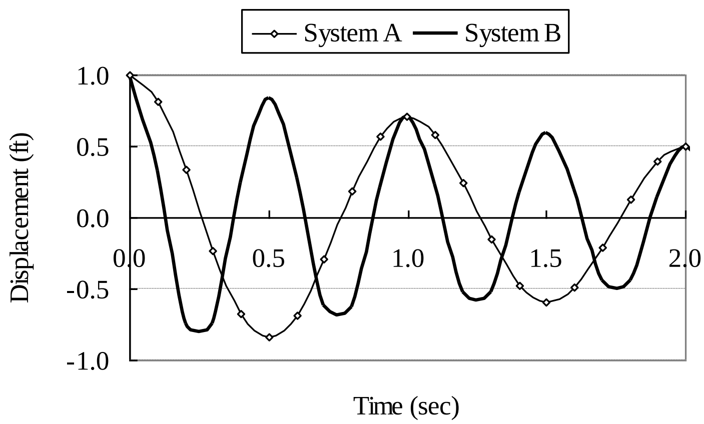
Fig.11.3 : Response of Damped 4-DOF System

Problems on the Dynamic Analysis of MDOF Systems

1. A small structure of stiffness 1 k/ft, natural frequency 1 rad/sec and damping 1 k-sec/ft is mounted on a larger undamped structure of stiffness 10 k/ft but the same natural frequency. Determine the (i) natural frequencies, (ii) natural modes of vibration, (iii) modal damping ratios of the system.

2. For a (20′ × 20′) floor system weighing 200 psf (including all dead and live loads) supported by four (10″ × 10″) square columns (each 12′ high) and a rigid massless footing, calculate the undamped natural period for horizontal vibration.
   Consider k for each column = $12EI/L^3$ and k_f for footing equal to (i) 2×10^6 lb/in, (ii) 2×10^4 lb/in.
   Assume the total weight of the system to be concentrated at the floor
   [Given: Modulus of elasticity of concrete = 3×10^6 psi].

3. A 2-DOF system is composed of two underdamped systems (A and B), whose free vibration responses are shown below. If each system weighs 100 lb, calculate the
   (i) undamped natural frequency and damping ratio of system A and B,
   (ii) first natural frequency and damping ratio of the 2-DOF system formed.



4. A lumped-mass 3-DOF dynamic system has the following properties
   $k_1 = k_2 = k_3 = 50$ k/ft, $c_1 = c_2 = c_3 = 1$ k-sec/ft, $m_1 = m_2 = m_3 = 2$ k-sec^2/ft.
   (i) Form the stiffness, damping and mass matrices of the system.
   (ii) Calculate the 1st natural frequency and damping ratio of the system, if the 1st modal vector for the system is given by $\boldsymbol{\phi_1} = \{0.445, 0.802, 1.000\}^T$.

5. The undamped 2-DOF system described in the class is subjected to harmonic load vectors of
   (i) $\mathbf{f}(t)= \{0, 50 \cos(3t)\}^T$, (ii) $\mathbf{f}(t)= \{0, 50 \cos(8t)\}^T$.
   In both cases, calculate the displacement vector $\mathbf{u}(t)$ of the system at time t = 0.1 seconds, if the system is initially at rest.

6. Calculate the maximum floor displacement of the system described in Question 2 when subjected to a horizontal step load of 10 kips at the floor level.

7. For the system defined in Question 5, calculate u(0.1) in each case using the CAA method.

Solution of Problems on the Dynamic Analysis of MDOF Systems

1. For this 2-DOF system

   $k_1 = 10$ k/ft, $k_2 = 1$ k/ft, $m_1 = 10$ k-sec^2/ft, $m_2 = 1$ k-sec^2/ft, $c_1 = 0$ k-sec/ft, $c_2 = 1$ k-sec/ft

   $$\mathbf{K} = \begin{pmatrix} 11 & -1 \\ -1 & 1 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

   $\therefore$ The eigenvalue problem $\Rightarrow (11-10\omega_n^2)(1-\omega_n^2) - (-1)^2 = 0 \Rightarrow 10\,\omega_n^4 - 21\omega_n^2 + 10 = 0$

   $\Rightarrow \omega_n = 0.854$ rad/sec, 1.171 rad/sec

   $\boldsymbol{\phi_1} = \{1, (11-10\omega_{n1}^2)\}^T = \{1, 3.702\}^T$, $\boldsymbol{\phi_2} = \{1, (11-10\omega_{n2}^2)\}^T = \{1, -2.702\}^T$

   $K_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{K}\ \boldsymbol{\phi_1} = 17.30$ k/ft, $K_2 = \boldsymbol{\phi_2}^{\mathbf{T}}\mathbf{K}\ \boldsymbol{\phi_2} = 23.70$ k/ft

   $M_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{M}\ \boldsymbol{\phi_1} = 23.70$ k-sec^2/ft, $M_2 = \boldsymbol{\phi_2}^{\mathbf{T}}\mathbf{M}\ \boldsymbol{\phi_2} = 17.30$ k-sec^2/ft

   $C_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{C}\ \boldsymbol{\phi_1} = 7.30$ k-sec/ft, $C_2 = \boldsymbol{\phi_2}^{\mathbf{T}}\mathbf{C}\ \boldsymbol{\phi_2} = 13.70$ k-sec/ft

   $\xi_1 = C_1/2\sqrt{(K_1M_1)} = 18.02\%$, $\xi_2 = C_2/2\sqrt{(K_2M_2)} = 33.83\%$

2. For this 2-DOF system

   $$\mathbf{K} = \begin{pmatrix} k_f + k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 0 & 0 \\ 0 & m_1 \end{pmatrix}$$

   $\therefore$ The eigenvalue problem $\Rightarrow (k_f + k_1)(k_1 - m_1\omega_n^2) - (-k_1)^2 = 0 \Rightarrow \omega_n^2 = \{k_f/(k_1 + k_f)\}(k_1/m_1)$

   $\Rightarrow \omega_{n1} = \sqrt{\{1/(1 + k_1/k_f)\}}\sqrt{(k_1/m_1)}$

   Here $k_1 = 4 \times 12\ EI/L^3 = 48 \times (3 \times 10^6) \times \{(10 \times 10^3)/12\}/144^3 = 4.02 \times 10^4$ lb/in,

   $m_1 = (20 \times 20)\ 200/386 = 207.25$ lb-sec^2/in

   (i) $k_f = 2 \times 10^6$ lb/in $\Rightarrow \omega_{n1} = \sqrt{\{1/(1 + 4.02 \times 10^4/2 \times 10^6)\}}\sqrt{(4.02 \times 10^4/207.25)} = 13.79$ rad/sec

   $\Rightarrow T_{n1} = 2\pi/\omega_{n1} = 0.456$ sec

   (ii) $k_f = 2 \times 10^4$ lb/in $\Rightarrow \omega_{n1} = 8.03$ rad/sec $\Rightarrow T_{n1} = 2\pi/\omega_{n1} = 0.783$ sec

3. For this 2-DOF system, $m_1 = m_2 = 100/32.2 = 3.11$ lb-sec^2/ft, $\xi_1 = 0.0552$, $\xi_2 = 0.0276$

   $\omega_{n1} = 6.285$ rad/sec, $\omega_{n2} = 12.571$ rad/sec (as found before)

   $k_1 = m_1\omega_{n1}^2 = 122.60$ lb/ft, $k_2 = m_2\omega_{n2}^2 = 490.42$ lb/ft

   $c_1 = 2\xi_1\, m_1\omega_{n1} = 2.15$ lb-sec/ft, $c_2 = 2\xi_1\, m_1\omega_{n1} = 2.15$ lb-sec/ft

   $$\mathbf{K} = \begin{pmatrix} 613.02 & -490.42 \\ -490.42 & 490.42 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 3.11 & 0 \\ 0 & 3.11 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 4.31 & -2.15 \\ -2.15 & 2.15 \end{pmatrix}$$

   $\therefore$ The eigenvalue problem $\Rightarrow (613.02-3.11\omega_n^2)(490.42-3.11\omega_n^2) - (-490.42)^2 = 0$

   $\Rightarrow \omega_n = 4.30$, 18.35 rad/sec $\Rightarrow \omega_{n1} = 4.30$ rad/sec

   $\therefore$ First mode of vibration is $\boldsymbol{\phi_1} = \{490.42: 613.02-3.11\omega_{n1}^2\}^T = \{1: 1.133\}^T$

   $K_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{K}\ \boldsymbol{\phi_1} = 131.25$ k/ft, $M_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{M}\ \boldsymbol{\phi_1} = 7.09$ k-sec^2/ft, $C_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{C}\ \boldsymbol{\phi_1} = 2.192$ k-sec/ft

   $\therefore \xi_1 = C_1/2(M_1\omega_{n1}) = 0.0359 = 3.59\%$

4. For the 3-DOF system,

   $$\mathbf{K} = \begin{pmatrix} 100 & -50 & 0 \\ -50 & 100 & -50 \\ 0 & -50 & 50 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

   $\boldsymbol{\phi_1} = \{0.445, 0.802, 1.000\}^T$

   $\therefore K_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{K}\ \boldsymbol{\phi_1} = 18.23$ k/ft, $M_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{M}\ \boldsymbol{\phi_1} = 3.682$ k-sec^2/ft, $C_1 = \boldsymbol{\phi_1}^{\mathbf{T}}\mathbf{C}\ \boldsymbol{\phi_1} = 0.365$ k-sec/ft

   $\therefore \omega_{n1} = \sqrt{(K_1/M_1)} = 2.225$ rad/sec, $\xi_1 = C_1/2\sqrt{(K_1M_1)} = 2.23\%$

5. As shown in the class,

The natural frequencies are ω_{n1} = 3.09 rad/sec, ω_{n2} = 8.09 rad/sec
The mode shapes are $\boldsymbol{\phi_1} = \{1.000, 1.618\}^T$, $\boldsymbol{\phi_2} = \{1.000, -0.618\}^T$
The modal masses are, M_1 = 3.618 k-sec^2/ft, M_2 = 1.382 k-sec^2/ft
The modal stiffnesses are, K_1 = 34.55 k/ft, K_2 = 90.45 k/ft
(i) The modal loads are, $f_1(t) = \boldsymbol{\phi_1}^T \mathbf{f}$ = 80.9 cos(3t), $f_2(t) = \boldsymbol{\phi_2}^T \mathbf{f}$ = –30.9 cos(3t)
$\therefore$ The uncoupled modal equations of motion are

$3.618\, d^2q_1/dt^2 + 34.55\, q_1 = 80.9 \cos(3t)$

$1.382\, d^2q_2/dt^2 + 90.45\, q_2 = -30.9 \cos(3t)$

$\therefore$ For an undamped system, $q_1(t) = [p_{01}/(K_1 - \omega^2 M_1)]\, [\cos(\omega t) - \cos(\omega_{n1} t)]$
$\Rightarrow q_1(0.1) = [80.9/(34.55 - 3^2 \times 3.618)]\, [\cos(0.3) - \cos(0.309)] = 0.110$ ft
$\therefore$ Also, $q_2(t) = [p_{02}/(K_2 - \omega^2 M_2)]\, [\cos(\omega t) - \cos(\omega_{n2} t)]$
$\Rightarrow q_2(0.1) = -[30.9/(90.45 - 3^2 \times 1.382)]\, [\cos(0.3) - \cos(0.809)] = -0.105$ ft
$\therefore\ u_1(0.1) = q_1(0.1)\, \phi_{1,1} + q_2(0.1)\, \phi_{1,2} = 0.110 \times 1.00 - 0.105 \times 1.00 = 0.005$ ft
$u_2(0.1) = q_1(0.1)\, \phi_{2,1} + q_2(0.1)\, \phi_{2,2} = 0.110 \times 1.618 - 0.105 \times (-0.618) = 0.243$ ft
(ii) $q_1(0.1) = [80.9/(34.55 - 8^2 \times 3.618)]\, [\cos(0.8) - \cos(0.309)] = 0.105$ ft
$\therefore$ Also, $q_2(0.1) = -[30.9/(90.45 - 8^2 \times 1.382)]\, [\cos(0.8) - \cos(0.809)] = -0.100$ ft
$\therefore\ u_1(0.1) = q_1(0.1)\, \phi_{1,1} + q_2(0.1)\, \phi_{1,2} = 0.105 \times 1.00 - 0.100 \times 1.00 = 0.005$ ft
$u_2(0.1) = q_1(0.1)\, \phi_{2,1} + q_2(0.1)\, \phi_{2,2} = 0.105 \times 1.618 - 0.100 \times (-0.618) = 0.232$ ft

6. As found in Question 2, the natural frequency is ω_{n1} = 13.79 rad/sec
$\therefore$ The mode shape is $\boldsymbol{\phi_1} = \{k_1, k_f + k_1\}^T = \{4.02, 204.02\}^T = \{1.00, 50.75\}^T$
The modal mass $M_1 = \boldsymbol{\phi_1}^T \mathbf{M}\, \boldsymbol{\phi_1} = 53.38 \times 10^4$ lb-sec^2/in
The modal stiffness $K_1 = M_1 \omega_{n1}^2 = 101.5 \times 10^6$ lb/in
The modal load is, $f_1(t) = \boldsymbol{\phi_1}^T \mathbf{f}$ = 507.5 kips = 50.75×10^4 lb
$\therefore$ The uncoupled modal equation of motion is, $53.38\, d^2q_1/dt^2 + 101.5 \times 10^2\, q_1 = 50.75$
$\Rightarrow q_{1,max} = 2 \times 50.75 \times 10^{-2}/101.5 = 0.010$ in
$\Rightarrow u_{2,max} = q_{1,max}\, \phi_{2,1} = 0.010 \times 50.75 = 0.5075$ in

7. The governing equations between time $t_0 = 0$ and $t_1 = 0.1$ sec [i.e., $\Delta t = 0.1$ sec] are
$(\mathbf{M} + \mathbf{C}\Delta t/2 + \mathbf{K}\Delta t^2/4)\, \mathbf{a}_1 = \mathbf{f}_1 - \mathbf{K}\, \mathbf{u}_0 - (\mathbf{C} + \mathbf{K}\Delta t)\, \mathbf{v}_0 - (\mathbf{C}\Delta t/2 + \mathbf{K}\Delta t^2/4)\, \mathbf{a}_0$
$\mathbf{u}_1 = \mathbf{u}_0 + \mathbf{v}_0\, \Delta t + (\mathbf{a}_0 + \mathbf{a}_1)\, \Delta t^2/4$
Since the structure was initially at rest, $\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$
$\therefore$ Initially, $\mathbf{M}\, \mathbf{a}_0 = \mathbf{f}_0 - \mathbf{K}\, \mathbf{u}_0 - \mathbf{C}\, \mathbf{v}_0 = \mathbf{f}_0$ ……………...……(1)
Also, $(\mathbf{M} + 0.0025\, \mathbf{K})\, \mathbf{a}_1 = \mathbf{f}_1 - 0.0025\, \mathbf{K}\, \mathbf{a}_0$, and $\mathbf{u}_1 = (\mathbf{a}_0 + \mathbf{a}_1)\, 0.0025$ ……...……(2), (3)

$$\mathbf{K} = \begin{pmatrix} 50 & -25 \\ -25 & 25 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{f}_0 = \begin{Bmatrix} 0 \\ 50 \end{Bmatrix}$$

$\Rightarrow \mathbf{a}_0 = \{0 \quad 50\}^T$

$\therefore$ Eq. (2) $\Rightarrow (\mathbf{M} + 0.0025\, \mathbf{K})\, \mathbf{a}_1 = \mathbf{f}_1 - 0.0025\, \mathbf{K}\, \mathbf{a}_0$

$\therefore$ Question (i), using $\mathbf{f}(t) = \{0, 50\cos(3t)\}^T \Rightarrow$

$$\begin{pmatrix} 1.125 & -0.0625 \\ -0.0625 & 1.0625 \end{pmatrix} \mathbf{a}_1 = \begin{Bmatrix} 0 \\ 47.77 \end{Bmatrix} - \begin{pmatrix} 0.125 & -0.0625 \\ -0.0625 & 0.0625 \end{pmatrix} \begin{Bmatrix} 0 \\ 50 \end{Bmatrix} = \begin{Bmatrix} 3.125 \\ 44.642 \end{Bmatrix}$$

$\Rightarrow \mathbf{a}_1 = \{5.13 \quad 42.32\}^T$

$\therefore$ Eq. (3) $\Rightarrow \mathbf{u}_1 = \{5.13 \quad 92.32\}^T \times 0.0025 = \{0.013 \quad 0.231\}^T$

Dynamic Equations of Motion for Continuous Systems

The basic concepts of Structural Dynamics discussed so far dealt with discrete dynamic systems; i.e., with Single-Degree-of-Freedom (SDOF) systems and Multi-Degree-of-Freedom (MDOF) systems. The fundamental equations of motion were derived using Newton's 2^{nd} law of motion. While this is useful for dealing with most problems involving point masses and forces, there are certain problems where such formulations are not convenient.

Method of Virtual Work

Another way of representing Newton's equations of static and dynamic equilibrium is by energy methods, which is based on the law of conservation of energy. According to the principle of virtual work, if a system in equilibrium is subjected to virtual displacements δu, the virtual work done by the external forces (δW_E) is equal to the virtual work done by the internal forces (δW_I)

$$\delta W_I = \delta W_E \qquad \text{…...…………………(12.1)}$$

where the symbol δ is used to indicate 'virtual'. This term is used to indicate hypothetical increments of displacements and works that are assumed to happen in order to formulate the problem.

Energy Formulation for Discrete SDOF System:

If a virtual displacement δu is applied on a SDOF system with a single mass m, a damping c and stiffness k undergoing displacement u(t) due to external load f(t),

the virtual internal work, $\delta W_E = f(t)\,\delta u$ ………………(12.2)

and virtual external work, $\delta W_I = m\, d^2u/dt^2\, \delta u + c\, du/dt\, \delta u + k\, u\, \delta u$ ………………(12.3)

$\therefore$ Combining $\Rightarrow m\, d^2u/dt^2\, \delta u + c\, du/dt\, \delta u + k\, u\, \delta u = f(t)\, \delta u \Rightarrow m\, d^2u/dt^2 + c\, du/dt + k\, u = f(t)$ …...…(12.4)

which is the same as Eq. (2.5), derived earlier from Newton's 2^{nd} law of motion.

So, the method of virtual work leads to the same conclusion as the equilibrium formulation. This method is not very convenient here, but its advantage is more apparent in the formulation for continuous systems.

Energy Formulation for Continuous SDOF Systems:

Most of the practical dynamic problems involve continuous structural systems. Unlike discrete MDOF systems, these continuous systems can only be defined properly by an infinite number of displacements; i.e., infinite degrees of freedom.

However several continuous systems are modeled as SDOF systems by assuming all the displacements as proportional to a single displacement (related by an appropriate deflected shape as a function of space). The governing equations from such assumed deflected shapes are similar to SDOF equations with the 'equivalent' mass m*, damping c*, stiffness k* and load f*(t) being the essential parameters instead of the respective discrete values m, c, k and f(t).

Therefore, once the appropriate defected shapes are assumed and the 'equivalent' parameters calculated, the solution of this continuous system is similar to any discrete SDOF system. One-dimensional continuous structural members undergoing axial deformations (e.g., columns or truss members) and flexural deformations (e.g., beams or frame members) are illustrated here as two such examples.

(1) Axially Loaded Bar

For an undamped member loaded axially by a load p(x,t) per unit length, the external virtual work due to virtual deformation δu is

$$\delta W_E = \int p(x,t)\, dx\, \delta u \qquad \text{……………………………(12.5)}$$

while the internal virtual work due to inertia and virtual axial strain $d(\delta u)/dx = \delta u'$ is

$$\delta W_I = \int m\, dx\, d^2u/dt^2\, \delta u + \int u'\, EA\, \delta u'\, dx \qquad \text{……………………………(12.6)}$$

where $\delta u'$ stands for differentiation of δu with respect to x (in general the symbol $'$ stands for differentiation with respect to x), E = modulus of elasticity and A = cross-sectional area of the axial member, m = mass of the member per unit length. E, A and m can vary with x.

$$\therefore \delta W_I = \delta W_E \Rightarrow \int m\, dx\, d^2u/dt^2\, \delta u + \int u'\, EA\, \delta u'\, dx = \int p(x,t)\, dx\, \delta u \qquad \text{…………....(12.7)}$$

If the displacements are assumed to be function of a single displacement u1, so that

$$u(x) = u_1\, \phi(x) \Rightarrow u' = u_1\, \phi'(x),\ d^2u/dt^2 = d^2u_1/dt^2\, \phi(x) \qquad \text{…...…………(12.8), (12.9), (12.10)}$$

$$\delta u = \delta u_1\, \phi(x) \Rightarrow \delta u' = \delta u_1\, \phi'(x) \qquad \text{……..………………(12.11), (12.12)}$$

$$\therefore \text{Eq. (12.7)} \Rightarrow \int m\, dx\, d^2u_1/dt^2\, \phi(x)\, \delta u_1\, \phi(x) + \int u_1\, \phi'(x)\, EA\, \delta u_1\, \phi'(x)\, dx = \int p(x,t)\, dx\, \delta u_1\, \phi(x)$$

$$\Rightarrow \{\int m\, [\phi(x)]^2\, dx\}\, d^2u_1/dt^2 + \{\int EA\, [\phi'(x)]^2\, dx\}\, u_1 = \int p(x,t)\, \phi(x)\, dx \qquad \text{……………..(12.13)}$$

$\therefore$ If the integrations are carried out after knowing $\phi(x)$, Eq. (12.13) can be rewritten as,

$$m^*\, d^2u_1/dt^2 + k^*\, u_1 = f^*(t) \qquad \text{……………….…...(12.14)}$$

where m*, k*, f*(t) are the 'effective' mass, stiffness and force of the SDOF system. The 'effective' damping of the system can be obtained by assuming a reasonable damping ratio.

(2) Transversely Loaded Beam

For an undamped member loaded transversely by a load q(x,t) per unit length, the external virtual work due to virtual deformation δu is

$$\delta W_E = \int q(x,t)\, dx\, \delta u \qquad \text{……………………………(12.15)}$$

while the internal virtual work due to inertia and virtual curvature $d(\delta u')/dx = \delta u''$ is

$$\delta W_I = \int m\, dx\, d^2u/dt^2\, \delta u + \int u''\, E\, I\, \delta u''\, dx \qquad \text{……………………………(12.16)}$$

where $\delta u''$ stands for double differentiation of δu with respect to x, E = modulus of elasticity and I = moment of inertia of the cross-sectional area of the flexural member, m = mass of the member per unit length. E, I and m can vary with x.

$$\therefore \delta W_I = \delta W_E \Rightarrow \int m\, dx\, d^2u/dt^2\, \delta u + \int u'' E\, I\, \delta u''\, dx = \int q(x,t)\, dx\, \delta u \qquad \text{……..…(12.17)}$$

If the displacements are assumed to be function of a single displacement u_2, so that

$$u(x) = u_2 \psi(x) \Rightarrow u'' = u_2\, \psi''(x),\ d^2u/dt^2 = d^2u_2/dt^2\, \psi(x) \qquad \text{……….….(12.18), (12.19), (12.20)}$$

$$\delta u = \delta u_2\, \psi(x) \Rightarrow \delta u = \delta u_2\, \psi(x) \qquad \text{…………………...…(12.21), (12.22)}$$

$\therefore$ Inserting these values in Eq. (12.17) $\Rightarrow$

$$\int m\, dx\, d^2u_2/dt^2\, \psi(x)\, \delta u_2\, \psi(x) + \int u_2\, \psi''(x)\, EI\, \delta u_2\, \psi''(x)\, dx = \int q(x,t)\, dx\, \delta u_2\, \psi(x)$$

$$\Rightarrow \{\int m\, [\psi(x)]^2\, dx\}\, d^2u_2/dt^2 + \{\int EI\, [\psi''(x)]^2\, dx\}\, u_2 = \int q(x,t)\, \psi(x)\, dx \qquad \text{…...…….(12.23)}$$

$\therefore$ If the integrations are carried out after knowing (or assuming) $\psi(x)$, Eq. (12.23) can be rewritten as,

$$m^*\, d^2u_2/dt^2 + k^*\, u_2 = f^*(t) \qquad \text{………………………….…...(12.24)}$$

where m*, k*, f*(t) are the 'effective' mass, stiffness and force of the SDOF system.

Once m*, c*, k* and f*(t) are calculated, Eq. (12.14) or (12.24) can be solved to obtain the deflection u_1 or u_2, from which the deflection u(x) at any point can be calculated using Eq. (12.8) or (12.18).

The accuracy of Eq. (12.14) or (12.24) depends on the accuracy of the shape functions $\phi(x)$ or $\psi(x)$. If the shape functions are not defined exactly (usually they are not), the solutions can only be approximate. These functions must be defined satisfying the natural boundary conditions; i.e., those involving displacements for axial deformation and displacements as well as rotations for flexural deformations. This method of solving dynamic problems is called the *Rayleigh-Ritz method*.

Example 12.1

For a cantilever rod, modulus of elasticity E = 450000 ksf, cross-sectional area A = 1 ft^2, moment of inertia I = 0.08 ft^4, length L = 10 ft, mass per unit length m = 0.0045 k-sec^2/ft^2.

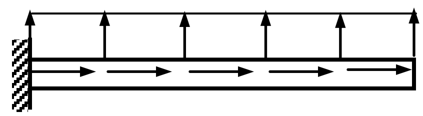
(i) Calculate the approximate natural frequency of the system in axial and flexural vibrations.

(ii) Calculate the approximate axial and flexural vibrations of the system for axial and transverse step loads of 1 k/ft.

Solution

(i) Assuming shape functions (satisfying natural boundary conditions)

$\phi(x) = x/L$, $\psi(x) = (x/L)^2$



[Note that: $\phi(0) = 0$, $\psi(0) = 0$, $\psi'(0) = 0$]

For axial deformations,

$m^* = \int m\,[\phi(x)]^2\,dx = mL/3 = 0.015$ k-sec^2/ft

$k^* = \int EA\,[\phi'(x)]^2\,dx = EA/L = 45000$ k/ft

$\therefore \omega_n = \sqrt{(k^*/m^*)} = (1.732/L)\sqrt{(EA/m)} = (1.732/10)\sqrt{(450000\times1/0.0045)} = 1732$ rad/sec

[The exact result is $\omega_n = (\pi/2L)\sqrt{(EA/m)} = 1571$ rad/sec]

For flexural deformations,

$m^* = \int m\,[\psi(x)]^2\,dx = mL/5 = 0.009$ k-sec^2/ft

$k^* = \int EI\,[\psi''(x)]^2\,dx = 4EI/L^3 = 144$ k/ft

$\therefore \omega_n = \sqrt{(k^*/m^*)} = (4.472/L^2)\sqrt{(EI/m)} = (4.472/100)\sqrt{(450000\times0.08/0.0045)} = 126.49$ rad/sec

[The exact result is $\omega_n = (3.516/L^2)\sqrt{(EI/m)} = 99.45$ rad/sec]

(ii) For axial deformations, effective force $f^* = \int p(x,t)\,\phi(x)\,dx = pL/2 = 5$ kips

$\therefore$ Equation for axial deformation is, $0.015\,d^2u_1/dt^2 + 45000\,u_1 = 5$

$\Rightarrow u_1(t) = 1.11\times10^{-4}\,[1-\cos(1732\,t)] \Rightarrow u(x,t) = 1.11\times10^{-4}\,[1-\cos(1732\,t)]\,(x/L)$

For flexural deformations, effective force $f^* = \int q(x,t)\,\psi(x)\,dx = qL/3 = 3.33$ kips

$\therefore$ Equation for flexural deformation is, $0.009\,d^2u_2/dt^2 + 144\,u_2 = 3.33$

$\Rightarrow u_2(t) = 0.02315\,[1-\cos(126.49\,t)] \Rightarrow u(x,t) = 0.02315\,[1-\cos(126.49\,t)]\,(x/L)^2$

Example 12.2

For the member properties mentioned in Example 12.1, calculate the approximate first natural frequencies for the flexural vibration of

(i) a cantilever beam, assuming $\psi(x) = 1-\cos(\pi x/2L)$,

(ii) a simply supported beam, assuming $\psi(x) = \sin(\pi x/L)$.

Solution

Both these shape functions satisfy the natural boundary conditions for the problems mentioned.

(i) For the cantilever beam,

$m^* = \int m\,[\psi(x)]^2\,dx = 0.2268\,mL = 0.0102$ k-sec^2/ft

$k^* = \int EI\,[\psi''(x)]^2\,dx = 3.044\,EI/L^3 = 109.59$ k/ft

$\therefore \omega_n = \sqrt{(k^*/m^*)} = (3.664/L^2)\sqrt{(EI/m)} = (3.664/100)\sqrt{(450000\times0.08/0.0045)} = 103.63$ rad/sec

This is a much better estimate of the first natural frequency of a uniform cantilever beam.

(ii) For the simply supported beam,

$m^* = \int m\,[\psi(x)]^2\,dx = mL/2 = 0.0225$ k-sec^2/ft

$k^* = \int EI\,[\psi''(x)]^2\,dx = (\pi/L)^4\,EI\,L/2 = 1753.36$ k/ft

$\therefore \omega_n = \sqrt{(k^*/m^*)} = (\pi/L)^2\sqrt{(EI/m)} = (9.870/100)\sqrt{(450000\times0.08/0.0045)} = 279.15$ rad/sec

This is the exact first natural frequency of a uniform simply supported beam.

These results show that the accuracy of the Rayleigh-Ritz method depends on the accuracy of the assumed shape function. Based on the shape function, this method can model the structure to be too stiff (i.e., over-estimate the 'effective' stiffness and natural frequency) or can reproduce the exact solution.

Stiffness and Mass Matrices of Continuous Systems

The Rayleigh-Ritz method presented for SDOF systems can estimate the first natural frequency of a continuous system, its accuracy depending on the type of shape function chosen for the analysis. The chosen shape function actually indicates the assumed first mode of vibration. In extending to MDOF systems, however, the Rayleigh-Ritz method has serious shortcomings. Once a shape function is chosen for the first mode, there is no automatic way of choosing the subsequent shape functions. Moreover, the choice of the function depends on the boundary conditions, thus needing a different formulation even if the structure remains the same otherwise.

The *Finite Element Method (FEM)*, on the other hand, is an extension of the *Stiffness Method* of structural analysis and has the advantage of a methodical formulation (very well suited to computers) and versatility in applying the boundary conditions for a large variety of linear and nonlinear problems. Like the Rayleigh-Ritz method, the formulation of FEM is based on energy principles. But rather than defining the displacement of the entire structure/structural member by a single function, the FEM divides the member into a number of small segments (called elements) and defines the displacements at any point in the member by interpolating between the displacements/rotations of the nodes at the ends of the member.

Based on the nature of the problem, the elements chosen in FEM can be one, two or three-dimensional and their interpolation functions can differ accordingly. For example, members of trusses or frames are formed by one-dimensional elements, plates, and shells by two-dimensional elements, and solid bodies by three-dimensional elements (Fig. 13.1).

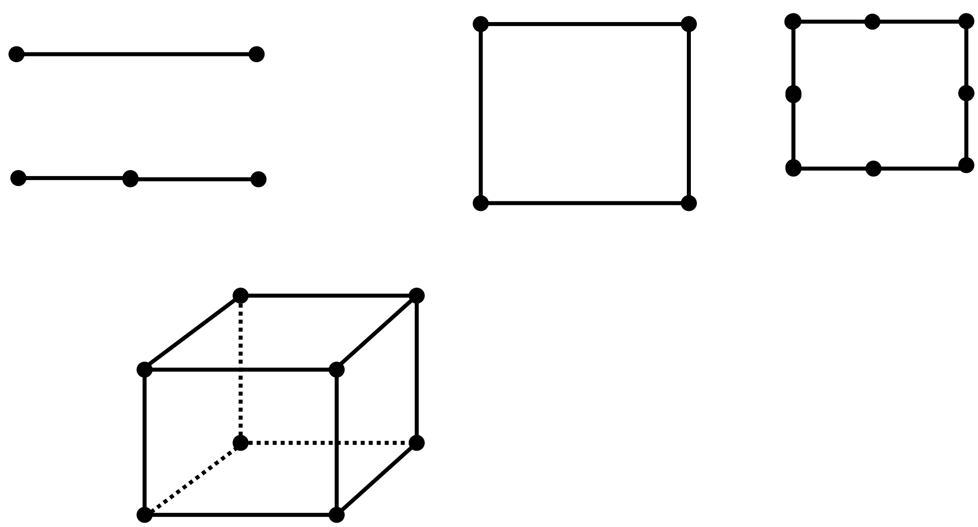


Fig. 13.1: Different type of elements used in FEM

Axial Members

Applying the method of virtual work to undamped members subjected to axial load of p(x,t) per unit length, $\delta W_I = \delta W_E \Rightarrow \int m\, dx\, d^2u/dt^2\, \delta u + \int u'\, E\, A\, \delta u'\, dx = \int p(x,t)\, dx\, \delta u$ ……………….…(12.7)

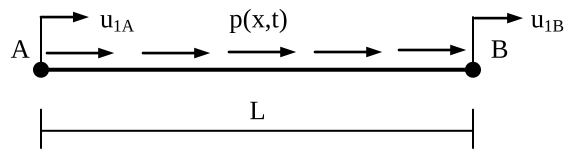


Fig. 13.2: Axially Loaded Member

If the displacements of a member AB (Fig. 13.2) are assumed to be interpolating functions [$\phi_1(x)$ and $\phi_2(x)$] of two nodal displacements u_{1A} and u_{1B},

$$u = u_{1A}\, \phi_1 + u_{1B}\, \phi_2 \Rightarrow u' = u_{1A}\, \phi_1' + u_{1B}\, \phi_2' \qquad \text{…………..………(13.1), (13.2)}$$

$$d^2u/dt^2 = d^2u_{1A}/dt^2\, \phi_1 + d^2u_{1B}/dt^2\, \phi_2 \qquad \text{……………………..……(13.3)}$$

$$\delta u = \delta u_{1A}\, \phi_1 + \delta u_{1B}\, \phi_2 \Rightarrow \delta u' = \delta u_{1A}\, \phi_1' + \delta u_{1B}\, \phi_2' \qquad \text{…………...……….(13.4), (13.5)}$$

$\therefore$ Eq. (12.7) can be written in matrix form as,

$$\begin{pmatrix} \int m\, \phi_1\, \phi_1 dx & \int m\, \phi_1\, \phi_2 dx \\ \int m\, \phi_2\, \phi_1 dx & \int m\, \phi_2\, \phi_2 dx \end{pmatrix} \begin{Bmatrix} d^2u_{1A}/dt^2 \\ d^2u_{1B}/dt^2 \end{Bmatrix} + \begin{pmatrix} \int EA\, \phi_1'\phi_1' dx & \int EA\, \phi_1'\phi_2' dx \\ \int EA\, \phi_2'\phi_1' dx & \int EA\, \phi_2'\phi_2' dx \end{pmatrix} \begin{Bmatrix} u_{1A} \\ u_{1B} \end{Bmatrix} = \begin{Bmatrix} \int p(x,t)\phi_1 dx \\ \int p(x,t)\phi_2\, dx \end{Bmatrix}$$

...…...…..(13.6)

For concentrated loads p(x,t) is a delta function of x, as mentioned before. If loads X_A and X_B are applied at joints A and B, they can be added to the right side of Eq. (13.6).

Eq. (13.6) can be rewritten as, $\mathbf{M_m}\, d^2\mathbf{u_m}/dt^2 + \mathbf{K_m}\, \mathbf{u_m} = \mathbf{f_m}$ ………………(13.7)

where $\mathbf{M_m}$ and $\mathbf{K_m}$ are the mass and stiffness matrices of the member respectively, while $d^2\mathbf{u_m}/dt^2$, $\mathbf{u_m}$ and $\mathbf{f_m}$ are the member acceleration, displacement and load vectors. They can be formed once the shape functions ϕ_1 and ϕ_2 are known or assumed. All of them are referred to the local (member) axes of the member. For the complete structure, these matrices and vectors should be transformed to the global (structural) axes and assembled for different members. The global damping matrix can be formed from modal damping ratios. The boundary conditions must be satisfied in the final form of these matrices and vector, after which the dynamic equations of motion for global displacement vector $\mathbf{u}(t)$ can be solved by time-step integration.

One-dimensional two-noded elements with linear interpolation functions are typically chosen in such cases, so that the shape functions ϕ_1 and ϕ_2 for axially loaded members are

$$\phi_1(x) = 1 - x/L, \text{ and } \phi_2(x) = x/L \qquad \text{………………………(13.8)}$$

Therefore, elements of the member mass and stiffness matrices are

$$M_{mij} = \int m\, \phi_i\, \phi_j\, dx, \text{ and } K_{mij} = \int EA\, \phi_i'\, \phi_j'\, dx \qquad \text{………………………(13.9)}$$

Fig. 13.3: Shape functions $\phi_1(x)$ and $\phi_2(x)$

Flexural Members

Applying the method of virtual work to undamped members subjected to flexural load of q(x,t) per unit length $\Rightarrow \int m\, dx\, d^2u/dt^2\, \delta u + \int u'' E I\, \delta u''\, dx = \int q(x,t)\, dx\, \delta u$ ……..………………(12.17)

Following the same type of formulation as for axial members, the member equations for undamped flexural members subjected to transverse load of q(x,t) per unit length (Fig. 13.4) can be written in matrix form like Eq. (13.10), but the member matrices are different here.

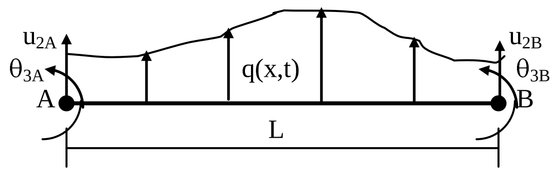


Fig. 13.4: Transversely Loaded Member

Two-noded elements with cubic interpolation functions for u_{2A}, θ_{3A}, u_{2B} and θ_{3B} are typically chosen in such cases, so that

$u(x) = u_{2A}\, \psi_1 + \theta_{3A}\, \psi_2 + u_{2B}\, \psi_3 + \theta_{3B}\, \psi_4$ ………..………………(13.10)

where $\psi_1(x) = 1-3(x/L)^2 + 2(x/L)^3$, $\psi_2(x) = x\{1-(x/L)\}^2$

$\psi_3(x) = 3(x/L)^2 - 2(x/L)^3$, $\psi_4(x) = (x-L)(x/L)^2$ ………………………(13.11)

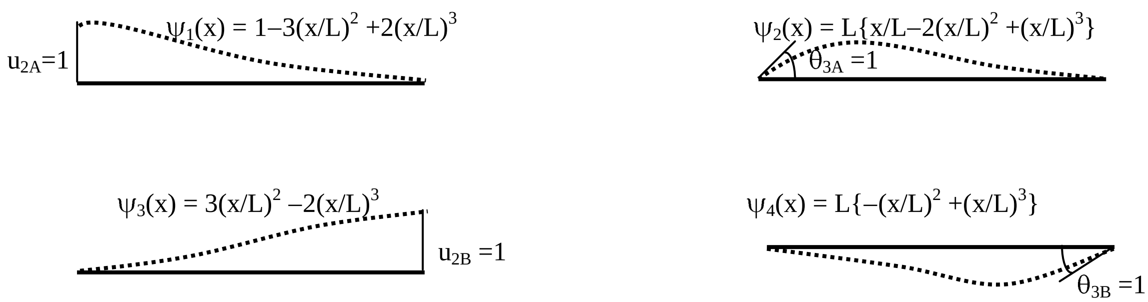


Fig. 13.5: Shape functions $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$ and $\psi_4(x)$

The size of the matrices is (4×4) here, due to transverse joint displacements (u_{2A}, u_{2B}) joint rotations (θ_{3A}, θ_{3B}) and their elements are given by

$M_{mij} = \int m\, \psi_i\, \psi_j\, dx$, and $K_{mij} = \int EI\, \psi_i''\, \psi_j''\, dx$ ………………………(13.12)

The equations of the mass matrix and stiffness matrix for axial members [Eq. (13.9)] as well as flexural members guarantee that for linear problems

(i) The mass and stiffness matrices are symmetric [i.e., element (i,j) = element (j,i)],

(ii) The diagonal elements of the matrices are positive [as element (i,i) involves square].

Example 13.1

For the member properties mentioned in Example 12.1, (modulus of elasticity E = 450000 ksf, cross-sectional area A = 1 ft^2, length L = 10 ft, mass per unit length m = 0.0045 k-sec^2/ft^2) calculate the approximate natural frequencies of the cantilever beam in axial direction, analyzing with (i) one linear element, (ii) two linear elements.

Solution

(i) For $\phi_1(x) = 1 – x/L$, $\phi_2(x) = x/L$, the following mass and stiffness matrices are obtained from Eq. (13.6)

$$\mathbf{M_m} = \begin{pmatrix} mL/3 & mL/6 \\ mL/6 & mL/3 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} EA/L & -EA/L \\ -EA/L & EA/L \end{pmatrix}$$

Assuming one linear element with properties of Example 12.1, mL/3 = 0.015 k-sec^2/ft, EA/L = 45000 k/ft

$$\therefore \mathbf{M_m} = \begin{pmatrix} 0.015 & 0.0075 \\ 0.0075 & 0.015 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} 45000 & -45000 \\ -45000 & 45000 \end{pmatrix}$$

Applying the boundary conditions that the only non-zero DOF is the axial deformation at B (u_{1B}), the mass and stiffness matrices are reduced to (1×1) matrices

$\mathbf{M} = 0.015$, $\mathbf{K} = 45000$

$\therefore |\mathbf{K} – \omega_n^2 \mathbf{M}| = 0 \Rightarrow 45000 – \omega_n^2\, 0.015 = 0 \Rightarrow \omega_n^2 = 3000000 \Rightarrow \omega_n = 1732$ rad/sec

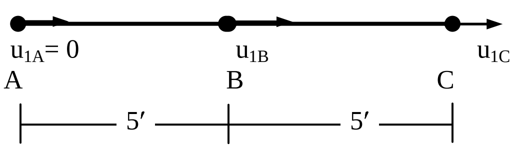
(ii) Assuming two linear elements (AB and BC) of length 5′ each,

mL/3 = 0.0075 k-sec^2/ft, EA/L = 90000 k/ft

$\therefore$ The following mass and stiffness matrices are obtained for each element

$$\mathbf{M_m} = \begin{pmatrix} 0.0075 & 0.00375 \\ 0.00375 & 0.0075 \end{pmatrix} \qquad \mathbf{K_m} = \begin{pmatrix} 90000 & -90000 \\ -90000 & 90000 \end{pmatrix}$$

Applying the boundary conditions that axial deformation at A (u_{1A}) is zero, only the axial deformations at B (u_{1B}) and C (u_{1C}) are non-zero, the mass and stiffness matrices are reduced to (2×2) matrices.



$$\mathbf{M} = \begin{pmatrix} 0.0075 & 0.00375 \\ 0.00375 & 0.015 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 90000 & -90000 \\ -90000 & 180000 \end{pmatrix}$$

$\therefore |\mathbf{K} - \omega_n^2 \mathbf{M}| = 0 \Rightarrow (90000 – \omega_n^2\, 0.0075)(180000 – \omega_n^2\, 0.015) – (–90000 – \omega_n^2\, 0.00375)^2 = 0$

$\Rightarrow \omega_n = 1612$ rad/sec, 5629 rad/sec

Therefore the first and second natural frequencies of the structure are calculated now. It should be mentioned here that the analytical solutions for the first two natural frequencies are 1571 rad/sec, 4712 rad/sec respectively. Therefore, the natural frequencies are again over-estimated but one can now get a better estimation of the first natural frequency, which is within 3% of the analytical value. The estimate of the second natural frequency is not as accurate, but it is still less than 20% greater than the exact value.

Example 13.2

For the member properties mentioned in Example 12.1 (E = 450000 ksf, I = 0.08 ft^4, L = 10 ft, m = 0.0045 k-sec^2/ft^2), calculate the approximate first natural frequency of the cantilever beam in transverse direction, analyzing with one element.

Solution

For $\psi_1(x)$ to $\psi_4(x)$ as shown in Eq. (13.11) and uniform m and EI, the following mass and stiffness matrices are obtained from Eq. (13.12)

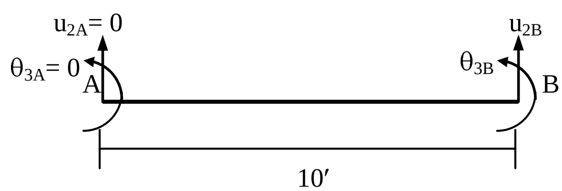
$$\mathbf{M_m} = (mL/420)\begin{pmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{pmatrix} \qquad \mathbf{K_m} = (EI/L^3)\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

$\therefore$ In this case, $mL/420 = 1.071\times 10^{-4}$ k-sec^2/ft, $EI/L^3 = 36$ k/ft

$$\mathbf{M_m} = 1.071\times10^{-4}\begin{pmatrix} 156 & 220 & 54 & -130 \\ 220 & 400 & 130 & -300 \\ 54 & 130 & 156 & -220 \\ -130 & -300 & -220 & 400 \end{pmatrix} \qquad \mathbf{K_m} = 36\begin{pmatrix} 12 & 60 & -12 & 60 \\ 60 & 400 & -60 & 200 \\ -12 & -60 & 12 & -60 \\ 60 & 200 & -60 & 400 \end{pmatrix}$$

Applying the boundary conditions that the only non-zero degrees of freedom are the vertical deflection and rotation at B (u_{2B} and θ_{3B}), the mass and stiffness matrices are reduced to (2×2) matrices

$$\mathbf{M} = 1.071\times10^{-4}\begin{pmatrix} 156 & -220 \\ -220 & 400 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 432 & -2160 \\ -2160 & 14400 \end{pmatrix}$$



$\therefore |\mathbf{K} - \omega_n^2\,\mathbf{M}| = 0 \Rightarrow (432 - \omega_n^2\, 0.01671)\,(14400 - \omega_n^2\, 0.04286) - (-2160 + \omega_n^2\, 0.02357)^2 = 0$

$\Rightarrow \omega_n = 99.92$ rad/sec, 984.49 rad/sec

The exact results for the first two natural frequencies are $\omega_{n1} = (3.516/L^2)\sqrt{(EI/m)} = 99.45$ rad/sec and $\omega_{n2} = (22.03/L^2)\sqrt{(EI/m)} = 623.10$ rad/sec. Therefore, the natural frequencies are over-estimated as was the case for the axial vibrations. The first natural frequency is only 0.5% higher than the analytical value, while the second natural frequency is more than 50% over-estimated.

Dynamic Analysis of Trusses

Two-dimensional Trusses

The mass and stiffness matrices derived for axially loaded members can be used for the dynamic analysis of two-dimensional trusses. The elements of the i^{th} row and j^{th} column of the mass and stiffness matrices are given by Eq. (13.9) in integral form and can be evaluated once the shape functions ϕ_i and ϕ_j are known or assumed, as shown in Eq. (13.8). One difference is that here the transverse displacements (u_{2A}, u_{2B}) are also considered in forming the matrices, so that the size of the matrices is (4×4) instead of (2×2).

If shape functions of Eq. (13.8) are assumed for truss members of uniform cross-section, the member (denoted by subscript **m**) mass and stiffness matrices take the following forms in the local (denoted by superscript **L**) axes system

$$\mathbf{M_m^L} = \begin{pmatrix} mL/3 & 0 & mL/6 & 0 \\ 0 & 0 & 0 & 0 \\ mL/6 & 0 & mL/3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K_m^L} = \begin{pmatrix} EA/L & 0 & -EA/L & 0 \\ 0 & 0 & 0 & 0 \\ -EA/L & 0 & EA/L & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{……..(14.1)}$$

The member matrices formed in the local axes system by Eq. (19.1) can be transformed into the global axes system by considering the angles they make with the horizontal. A transformation matrix, $\mathbf{T_m}$ [shown in Eq. (14.2)] is used for to represent the relation between the local vectors (e.g., displacement, velocity, acceleration, force) and global vectors (Fig. 19.1).

The local vectors and global vectors are related by the following equations.

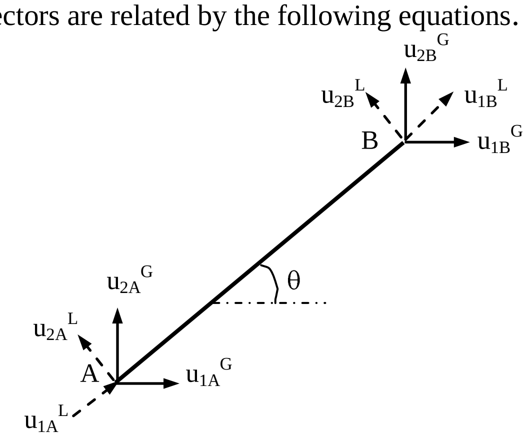


Fig. 19.1: Local and global joint displacements of a truss member

$$\mathbf{T_m} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{pmatrix} \quad \text{………....(14.2)}$$

where C = cosθ, and S = sinθ

This matrix can be used with further mathematical manipulations to obtain the global mass matrix and global stiffness matrix.

$$\mathbf{M_m^G = T_m^T\, M_m^L\, T_m} \quad \text{………………..(14.3)}$$
$$\mathbf{K_m^G = T_m^T\, K_m^L\, T_m} \quad \text{………………..(14.4)}$$
$$\mathbf{f_m^G = T_m^T\, f_m^L} \quad \text{………………..(14.5)}$$

where $\mathbf{T_m^T}$ is the transpose of the transformation matrix $\mathbf{T_m}$.

If the structural member is uniform (i.e., same area A and unit mass m) throughout its length, the calculations of Eqs. (14.3) and (14.4) can be carried out explicitly. This gives the following $\mathbf{M_m^G}$ and $\mathbf{K_m^G}$ matrices

$$\mathbf{M_m}^{\mathbf{G}} = (mL/3)\begin{pmatrix} \vartheta & \vartheta/2 \\ \vartheta/2 & \vartheta \end{pmatrix} \qquad \mathbf{K_m}^{\mathbf{G}} = (EA/L)\begin{pmatrix} \vartheta & -\vartheta \\ -\vartheta & \vartheta \end{pmatrix} \quad \text{…….(14.6)}$$

where ϑ is a (2×2) matrix of coefficients given by

$$\vartheta = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \quad \text{………………..(14.7)}$$

The mass and stiffness matrices and load vector of the whole structure can be assembled from the member matrices and vector ($\mathbf{M_m}^{\mathbf{G}}$, $\mathbf{K_m}^{\mathbf{G}}$ and $\mathbf{f_m}^{\mathbf{G}}$). However, they are obtained in their final forms only after applying appropriate boundary conditions.

Denoting the global structural matrices by **M** and **K** respectively and assuming appropriate damping ratios, the damping matrix **C** can be obtained as,

$$\mathbf{C} = a_0\,\mathbf{M} + a_1\,\mathbf{K} \quad \text{………………..(10.8)}$$

The dynamic analysis can be carried out once these matrices and vector are formed.

Three-dimensional Trusses

The formulation of mass and stiffness matrices for the dynamic analysis of three-dimensional trusses is very similar to the formulations for two-dimensional trusses discussed before. Since the joint displacements in the x, y and z axes (u_{1A}, u_{2A}, u_{3A}, u_{1B}, u_{2B}, u_{3B}) are considered in forming the matrices, the size of the matrices is (6×6).

For truss members of uniform cross-section, the member mass and stiffness matrices take the following forms in the local axes system

$$\mathbf{M_m}^{\mathbf{L}} = \begin{pmatrix} mL/3 & 0 & 0 & mL/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ mL/6 & 0 & 0 & mL/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{K_m}^{\mathbf{L}} = \begin{pmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{...…...(14.8)}$$

After making the transformation from local to global axes system with an appropriate transformation matrix $\mathbf{T_m}$ and using Eqs. (14.3)-(14.5), the following global mass matrix and global stiffness matrix are obtained for a uniform structural member.

$$\mathbf{M_m}^{\mathbf{G}} = (mL/3)\begin{pmatrix} \vartheta & \vartheta/2 \\ \vartheta/2 & \vartheta \end{pmatrix} \qquad \mathbf{K_m}^{\mathbf{G}} = (EA/L)\begin{pmatrix} \vartheta & -\vartheta \\ -\vartheta & \vartheta \end{pmatrix} \quad \text{…....(14.9)}$$

They are identical to the forms given by Eq. (14.6), but here ϑ is a (3×3) matrix of coefficients given by

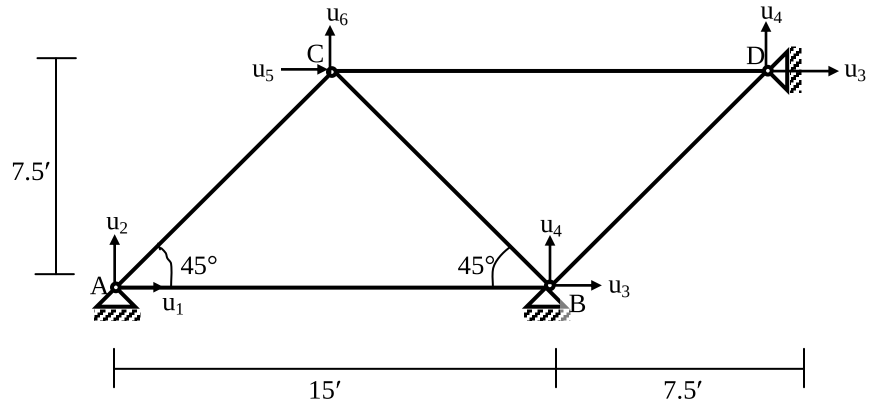
$$\vartheta = \begin{pmatrix} C_x^2 & C_xC_y & C_xC_z \\ C_yC_x & C_y^2 & C_yC_z \\ C_zC_x & C_zC_y & C_z^2 \end{pmatrix} \quad \text{……..………..(14.10)}$$

C_x, C_y and C_z are the direction cosines of the member in the x, y and z axes respectively. The subsequent matrix assembly, setting boundary conditions, forming **C** matrix and carrying out the numerical integration follow the usual procedures mentioned for 2D trusses.

Example 14.1

For the truss shown below, modulus of elasticity E = 30000 ksi, cross-sectional area A = 2 in^2, mass per unit length m = 1.5×10^{-6} k-sec^2/in^2 for each member. Calculate its approximate natural frequencies.

Solution



The truss has 4 joints, therefore 8 DOF. The displacements u_1~u_4 and u_7, u_8 are restrained, so that only two DOF (u_5, u_6) can possibly be non-zero. There are five members in the truss, all with the same cross-sectional properties, but different lengths. The member mass and stiffness matrices can be obtained from

$$\mathbf{M_m}^{\mathbf{G}} = (mL/3)\begin{pmatrix} C^2 & CS & C^2/2 & CS/2 \\ CS & S^2 & CS/2 & S^2/2 \\ C^2/2 & CS/2 & C^2 & CS \\ CS/2 & S^2/2 & CS & S^2 \end{pmatrix} \qquad \mathbf{K_m}^{\mathbf{G}} = (EA/L)\begin{pmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{pmatrix}$$

For member AB, C = 1, S = 0, L = 15′ = 180″, $\therefore$ mL/3 = 9.0×10^{-5} k-sec^2/in, EA/L = 333.33 k/in

$$\mathbf{M_{AB}}^{\mathbf{G}} = 9.0\times10^{-5}\begin{pmatrix} 1.0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K_{AB}}^{\mathbf{G}} = 333.33\begin{pmatrix} 1.0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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The matrices for AB and CD are the same, but the latter connects displacements 5, 6, 7 and 8

For member AC, C = 0.707, S = 0.707, L = 10.607′ = 127.28″

$\therefore$ mL/3 = 6.37×10^{-5} k-sec^2/in, EA/L = 471.41 k/in

$$\mathbf{M_{AC}}^{\mathbf{G}} = 6.37\times10^{-5}\begin{pmatrix} 0.5 & 0.5 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 & 0.5 \end{pmatrix} \qquad \mathbf{K_{AC}}^{\mathbf{G}} = 471.41\begin{pmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{pmatrix}$$

[1 2 5 6]

The matrices for AC and BD are the same, but the latter connects displacements 3, 4, 7 and 8

For member BC, C = -0.707, S = 0.707, L = 10.607′ = 127.28″

$\therefore$ mL/3 = 6.37×10^{-5} k-sec^2/in, EA/L = 471.41 k/in

$$\mathbf{M_{BC}}^{\mathbf{G}} = 6.37\times10^{-5}\begin{pmatrix} 0.5 & -0.5 & 0.25 & -0.25 \\ -0.5 & 0.5 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.5 & -0.5 \\ 0.25 & -0.25 & -0.5 & 0.5 \end{pmatrix} \qquad \mathbf{K_{BC}}^{\mathbf{G}} = 471.41\begin{pmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix}$$

[3 4 5 6]

The mass and stiffness matrices for the whole truss can be assembled from the member matrices. In assembling, the elements denoted by corresponding row and column number will be added. After applying boundary conditions, only the elements corresponding to the 5th and 6th DOF will remain, and the matrices will be reduced to (2×2) of the form

$$\mathbf{M} = 10^{-5} \begin{pmatrix} 15.37 & 0 \\ 0 & 6.37 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 804.74 & 0 \\ 0 & 471.41 \end{pmatrix}$$

$\therefore |\, \mathbf{K} - \omega_n^2\, \mathbf{M}\, | = 0 \Rightarrow (804.74 - \omega_n^2\, 0.0001537)\,(471.41 - \omega_n^2\, 0.0000637) = 0$

$\Rightarrow \omega_n = 2288$ rad/sec, 2720 rad/sec

Dynamic Analysis of Frames

Two-dimensional Frames

The matrices formed for flexural members and already used for a cantilever beam can be used for the dynamic analysis of two-dimensional frames. The elements of the i^{th} row and j^{th} column of the mass and stiffness matrices are given by Eq. (13.12) in integral form and can be evaluated once the shape functions ψ_i and ψ_j are known or assumed [as shown in Eq. (13.11)]. However, the axial displacements of joints (u_{1A}, u_{1B}) are also considered for frames in addition to the transverse displacements (u_{2A}, u_{2B}) and rotations (θ_{3A}, θ_{3B}) about the out-of-plane axis considered in forming the matrices for beams, so that the size of the matrices is (6×6) instead of the (4×4) matrices shown for beams.

If shape functions of Eq. (13.11) are assumed for frame members of uniform cross-section, the member mass and stiffness matrices take the following forms in the local axes system

$$\mathbf{M_m}^{\mathbf{L}} = (mL/420)\begin{pmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{pmatrix} \qquad \mathbf{K_m}^{\mathbf{L}} = \begin{pmatrix} S_x & 0 & 0 & -S_x & 0 & 0 \\ 0 & S_1 & S_2 & 0 & -S_1 & S_2 \\ 0 & S_2 & S_3 & 0 & -S_2 & S_4 \\ -S_x & 0 & 0 & S_x & 0 & 0 \\ 0 & -S_1 & -S_2 & 0 & S_1 & -S_2 \\ 0 & S_2 & S_4 & 0 & -S_2 & S_3 \end{pmatrix} \quad \text{……..(15.1)}$$

where $S_x = EA/L$, $S_1 = 12EI/L^3$, $S_2 = 6EI/L^2$, $S_3 = 4EI/L$, $S_4 = 2EI/L$

The member matrices formed in the local axes system by Eq. (15.1) can be transformed into the global axes system by considering the angles they make with the horizontal. A transformation matrix, $\mathbf{T_m}$ [shown in Eq. (15.2)] is used for to represent the relation between the local vectors (e.g., displacement, velocity, acceleration, force) and global vectors (Fig. 15.1). The transformation matrix is also (6×6) instead of the (4×4) matrix for truss members.

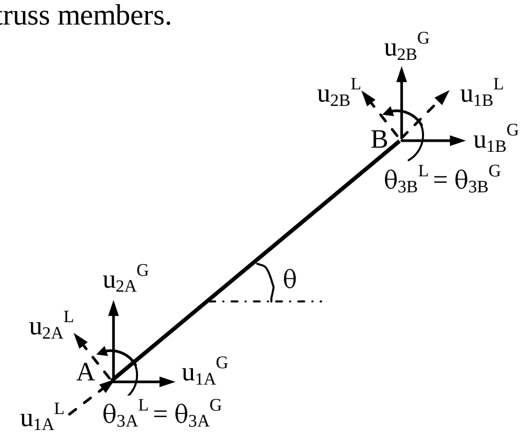


Fig. 15.1: Local and global joint displacements of a frame member

$$\mathbf{T_m} = \begin{pmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{[where } C = \cos\theta,\ S = \sin\theta\text{]} \quad \text{..……....(15.2)}$$

This matrix can be used with further mathematical manipulations to obtain the global mass matrix and global stiffness matrix as was the case with truss members [Given by Eqs. (14.3)~(14.5)]. If the structural member is uniform (i.e., same area A and unit mass m) throughout its length, the calculations of Eqs. (14.3) and (14.4) can again be carried out explicitly. However, the explicit expressions are not shown here because they are not as convenient to write as were the matrices for truss members.

The mass and stiffness matrices and load vector of the whole structure can be assembled from the member matrices and vector ($\mathbf{M_m}^{\mathbf{G}}$, $\mathbf{K_m}^{\mathbf{G}}$ and $\mathbf{f_m}^{\mathbf{G}}$) and obtained in their final forms (along with matrix $\mathbf{C}$) after applying appropriate boundary conditions.

Three-dimensional Frames

The formulation of mass and stiffness matrices for the dynamic analysis of three-dimensional frames follows the same procedure as the formulation for two-dimensional frames discussed before. Since the joint displacements and rotations in the x, y and z-axis (u_{1A}, u_{2A}, u_{3A}, θ_{1A}, θ_{2A}, θ_{3A}, u_{1B}, u_{2B}, u_{3B}, θ_{1B}, θ_{2B}, θ_{3B}) are considered in forming the matrices, the size of the matrices is (12×12).

A new feature of the displacements of three-dimensional frame element is the presence of torsional rotations θ_{1A} and θ_{1B}. In addition to the biaxial transverse displacements (u_{2A}, u_{3A}, u_{2B} and u_{3B}) and rotations (θ_{2A}, θ_{3A}, θ_{2B} and θ_{3B}), they add to the complications in solving the three-dimensional frame problem. The member mass and stiffness matrices in the local axes system are shown in Table 15.1 and Table 15.2 respectively. The elements in Table 15.1 have a multiplying factor (mL/420) with them.

Table 15.1: Elements of member Mass Matrix in the local axis system

(mL/420)

u_{1A}	u_{2A}	u_{3A}	θ_{1A}	θ_{2A}	θ_{3A}	u_{1B}	u_{2B}	u_{3B}	θ_{1B}	θ_{2B}	θ_{3B}
140						70					
	156				22L		54				-13L
		156		22L				54		-13L	
			$140r^2$						$70r^2$		
		22L		$4L^2$				13L		$-3L^2$	
	22L				$4L^2$		13L				$-3L^2$
70						140					
	54				13L		156				-22L
		54		13L				156		-22L	
			$70r^2$						$140r^2$		
		-13L		$-3L^2$				-22L		$4L^2$	
	-13L				$-3L^2$		-22L				$4L^2$

[where r is the polar radius of gyration of the cross-section]

Table 15.2: Elements of member Stiffness Matrix in the local axis system

u_{1A}	u_{2A}	u_{3A}	θ_{1A}	θ_{2A}	θ_{3A}	u_{1B}	u_{2B}	u_{3B}	θ_{1B}	θ_{2B}	θ_{3B}
S_x						$-S_x$					
	S_{1z}				S_{2z}		$-S_{1z}$				S_{2z}
		S_{1y}		$-S_{2y}$				$-S_{1y}$		$-S_{2y}$	
			T_x						$-T_x$		
		$-S_{2y}$		S_{3y}				S_{2y}		S_{4y}	
	S_{2z}				S_{3z}		$-S_{2z}$				S_{4z}
$-S_x$						S_x					
	$-S_{1z}$				$-S_{2z}$		S_{1z}				$-S_{2z}$
		$-S_{1y}$		S_{2y}				S_{1y}		S_{2y}	
			$-T_x$						T_x		
		$-S_{2y}$		S_{4y}				S_{2y}		S_{3y}	
	S_{2z}				S_{4z}		$-S_{2z}$				S_{3z}

[where $S_x = EA/L$, $S_{1z} = 12EI_z/L^3$, $S_{2z} = 6EI_z/L^2$, $S_{3z} = 4EI_z/L$, $S_{4z} = 2EI_z/L$,
$T_x = GJ/L$, $S_{1y} = 12EI_y/L^3$, $S_{2y} = 6EI_y/L^2$, $S_{3y} = 4EI_y/L$, $S_{4z} = 2EI_y/L$]

The transformation matrix for 3D frames is quite complicated and is not shown here. It can be derived by three-dimensional vector algebra or by applying the axes rotations of the global axis system one by one.

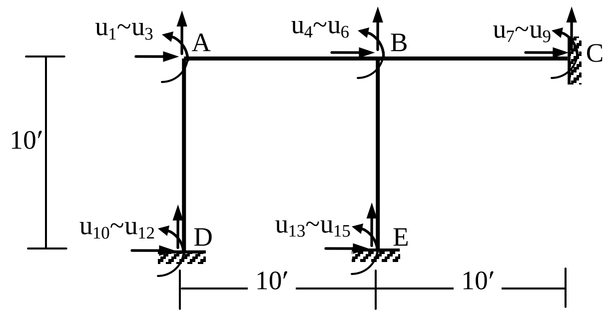
After making the transformation from local to global axes system with an appropriate transformation matrix $\mathbf{T_m}$ and using Eqs. (14.3)~(14.5), the global mass matrix and global stiffness matrix are obtained for a three-dimensional frame member.

The subsequent matrix assembly, setting boundary conditions, forming **C** matrix and carrying out the numerical integration follow the usual procedures.

Example 15.1

For the 2-dimensional concrete frame structure shown below, modulus of elasticity E = 450000 ksf, cross-sectional area A = 1 ft^2, moment of inertia I = 0.08 ft^2, mass per unit length m = 0.045 k-sec^2/ft^2 for all the members. Formulate the mass matrix and stiffness matrix of the frame.

Solution



The frame has 5 joints, therefore 15 DOF.
The displacements u_7~u_{15} are restrained, so that only six DOF (u_1~u_6) can possibly be non-zero. There are four members in the frame, all with the same cross-sectional properties and lengths.

The member mass and stiffness matrices are

$$\mathbf{M_{AB}}^{\mathbf{G}} = 1.071\times10^{-4}\begin{pmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 220 & 0 & 54 & -130 \\ 0 & 220 & 400 & 0 & 130 & -300 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 130 & 0 & 156 & -220 \\ 0 & -130 & -300 & 0 & -220 & 400 \end{pmatrix} \qquad \mathbf{K_{AB}}^{\mathbf{G}} = \begin{pmatrix} 45000 & 0 & 0 & -45000 & 0 & 0 \\ 0 & 432 & 2160 & 0 & -432 & 2160 \\ 0 & 2160 & 14400 & 0 & -2160 & 7200 \\ -45000 & 0 & 0 & 45000 & 0 & 0 \\ 0 & -432 & -2160 & 0 & 432 & -2160 \\ 0 & 2160 & 7200 & 0 & -2160 & 14400 \end{pmatrix}$$

DOF [1 2 3 4 5 6] DOF [1 2 3 4 5 6]

The same matrices for BC, with DOF u_4~u_9

$$\mathbf{M_{AD}}^{\mathbf{G}} = 1.071\times10^{-4}\begin{pmatrix} 156 & 0 & 220 & 54 & 0 & -130 \\ 0 & 140 & 0 & 0 & 70 & 0 \\ 0 & 220 & 400 & 0 & -130 & -300 \\ 54 & 0 & -130 & 156 & 0 & -220 \\ 0 & 70 & 0 & 0 & 140 & 0 \\ 0 & -130 & -300 & 0 & -220 & 400 \end{pmatrix} \qquad \mathbf{K_{AD}}^{\mathbf{G}} = \begin{pmatrix} 432 & 0 & 2160 & -432 & 0 & 2160 \\ 0 & 45000 & 0 & 0 & -45000 & 0 \\ 0 & 2160 & 14400 & 0 & -2160 & 7200 \\ -432 & 0 & -2160 & 432 & 0 & -2160 \\ 0 & -45000 & 0 & 0 & 45000 & 0 \\ 0 & 2160 & 7200 & 0 & -2160 & 14400 \end{pmatrix}$$

DOF [1 2 3 10 11 12] DOF [1 2 3 10 11 12]

The same matrices for BE, with DOF u_4~u_{15}.
The structural mass and stiffness matrices (15×15) can be assembled from the member matrices by locating the elements at appropriate rows and columns. If only six DOF (u_1~u_6) are active, the final mass and stiffness matrices become

$$\mathbf{M} = 1.071\times10^{-4}\begin{pmatrix} 296 & 0 & 220 & 70 & 0 & 0 \\ 0 & 296 & 220 & 0 & 54 & -130 \\ 220 & 220 & 800 & 0 & 130 & -300 \\ 70 & 0 & 0 & 436 & 0 & 220 \\ 0 & 54 & 130 & 0 & 452 & 0 \\ 0 & -130 & -300 & 220 & 0 & 1200 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 45432 & 0 & 2160 & -45000 & 0 & 0 \\ 0 & 45432 & 2160 & 0 & -432 & 2160 \\ 2160 & 2160 & 28800 & 0 & -2160 & 7200 \\ -45000 & 0 & 0 & 90432 & 0 & 2160 \\ 0 & -432 & -2160 & 0 & 45864 & 0 \\ 0 & 2160 & 7200 & 2160 & 0 & 43200 \end{pmatrix}$$

DOF [1 2 3 4 5 6] DOF [1 2 3 4 5 6]

If axial deformations are neglected, the matrices reduce to (2×2) corresponding to 2 DOF only (u_3 and u_6). In that case, the matrices are

$$\mathbf{M} = 1.071\begin{pmatrix} 0.08 & -0.03 \\ -0.03 & 0.12 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 28800 & 7200 \\ 7200 & 43200 \end{pmatrix}$$

$\therefore |\,\mathbf{K}\text{-}\omega_n^2\,\mathbf{M}\,| = 0 \Rightarrow (28800 - \omega_n^2\, 0.0857)\,(43200 - \omega_n^2\, 0.1286) - (7200 + \omega_n^2\, 0.0321)^2 = 0$
$\Rightarrow \omega_n$ = 453 rad/sec, 763 rad/sec

Effect of Material and Geometric Nonlinearity

In many practical situations, the structural properties cannot be assumed to remain constant; e.g., yielding of the structural materials, a likely situation in a severe earthquake, may alter the stiffness properties. Other possibilities include changes in the geometric stiffness of members due to significant axial forces, the mass or damping may undergo changes during dynamic response.

Structural Analysis of Linearly Elastic and Nonlinear Dynamic Systems

For a linearly elastic system the relationship between the applied force f_s and the resulting deformation u is linear, that is, $f_s = k\,u$ …………...(16.1)

where k is the uniform stiffness of the system. This is however not valid when the load-deformation relationship is nonlinear, i.e., when the stiffness itself is not constant but a function of u. Moreover, if a structural component undergoes cyclic deformation (due to dynamic excitation) and the initial loading curve is nonlinear at the large amplitudes of deformation, the unloading and reloading curves differ from the initial loading branch. This implies that the force f_s corresponding to deformation u is not single valued and depends on the history of the deformations and whether the deformation is increasing (positive velocity) or decreasing (negative velocity). Thus the resisting force can be expressed as

$$f_s = f_s(u,v) \qquad \text{..……………………....(16.2)}$$

Following are some significant differences between the analyses of linear and nonlinear systems.

(1) For a linearly elastic system, the total forces can be determined by combining the results of two separate analyses. However, such direct superposition is not valid for nonlinear systems.
(2) One cannot correctly predict the type of failure by linear analysis; e.g., whether the failure is caused by shear, which is only possible by nonlinear analysis.
(3) As the sectional and structural properties depend on the deformations, nonlinear analysis is only done by iteration method. Such iteration schemes are computationally demanding and not guaranteed to converge unless a suitable numerical scheme is chosen.

Solution of the governing Equations of Motion by Incremental time-step Integration

For an inelastic SDOF system the equation of motion to be solved numerically is

$$m\,a + c\,v + f_s(u,v) = f(t) \qquad \text{………………..(16.3)}$$

subject to specified initial conditions. Since the structural properties depend on the value of displacement and velocity, it is more convenient to assume the equation in incremental form, i.e.,

$$m\,\Delta a_i + c\,\Delta v_i + (\Delta f_s)_i = \Delta f_i \qquad \text{…….………(16.4)}$$

The incremental resisting force, $(\Delta f_s)_i = (k_i)_{sec}\,\Delta u_i$ ……...………(16.5)

where the secant stiffness $(k_i)_{sec}$, as shown in Fig. 16.1, cannot be determined because u_{i+1} is not known. Making the assumption that over a small time step Δt, the secant stiffness $(k_i)_{sec}$ could be replaced by the tangent stiffness $(k_i)_T$ shown in Fig. 16.1, Eq. 16.5 is approximated by

$$(\Delta f_s)_i = (k_i)_T\,\Delta u_i \qquad \text{…….………(16.6)}$$

Dropping the subscript T gives, $m\,\Delta a_i + c\,\Delta v_i + (k_i)\,\Delta u_i = \Delta f_i$ …….………(16.7)

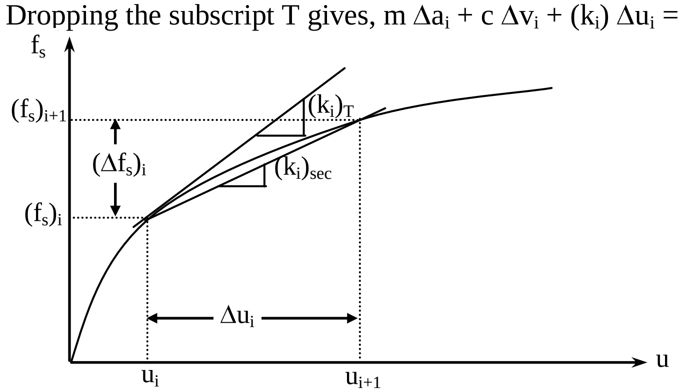


Fig. 16.1: Force-displacement relationship

The similarity between this equation and the corresponding equation for linear systems suggests that the non-iterative formulation used for linear systems may also be used in the analysis of nonlinear response, replacing k by the tangent stiffness k_i to be evaluated at the beginning of each time step.

Effect of Material Nonlinearity

Fig. 16.2(a) shows the properties on an undamped SDOF system with m = 1 k-sec^2/ft, c = 0, if the system is nonlinear elastic with stiffness k_T = 25 k/ft, when $|u| \leq u_y$, and = 0 if $|u| \geq u_y$. Fig. 16.2(b) shows its dynamic displacements for a step loading with p_0 = 25 k. Four cases are considered here for study according to increasing nonlinearity; the first results are for the linear system, while the other three are for nonlinear systems with 'yield displacement' u_y = 1.8, 1.5 and 1.2 ft respectively.

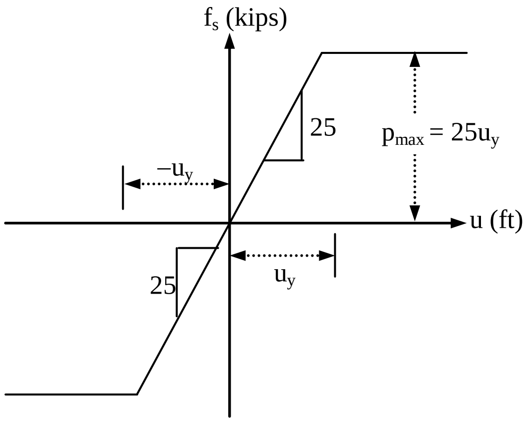


Fig. 16.2(a): Nonlinear Elastic Spring property

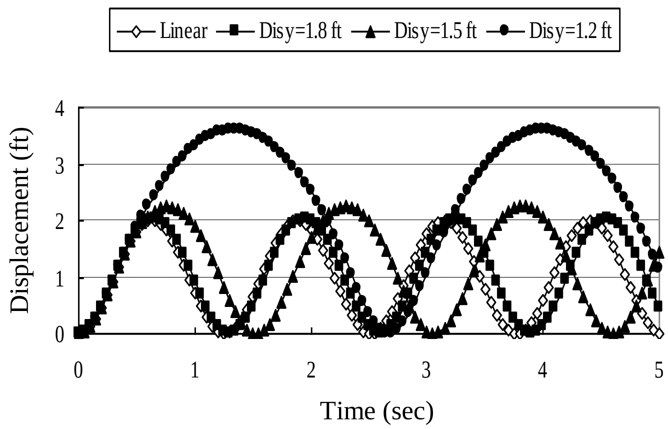


Fig. 16.2(b): Response of Nonlinear SDOF System

The main observations from these results are

(1) The displacements increase for the more nonlinear systems because of the reduced 'equivalent stiffness'. The maximum displacements for the four systems are 2.0, 2.02, 2.25 and 3.60 ft. The nonlinearity is not prominent in the second and third cases since it only started beyond 1.8 and 1.5 ft respectively, out maximum displacements of around 2.0′. But it is pronounced for the fourth case.

(2) However the respective maximum forces in the spring in the four cases are 50, 45, 37.5 and 30 kips. It shows that despite the magnified displacements, the forces are reduced due to material nonlinearity.

(3) Natural period has little meaning for a nonlinear system. However the dynamic responses still indicate periodicity, which can be interpreted in terms of natural frequency or period. The natural period of the linear system was 1.257 sec, which increases to 1.30, 1.52 and 2.67 sec for the three nonlinear systems due to the decreased stiffness of the system.

(4) Since the material is nonliearly elastic (and not inelastic) and does not fail, the complete implication of nonliearity (i.e., residual displacement, structural failure) cannot be shown.

More realistic conclusions can be drawn from Figs. 16.3(a) and 16.3(b), which show the shear forces (obtained from linear and nonlinear dynamic analysis respectively) in a first floor beam of a two-dimensional 5-storied RC building subjected to the Kobe earthquake ground motion. In addition to the fact that the nonlinear forces are smaller overall, material nonlinearity leaves the beam with a residual shear force (due to permanent deformation) at the end of the earthquake motion.

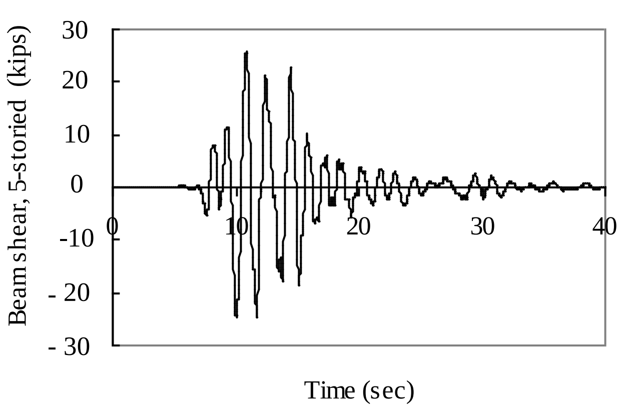


Fig. 16.3(a): 2D linear Kobe response

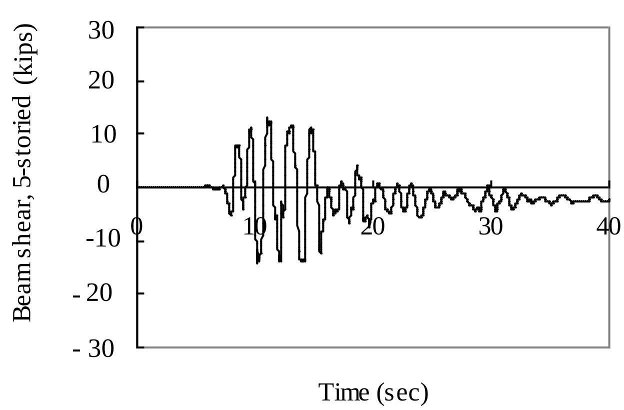


Fig. 16.3(b): 2D nonlinear Kobe response

Geometric Nonlinearity

Geometric nonlinearity is the change in the elastic load-deformation characteristics of the structure caused by the change in the structural shape due to large deformations. Among the various types of geometric nonlinearity in typical engineering structures, structural instability or moment magnification due to large compressive forces, stiffening of structures due to large tensile forces, change in structural parameters due to applied loads (e.g., leading to changed damping or parametric resonance) are significant.

Buckling of Beams-columns and Two-dimensional Frames

Buckling of columns and magnification of the internal forces obtained by linear analysis are among the most important types of geometric nonlinearities in typical building structures. The Euler formulation for buckling of perfectly straight, axially loaded linearly elastic column is well known, as well as its extensions considering different support conditions, column imperfection, load eccentricity, material nonlinearity as well as residual stresses in steel. Other than buckling, another significant effect of compressive load is the magnification of internal stresses calculated from linear (1^{st} order) analysis due to the added deformations from geometric nonliearity, an effect called moment magnification.

For structural analysis the effect of axial load on flexural behavior can be approximated by simplified formulations of the geometric nonlinearity problem. For this purpose, a new matrix called the geometric stiffness matrix (**G**) has been added to the original stiffness matrix **K** obtained from linear analysis of the undeformed deflected shape of the structure. Therefore, the total stiffness matrix of a flexural member is

$$\mathbf{K_{total}} = \mathbf{K} + \mathbf{G} \qquad \text{..........................(16.8)}$$

For beams and columns, the geometric stiffness matrix can be obtained from

$$G_{mij} = P\int \psi_i' \, \psi_j' \, dx \qquad \text{..........................(16.9)}$$

where P is the tensile force on the member. For compressive force, P will be negative. Using the same shape functions ψ_i (i = 1~4) as done for the linear analyses of beams and frames, the following geometric stiffness matrix is formed in the local axes system of a member of length L.

$$\mathbf{G_m^L} = (P/30L) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3L & 0 & -36 & 3L \\ 0 & 3L & 4L^2 & 0 & -3L & -L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3L & 0 & 36 & -3L \\ 0 & 3L & -L^2 & 0 & -3L & 4L^2 \end{pmatrix} \qquad \text{.......................(16.10)}$$

However it should be noted that the presence of axial force P in the geometric stiffness matrix makes the problem nonlinear because P is obtained from member deformations, which cannot be found (other than in special cases) before performing the structural analysis.

Buckling occurs when the structure loses its stiffness, i.e., when the total stiffness matrix $\mathbf{K_{total}}$ becomes singular. Therefore, the buckling load can be obtained by solving the eigenvalue problem

$$|\mathbf{K_{total}}| = 0 \Rightarrow |\mathbf{K}+\mathbf{G}| = 0 \qquad \text{.........................(16.11)}$$

If a beam is modeled as a SDOF system with an assumed deflected shape given by $\psi(x)$, its geometric stiffness using the Rayleigh-Ritz method is

$$g^* = \int P[\psi'(x)]^2 \, dx \qquad \text{.........................(16.12)}$$

so that the governing equation of motion can be rewritten as,

$$m^* \, d^2u_2/dt^2 + c^* \, du_2/dt + (k^*+g^*) \, u_2 = f^*(t) \qquad \text{.........................(16.13)}$$

Eq. (16.11) shows that the stiffness (k*+g*) of the SDOF system vanishes when P reaches the buckling load of the column. Therefore, the natural frequency of the structure is also zero; i.e., the natural periods are infinity. In other words, the structure loses its ability to oscillate and therefore its deformations tend to diverge in a direction. This eventually leads to structural failure if material nonlinearity is also considered.

Example 16.1

For the member properties mentioned in Example 12.1 ($E = 450000$ ksf, $I = 0.08$ ft^4, $L = 10$ ft, $m = 0.0045$ k-sec^2/ft^2, damping ratio $\xi = 0$), calculate

(i) The approximate first buckling load for a cantilever beam,

(ii) The natural frequency and dynamic response of the beam when subjected to a uniformly distributed transverse load of 1 k/ft with a compressive load of 400 kips.

Solution

(i) For the cantilever beam, results from Example 12.2 show that assuming $\psi(x) = 1-\cos(\pi x/2L)$,
$m^* = 0.2268\ mL = 0.0102$ k-sec^2/ft, $k^* = 3.044\ EI/L^3 = 109.59$ k/ft
Also, $g^* = \int P\ [\psi'(x)]^2\ dx = P\ \pi^2/8L = P\ (1.2337/L)$ k/ft
$k^*+g^* = 0 \Rightarrow 3.044\ EI/L^3 + P_{cr}\ (1.2337/L) = 0$
$\Rightarrow P_{cr} = -2.467\ EI/L^2 = -888.25$ kips; i.e., a compressive load of 888.25 kips.
This is in fact the exact buckling load of the beam, because the assumed deflected shape is the first buckling mode for cantilever beams.

(ii) Natural frequency without considering g^*, $\omega_n = 103.63$ rad/sec, as found in Example 12.2
For $P = -400$ kips (i.e., compressive load of 400 kips), $g^* = P\ (1.2337/L) = -49.35$ k/ft
$\therefore k_{Total} = k^*+g^* = 109.59 - 49.35 = 60.24$ k/ft
Changed natural frequency of the system, $\omega_n = \sqrt{\{(k^*+g^*)/m^*\}} = \sqrt{(60.24/0.0102)}$
$= 76.85$ rad/sec, which is about 26% smaller than the original natural frequency.

The effective force $f^* = \int q(x,t)\ \psi(x)\ dx = qL(0.363) = 3.63$ kips
$\therefore$ Equation for flexural deformation is, $0.0102\ d^2u_2/dt^2 + 60.24\ u_2 = 3.63$
$\Rightarrow u_2(t) = 0.0603\ [1-\cos(76.85\ t)]$, which reaches a peak value of 0.1206 ft, instead of the maximum for the original (linear) case of $2\times3.63/109.59 = 0.0663$ ft (Fig. 16.4).

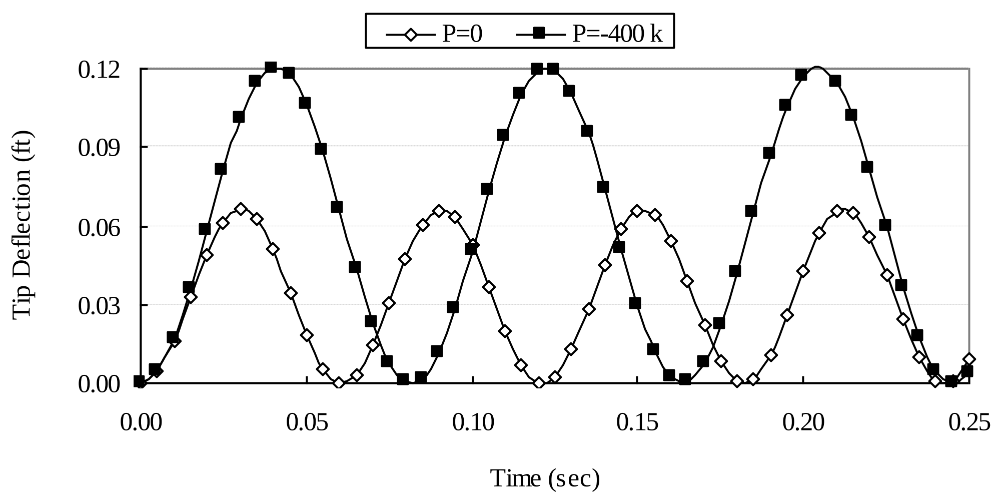
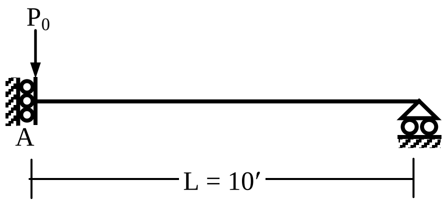
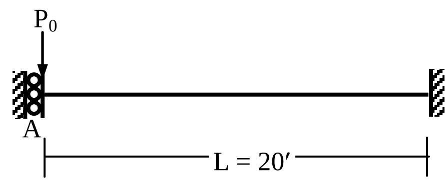


Fig. 16.4: Dynamic Response of Cantilever Beam

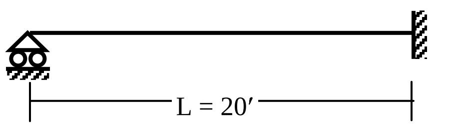
Problems on the Dynamic Analysis of Continuous Systems

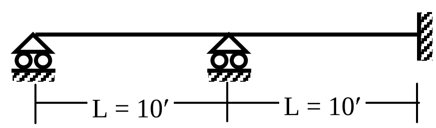
1. For the undamped beams shown below
   (a) choose an appropriate shape function (satisfying the essential boundary conditions) among (i) $\psi(x) = \cos(\pi x/2L)$, (ii) $\psi(x) = [1 + \cos(\pi x/L)]/2$ and (iii) $\psi(x) = \sin(\pi x/L)$
   (b) use the chosen shape function to calculate their natural frequencies [Given: $EI = 40 \times 10^6$ lb-ft^2, mass per unit length $m = 5$ lb-sec^2/ft^2].
   (c) use the chosen shape function to calculate the deflections at A if $P_0 = 10 \cos (100t)$

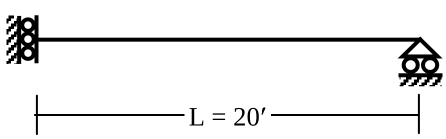


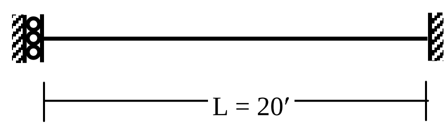


2. For the member properties mentioned in Example 12.1, (modulus of elasticity E = 450000 ksf, cross-sectional area A = 1 ft^2, mass per unit length m = 0.0045 k-sec^2/ft^2) calculate the approximate natural frequencies of the beams shown below for (i) axial vibration, (ii) transverse vibration.

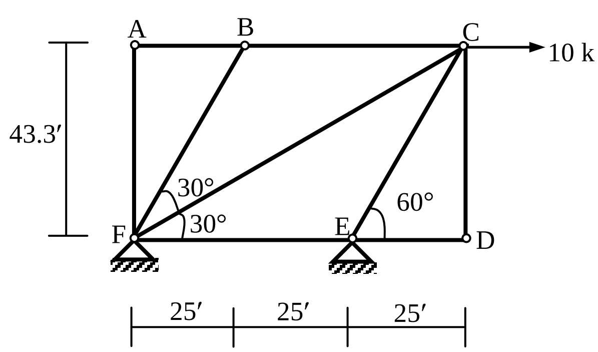








3. For the truss shown below, modulus of elasticity E = 30000 ksi, cross-sectional area A = 2 in^2, mass per unit length m = 1.5×10^{-6} k-sec^2/in^2 for each member.
   Calculate its approximate natural frequencies.



4. For the beams shown in Question 1, calculate
   (i) the approximate first buckling load,
   (ii) the natural frequency and dynamic vibration of point A when subjected to a uniformly distributed transverse load of 1 k/ft with a compressive load of 400 kips (neglect P_0).

Nature of Earthquake Vibration

Earthquakes are one of the most powerful natural forces that can disrupt our daily lives. Few natural phenomena can wreak as much havoc as earthquakes. Over the centuries they have been responsible for millions of deaths and an incalculable amount of damage to property. While earthquakes have inspired dread and superstitious awe since ancient times, little was understood about them until the emergence of seismology at the beginning of the 20th century. Seismology, that involves scientific study of all aspects of earthquakes, has yielded answers to long-standing questions on why and how earthquakes occur.

Cause of Earthquake

According to the *Elastic Rebound Theory* (Reid 1906), earthquakes are caused by pieces of the crust of the earth that suddenly shift relative to each other. The most common cause of earthquakes is faulting. A fault is a break in the earth's crust along which movement occurs.

Most earthquakes occur in narrow belts along the boundaries of crustal plates, particularly where the plates push together or slide past each other. At times, the plates are locked together, unable to release the accumulating energy. When this energy grows strong enough, the plates break free. When two pieces that are next to each other get pushed in different directions, they will stick together for a long time (many years), but eventually the forces pushing on them will force them to break apart and move. This sudden shift in the rock shakes all of the ground around it.

Earthquake Terminology

The point beneath the earth's surface where the rocks break and move is called the *focus* of the earthquake. The focus is the underground point of origin of an earthquake. Directly above the focus, on earth's surface, is the *epicenter*. Earthquake waves reach the epicenter first. During an earthquake, the most violent shaking is found at the epicenter.

During earthquakes, the strain energy stored within the crustal plates is released through 'seismic waves'. There are three main types of seismic waves. *Primary or P-waves* vibrate particles along the direction of wave, *Secondary or S-waves* that vibrate particles perpendicular to the direction of wave and *Love wave or L-waves* move along the surface and cause maximum damage.

Earthquake Magnitude

Earthquakes range broadly in size. Modern seismographic systems precisely amplify and record ground motion (typically at periods of between 0.1 and 100 seconds) as a function of time. This amplification and recording as a function of time is the source of instrumental amplitude and arrival-time data on near and distant earthquakes. Based on these data, *Charles F. Richter (1935)* introduced the concept of earthquake magnitude. His original definition was

$$M_L = \log (A/A_0) \qquad \text{.....................(17.1)}$$

where A is the maximum trace amplitude in micrometers recorded on a standard short-period seismometer and A_0 is a standard value as a function of distance <= 600 kilometers. However it held only for California earthquakes occurring within 600 km of a particular type of seismograph. His basic idea was quite simple: by knowing the distance from a seismograph to an earthquake and observing the maximum signal amplitude recorded on the seismograph, an empirical quantitative ranking of the earthquake's inherent size or strength could be made. Most California earthquakes occur within the top 16 km of the crust; to a first approximation, corrections for variations in earthquake focal depth were, therefore, unnecessary.

Richter's original magnitude scale was then extended to observations of earthquakes of any distance and of focal depths ranging between 0 and 700 km. Because earthquakes excite both body waves, which travel into and through the Earth, and surface waves, which are constrained to follow the natural wave guide of the Earth's uppermost layers, two magnitude scales evolved; i.e., the m_b and M_S scales.

The standard body-wave magnitude formula is

$$m_b = \log_{10}(A/T) + Q(D,h) \qquad \text{.....................(17.2)}$$

where A is the amplitude of ground motion (in microns); T is the corresponding period (in seconds); and Q(D,h) is a correction factor that is a function of distance, D (degrees), between epicenter and station and focal depth, h (in kilometers), of the earthquake.

The standard surface-wave formula is

$$M_S = \log_{10} (A/T) + 1.66 \log_{10} (D) + 3.30 \qquad \text{………………..(17.3)}$$

There are many variations of these formulae (e.g., the Moment Magnitude M_W, the Energy Magnitude M_e) that take into account effects of specific geographic regions, so that the final computed magnitude is reasonably consistent with Richter's original definition of M_L.

The Modified Mercalli Intensity Scale

The Richter Scale is not used to express damage. An earthquake in a densely populated area which results in many deaths and considerable damage may have the same magnitude as a shock in a remote area that does nothing more than frighten the wildlife. Large-magnitude earthquakes that occur beneath the oceans may not even be felt by humans.

The intensity scale, on the other hand, is based mainly on the effects of earthquake rather than its magnitude. It consists of a series of certain key responses such as people awakening, movement of furniture, damage to chimneys and finally total destruction. Although numerous intensity scales have been developed to evaluate the effects of earthquakes, the one currently used most is the *Modified Mercalli (MM) Intensity Scale* (Wood and Neumann, 1931). This scale, composed of 12 increasing levels of intensity that range from imperceptible shaking (I: Not felt except by a very few under especially favorable conditions) to catastrophic destruction (XII: Damage total. Lines of sight and level are distorted. Objects thrown into the air), is designated by Roman numerals. It has no mathematical basis; instead it is an arbitrary ranking based on observed effects.

Frequency of Earthquakes Worldwide

A rough idea of frequency of occurrence of large earthquakes is given by the following tables (Table 17.1 and Table 17.2). These are collected from Internet sources as data reported by the National Earthquake Information Center (NEIC) of the United States Geological Survey (USGS).

Table 17.1: Frequency of Occurrence of Earthquakes (Based on Observations since 1900)

Descriptor	Magnitude	Average Annually
Great	8 and higher	1
Major	7 - 7.9	18
Strong	6 - 6.9	120
Moderate	5 - 5.9	800
Light	4 - 4.9	6,200 (estimated)
Minor	3 - 3.9	49,000 (estimated)
Very Minor	< 3.0	Magnitude 2 - 3: about 1,000 per day Magnitude 1 - 2: about 8,000 per day

Table 17.2: The Number of Earthquakes Worldwide for 1992 - 2001 (Located by the USGS-NEIC)

Magnitude	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
8.0 to 9.9	0	1	2	3	1	0	2	0	4	1
7.0 to 7.9	23	15	13	22	21	20	14	23	14	6
6.0 to 6.9	104	141	161	185	160	125	113	123	157	45
5.0 to 5.9	1541	1449	1542	1327	1223	1118	979	1106	1318	382
4.0 to 4.9	5196	5034	4544	8140	8794	7938	7303	7042	8114	2127
3.0 to 3.9	4643	4263	5000	5002	4869	4467	5945	5521	4741	1624
2.0 to 2.9	3068	5390	5369	3838	2388	2397	4091	4201	3728	1319
1.0 to 1.9	887	1177	779	645	295	388	805	715	1028	225
0.1 to 0.9	2	9	17	19	1	4	10	5	6	0
No Magnitude	4084	3997	1944	1826	2186	3415	2426	2096	3199	749
Total	19548	21476	19371	21007	19938	19872	21688	20832	22309	6478
Estimated Deaths	3814	10036	1038	7949	419	2907	8928	22711	231	14923

History of Earthquakes in Bangladesh

During the last 150 years, seven major earthquakes (with M>7.0) have affected the zone that is now within the geographical borders of Bangladesh. Out of these, three had epicenters within Bangladesh. The earthquakes and their effects are described in Table 17.3.

However, the recent awareness in Bangladesh of possible earthquake risks and necessary preparations has been due mainly to the devastating earthquakes that hit India in 2001, South-East Asia (particularly Indonesia, Sri Lanka and India) in 2004 and Pakistan in 2005. Reports by Bilham and co-workers have indicated that Bangladesh is under major seismic risk, with earthquakes of magnitude 8.0 or more already overdue in the Eurasian and Indian plates.

Table 17.3: List of major Earthquakes affecting Bangladesh

Date	Name of Earthquake	Magnitude (Richter)	Epicentral distance from Dhaka (km)	Affected zone
10th Jan, 1869	Cachar Earthquake	7.5	250	Tremor mainly in Sylhet
14th July, 1885	Bengal Earthquake	7.0	170	Damage in Jamalpur Sherpuur, Bogra
12th June, 1897	Great Indian Earthquake	8.7	230	Damage in Sylhet, Mymensingh
8th July, 1918	Srimongal Earthquake	7.6	150	Tremor in Sylhet
2nd July, 1930	Dhubri Earthquake	7.1	250	Damage in Eastern part of Rangpur
15th Jan, 1934	Bihar-Nepal Earthquake	8.3	510	None
15th Aug, 1950	Assam Earthquake	8.5	780	Tremor throughout the country

Earthquake Ground Motion

Earthquake involves vibration of the ground typically for durations of 10-40 seconds, which increases gradually to the peak amplitude and then decays. It is primarily a horizontal vibration, although some vertical movement is also present. Since the vibrations are time-dependent, earthquake is essentially a dynamic problem and the proper way to deal with it is through dynamic analysis of the structure, including its foundation and the surrounding soil. The dynamic analysis is done for time dependent ground motion.

Figs. 17.1-17.4 show the temporal variation of ground accelerations recorded during some of the best known and widely studied earthquakes of the 20th century. Among them, only the *El Centro earthquake (1940)* dates more than ten years back. The El Centro earthquake data has over the last fifty years been the most used seismic data. However, Fig. 17.1 and 17.2 show that the ground accelerations recorded during this earthquake were different at different stations. It is about 6.61 ft/sec^2 for the first station and 9.92 ft/sec^2 for the second, which shows that the location of the recording station should be mentioned while citing the peak acceleration in an earthquake. The earthquake magnitudes calculated from these data are also different.

Figs. 17.3 and 17.4 show the ground acceleration from the Kobe (1995) and Northridge (1994) earthquake, both of which had caused major destructions in two of the 'best prepared countries' over the last decade. The maximum ground accelerations they represent can only provide a rough estimate of their nature. The Fourier amplitude spectra can provide a better insight into their nature.

The Fourier amplitude spectra for the El Centro2 (henceforth called only El Centro) and Kobe earthquake ground acceleration are shown in Figs. 17.5 and 17.6 respectively. The spectra show the higher values of the spectral ordinates for the Kobe earthquake, which is understandable because of the larger accelerations recorded. But they also show that the spectral peaks for the El Centro data occur at larger frequencies (1.0~1.5 cycle/sec) than the peaks of Kobe (0.5~1.0 cycle/sec), which makes the latter dangerous for the more flexible structures.

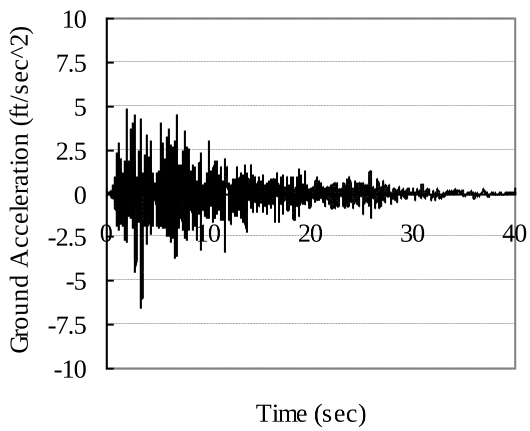


Fig. 17.1 : El Centro1 Ground Acceleration

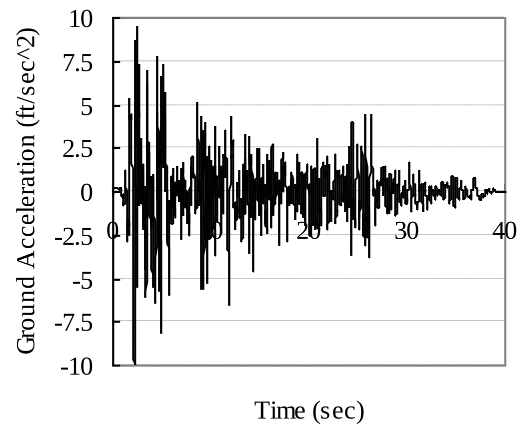


Fig. 17.2 : El Centro2 Ground Acceleration

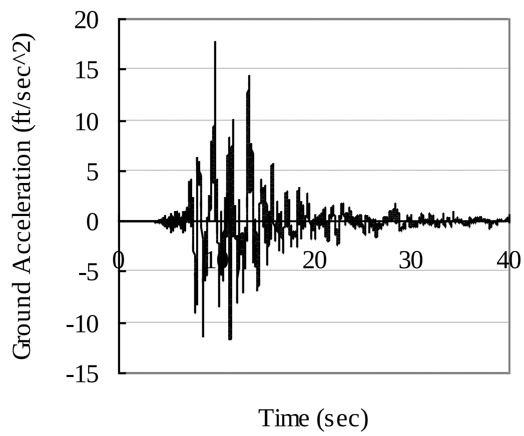


Fig. 17.3 : Kobe Ground Acceleration

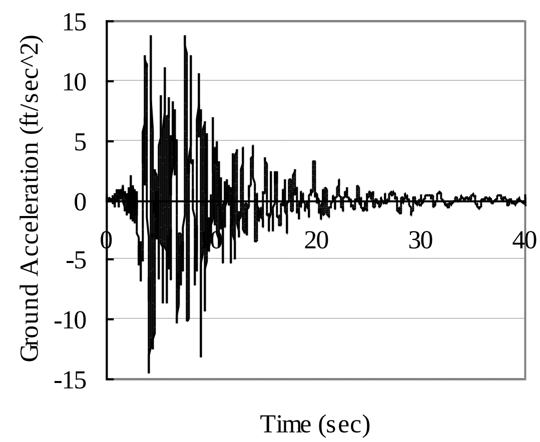


Fig. 17.4 : Northridge Ground Acceleration

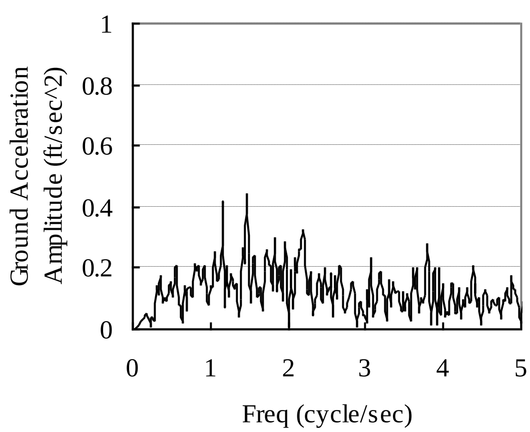


Fig. 17.5 : El Centro Ground Acceleration Spectrum

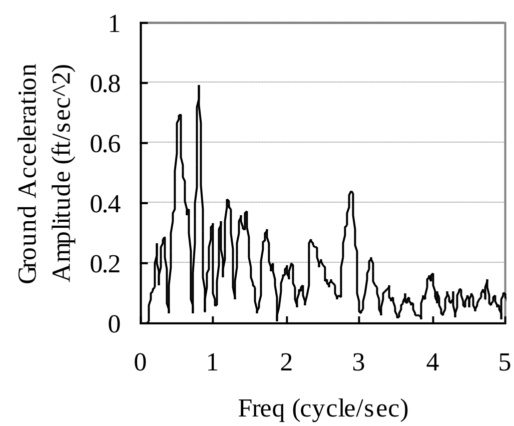


Fig. 17.6 : Kobe Ground Acceleration Spectrum

Governing Equation of Motion for Systems under Seismic Vibration

The loads induced by earthquake are not body-forces; rather it is a ground vibration that induces certain forces in the structure. To illustrate that, the dynamic equations of motion for a SDOF system and a 2-DOF system are derived due to ground motion.

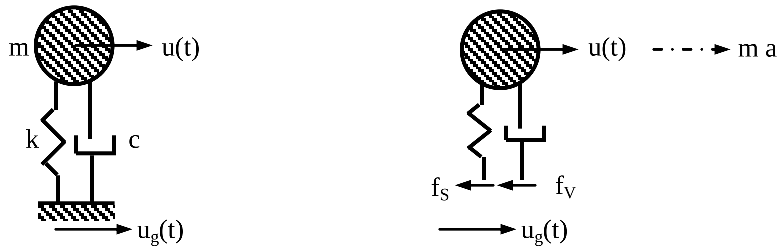


Fig. 17.7: Dynamic SDOF system subjected to ground displacement $u_g(t)$

For the SDOF system subjected to ground displacement $u_g(t)$,

f_S = Spring force = Stiffness times the relative displacement = $k\,(u-u_g)$ ……………………(17.4)

f_V = Viscous force = Viscous damping times the relative velocity = $c\,(du/dt-du_g/dt)$ …………(17.5)

f_I = Inertia force = Mass times the acceleration = $m\,d^2u/dt^2$ ……………………(17.6)

Combining the equations, the equation of motion for a SDOF system is derived as follows,

$$m\,d^2u/dt^2 + c\,(du/dt-du_g/dt) + k\,(u-u_g) = 0$$

$$\Rightarrow m\,d^2u/dt^2 + c\,du/dt + k\,u = c\,du_g/dt + k\,u_g \quad \text{………………………(17.7)}$$

$$\Rightarrow m\,d^2u_r/dt^2 + c\,du_r/dt + k\,u_r = -m\,d^2u_g/dt^2 \quad \text{………………………(17.8)}$$

where $u_r = u-u_g$ is the relative displacement of the SDOF system with respect to the ground displacement. Eqs. (17.7) and (17.8) show that the ground motion appears on the right side of the equation of motion just like a time-dependent load. Therefore, although there is no body-force on the system, it is still subjected to dynamic excitation by the ground displacement.

For a lumped 2-DOF system subjected to ground displacement $u_g(t)$, velocity $v_g(t)$ and acceleration $a_g(t)$, the following equations are obtained in matrix form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{Bmatrix} d^2u_1/dt^2 \\ d^2u_2/dt^2 \end{Bmatrix} + \begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix}\begin{Bmatrix} du_1/dt \\ du_2/dt \end{Bmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} k_1u_g + c_1v_g \\ 0 \end{Bmatrix}$$

…………(17.9)

Eq. (17.9) can also be written as

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{Bmatrix} d^2u_{1r}/dt^2 \\ d^2u_{2r}/dt^2 \end{Bmatrix} + \begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix}\begin{Bmatrix} du_{1r}/dt \\ du_{2r}/dt \end{Bmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{Bmatrix} u_{1r} \\ u_{2r} \end{Bmatrix} = -\begin{Bmatrix} m_1\,a_g \\ m_2\,a_g \end{Bmatrix}$$

…………(17.10)

Eqs. (17.9) and (17.10) can be easily extended to MDOF systems.

Response Spectrum Analysis and Equivalent Static Method

Response Spectrum Analysis

The time domain *Response History Analysis (RHA)* procedure presented so far in this course (analytically or numerically for SDOF and MDOF systems) provides a structural response with time, but the design of structural members is usually based on the peak response; i.e., the maximum values of the design forces. Therefore the main objective of seismic design methods is to conveniently calculate the peak displacements and forces resulting from a particular design ground motion. The *Response Spectrum Analysis (RSA)* is an approximate method of dynamic analysis that can be readily used for a reasonably accurate prediction of dynamic response due to seismic ground motion.

As shown in the previous lecture, the governing equation of motion for a SDOF system subjected to ground motion $u_g(t)$ is given by

$$m\, d^2u/dt^2 + c\, du/dt + k\, u = c\, du_g/dt + k\, u_g \qquad \text{……………………………(17.7)}$$

$$\Rightarrow m\, d^2u_r/dt^2 + c\, du_r/dt + k\, u_r = -\, m\, d^2u_g/dt^2 \qquad \text{……………………………(17.8)}$$

Since the loads themselves (on the right side of the equations) are proportional to the structural properties, each of these equations can be normalized in terms of the system properties (natural frequency ω_n and damping ratio ξ) and the ground motion (acceleration or displacement and velocity). For example, Eq. (17.8) becomes

$$d^2u_r/dt^2 + 2\omega_n\xi\, du_r/dt + \omega_n^2\, u_r = -\, d^2u_g/dt^2 \qquad \text{……………………………(18.1)}$$

For a specified ground motion data (e.g., the El Centro2 data shown in Fig. 17.2, or the Kobe data of Fig. 17.3) the temporal variation of structural displacement, velocity and acceleration depends only on its natural frequency ω_n and the damping ratio ξ. From the time series thus obtained, the maximum parameters can be identified easily as the maximum design criteria for that particular structure (and that particular ground motion). Such maximum values can be similarly obtained for structures with different natural frequency (or period) and damping ratio. Since natural period (T_n) is a more familiar concept than ω_n, the peak responses can be represented as functions of T_n and ξ for the ground motion under consideration.

If a 'standard' ground motion data is chosen for the design of all SDOF structures, the maximum responses thus obtained will depend on the two structural properties only. A plot of the peak value of the response quantity as a function of natural T_n and ξ is called the response spectrum of that particular quantity. If such curves can be obtained for a family of damping ratios (ξ), they can provide convenient curves for seismic analysis of SDOF systems.

The following peak responses for the displacement (u), velocity (v) and acceleration (a) are called the response spectra for the relative deformation, velocity and acceleration.

$$u_{r0}(T_n, \xi) = \text{Max}\left| u(t, T_n, \xi) \right| \qquad \text{…………………(18.2)}$$

$$v_{r0}(T_n, \xi) = \text{Max}\left| v(t, T_n, \xi) \right| \qquad \text{…………………(18.3)}$$

$$a_{r0}(T_n, \xi) = \text{Max}\left| a(t, T_n, \xi) \right| \qquad \text{…………………(18.4)}$$

Such response spectra have long been used as useful tools for the seismic analysis of structures. The same spectra can be used for MDOF systems, which can be decomposed into several SDOF systems by Modal Analysis. Once the peak responses for all the modes are calculated from the response spectra, they can be combined statistically to obtain the approximate maximum response for the whole structure.

In order to account for the amplification of waves while propagating through soft soils, some simplified wave propagation analyses can be performed. Such works, performed statistically for a variety of soil conditions, provide the response spectra as shown in Fig. 18.1, which leads to the code-specified response spectrum in Fig. 18.2.

However, the RSA method suffers from some serious shortcomings when compared to the numerical time domain analysis. Although one can obtain the response spectra for any given ground motion data, the spectra used in all the building codes were all derived from the El Centro data and thus they may not represent the more severe earthquakes that have since occurred. Besides the method needs Modal Analysis to obtain the structure's natural frequencies, which can be a laborious task. Moreover the method cannot be applied for nonlinear structures or it cannot predict structural failures. It is essentially a linear analysis but is approximately used for nonlinear structures applying a 'ductility factor' to reduce the amplitudes.

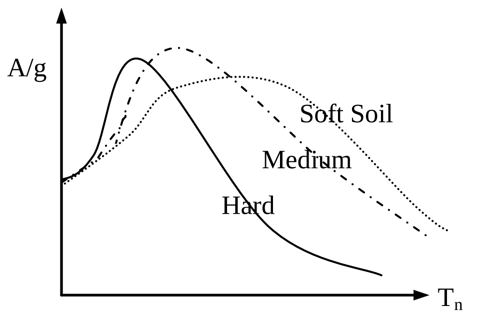


Fig. 18.1: Acceleration Spectra for different sites

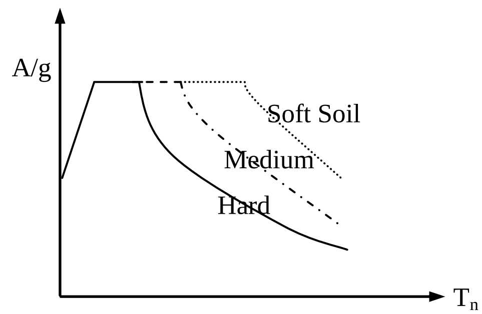


Fig. 18.2: Code Specified Acceleration Spectra

<u>Equivalent Static Force Method</u>
This 'Equivalent Static Analysis' of seismic vibration is based on the concept of replacing the inertia forces at various 'lumped masses' (i.e., story levels) by equivalent horizontal forces that are proportional the weight of the body (therefore its mass) and its displacement (therefore its acceleration). The summation of these concentrated forces is balanced by a 'base shear' at the base of the structure.

This method may be used for calculation of seismic lateral forces for all structures specified in the building codes. The following provisions are taken from the *Uniform Building Code* of USA *(UBC, 1994)*, and is also valid for *Bangladesh National Building Code (BNBC, 1993)* for most part.

(1) Design Base Shear
The total design base shear in a given direction is determined from the following relation:

$$V = (ZIC/R)\ W \qquad \text{……………….(18.5)}$$

where, Z = Seismic zone coefficient given in Table 24.1 (for Bangladesh Code)
I = Structure importance coefficient given in Table 24.2
R = Response modification coefficient for structural systems given in Table 24.3
W = The total seismic dead load

The 'Seismic Dead Load' is not only the dead load of the structure but also has to include some live loads as and when they superimpose on the dead loads. Seismic dead load W, is the total dead load of a building or structure, including permanent partitions, and applicable portions of other loads.

C = Numerical coefficient given by the relation: $C = 1.25\ S/T^{2/3}$ …………….(18.6)
S = Site coefficient for soil characteristics as provided in Table 24.4
T = Fundamental period of vibration of the structure for the direction under consideration (in seconds)

The value of C need not exceed 2.75. Except for those requirements where Code prescribed forces are scaled up by 0.375R, the minimum value of the ratio C/R is 0.075.

(2) Structural Period
The value of the fundamental period T of the structure can be reasonably calculated using one of the following simplified methods:

a) Method A: The value of T may be approximated by the following formula

$$T = C_t\,(h_n)^{3/4} \qquad \text{………………………….(18.7)}$$

where, C_t = 0.083 for steel moment resisting frames
= 0.073 for RCC moment resisting frames, and eccentric braced steel frames
= 0.049 for all other structural systems
h_n = Height (in meters) above the base to level n.

There are alternative ways of calculating T and C_t.

b) Method B: The fundamental period T may be calculated using the structural properties and deformational characteristics of the resisting elements in a properly substantiated analysis. This requirement may be satisfied by using the following formula:

$$T = 2\pi\sqrt{[\sum w_i u_i^2/g \sum w_i u_i]} \qquad \text{………………..………(18.8)}$$

Here, w_i represents the weight and u_i the displacement of the i^{th} floor

(3) Vertical Distribution of Lateral Forces

In the absence of a more rigorous procedure, the total lateral force which is the base shear V, is distributed along the height of the structure in accordance with Eq (18.9)-(18.11).

$$V = F_t + \sum F_i \qquad \text{………………..………(18.9)}$$

where, F_i = Lateral force applied at storey level i and

F_t = Concentrated lateral force considered at the top of the building in addition to F_n.

The concentrated force, F_t acting at the top of the building is determined as follows:

$$F_t = 0.07\ TV \le 0.25V, \text{ when } T > 0.7 \text{ second}$$
$$F_t = 0.0 \qquad \text{when } T \le 0.7 \text{ second} \qquad \text{………………(18.10)}$$

The remaining portion of the base shear $(V–F_t)$, is distributed over the height of the building, including level n, according to the relation

$$F_j = (V–F_t)\ [\sum w_j h_j/\sum w_i h_i] \qquad \text{...………..……(18.11)}$$

The design story shear V_x in any story x is the sum of the forces F_x and F_t above that story. V_x is distributed to the various elements of the vertical lateral force resisting system in proportion to their rigidities, considering the rigidity of the floor or roof diaphragm.

Other Building Codes

The National Building Code of Canada (NBCC, 1995) predicts the base shear with an equation similar to Eq. (18.5), with slightly different coefficients. An additional feature of the NBCC is that it introduces an 'over-strength factor U' to account for the fact that the actual strength of the building is expected to be larger than the calculated strength.

The Mexico Federal District Code (MFDC 1987) introduces a building resistance factor Q′ instead of the factor R in the UBC. One feature of the factor Q′ is that it changes with the natural period of the structure, which is consistent with real elastoplastic systems.

Structural Dynamics in Building Codes

The Equivalent Static Force Method (ESFM) tries to model the dynamic aspects of seismic loads in an approximate manner. Therefore it is natural that the ESFM includes several equations that are derived from Structural Dynamics. The following are worth noting

(1) The zone factor Z can be interpreted as the ratio of the maximum ground acceleration and g, while the factor C is the amplitude of the Response Spectra. Ignoring I (the over-design factor for essential facilities), the V_e = (ZIC) W gives the maximum elastic base shear for the building. Therefore the factor R is the building resistance factor that accounts for the ductility of the building, i.e., its ability to withstand inelastic deformations.

(2) The distribution of story shear [Eq. (18.11)] in proportion to the mass and height of the story is an approximation of the 1^{st} modal shape, which is almost linear for shorter buildings but tends to be parabolic to include higher modes of vibration. Therefore, a concentrated load is added at the top to approximately add the 2^{nd} mode of vibration for taller buildings.

(3) The equation of the natural frequency [Eq. (18.8)] is very similar to the equation of natural frequency of continuous dynamic systems.

(4) The factor S is introduced in the factor C to account for amplification of seismic waves in soft soils.

Elastic Dynamic Analysis and Equivalent Static Force Method

Having pointed out the analogy between the Response Spectrum Analysis (RSA) and Equivalent Static Force Method (ESFM), this section compares some numerical results between the two methods. Of central importance is the term 'base shear' used in ESFM, which is the static force at the base of the ground floor column developed due to ground motion. For a SDOF system, this force is given by

$$f_s = k\,u_r = k\,(u-u_g) \qquad \text{......................(19.1)}$$

Using $k = m\omega_n^2 \Rightarrow f_s = k\,u_r = m(\omega_n^2 u_r) = m\,a_0$ ....(19.2)

where the term $a_0 = \omega_n^2 u_r$ is called the 'pseudo' acceleration. Therefore, the base shear is the mass times the 'pseudo' acceleration. Using the ESFM for a linearly elastic system, the base shear is also given by

$$f_s = ZCW \qquad \text{......................(19.3)}$$

Equating the two $\Rightarrow a_0 = Zg\,C = a_{g(max)}\,C \Rightarrow C = a_0/a_{g(max)}$ ....(19.4)

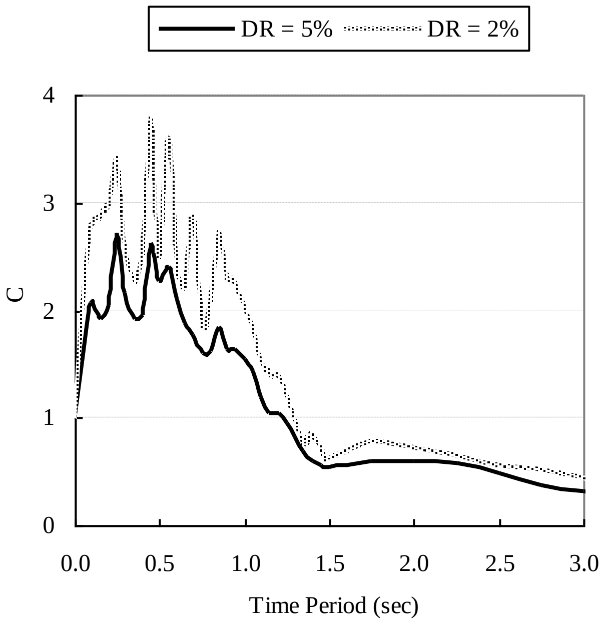


Fig. 19.1: Response Spectra (El Centro Earthquake)

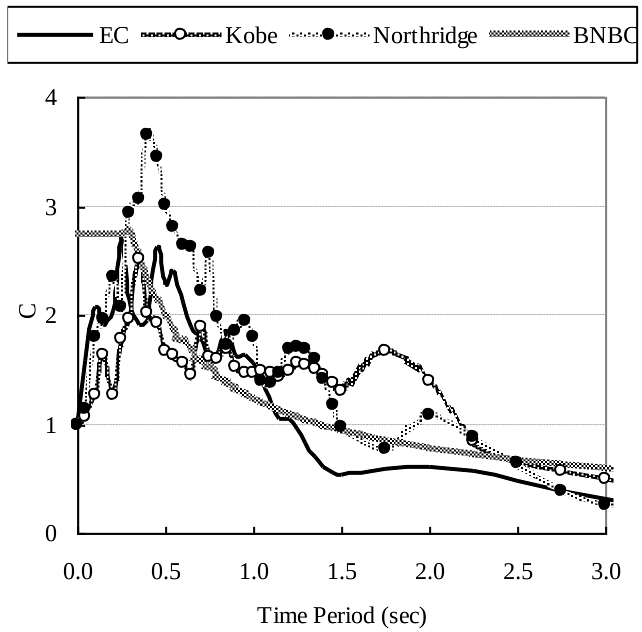


Fig. 19.2: Response Spectra for 5% Damping

Fig. 19.1 shows the variation of C for the El Centro earthquake data (for damping ratios 2% and 5%), while and Fig. 19.2 shows the variation of C for the El Centro, Kobe and Northridge earthquake data (for damping ratio 5%) as well as the design values suggested by BNBC (for very hard soil).

<u>Example 19.1</u>

For the SDOF system described in Example 3.1, calculate the base shear using
(i) El Centro data, (ii) Kobe data, (iii) BNBC (using Z for El Centro, Kobe and Dhaka).

<u>Solution</u>

For the SDOF system, $m = 1$ k-sec^2/ft, $k = 25$ k/ft, $c = 0.5$ k-sec/ft

$\therefore$ Natural frequency $\omega_n = 5$ rad/sec $\Rightarrow$ Time period $T_n = 2\pi/\omega_n = 1.257$ sec, Damping ratio, $\xi = 0.05$

(i) For El Centro data, Z = 0.313, C = 0.933

$\therefore$ Maximum Relative Displacement $u_{max} = ZgC/\omega_{n1}^2 = 0.313 \times 32.17 \times 0.933/5^2 = 0.376$ ft

$\therefore$ Base shear $V_b = ZCW = 0.313 \times 0.933 \times (1 \times 32.17) = 9.40$ kips

(ii) For Kobe data, Z = 0.553, C = 1.560

$\therefore$ Maximum Relative Displacement $u_{max} = ZgC/\omega_{n1}^2 = 0.553 \times 32.17 \times 1.560/5^2 = 1.110$ ft

$\therefore$ Base shear $V_b = ZCW = 0.553 \times 1.560 \times (1 \times 32.17) = 27.74$ kips

(iii) Using BNBC for hard soils, $C = 1.25/T_n^{2/3} \le 2.75 \Rightarrow C = 1.073$

$\therefore$ For El Centro data, Base shear $V_b = 0.313 \times 1.073 \times (1 \times 32.17) = 10.81$ kips

For Kobe data, Base shear $V_b = 0.553 \times 1.073 \times (1 \times 32.17) = 19.10$ kips

For Dhaka, $Z = 0.15 \Rightarrow$ Base shear $V_b = 0.15 \times 1.073 \times (1 \times 32.17) = 5.18$ kips

$\therefore$ The corresponding maximum displacements are 0.432, 0.764 and 0.207 ft respectively.

Example 19.2

For the 2-DOF system described in Examples 10.1 and 10.2, calculate the base shear using (i) El Centro data, (ii) Kobe data, (iii) BNBC (Dhaka), (iv) Equivalent Static Force Method for all three.

Solution

For the MDOF system, the modal masses, stiffnesses and damping ratios are,

$M_1 = 3.618$ k-sec^2/ft, $M_2 = 1.382$ k-sec^2/ft, $K_1 = 34.55$ k/ft, $K_2 = 90.45$ k/ft, $\xi_1 = 0.0309$, $\xi_2 = 0.0809$

$\Rightarrow$ Natural frequencies $\omega_{n1} = 3.09$ rad/sec, $\omega_{n2} = 8.09$ rad/sec

$\therefore$ Time periods $T_{n1} = 2\pi/\omega_{n1} = 2.033$ sec, $T_{n2} = 2\pi/\omega_{n2} = 0.777$ sec

The modal loads are, $f_1(t) = \boldsymbol{\phi_1}^{\mathbf{T}}\, \mathbf{f} = -2.618\, a_g$, $f_2(t) = \boldsymbol{\phi_2}^{\mathbf{T}}\, \mathbf{f} = -0.382\, a_g$

(i) For El Centro, $Z_1 = 0.313 \times 2.618/3.618 = 0.227$, and RSA for El Centro $\Rightarrow C_1 = 0.665$

$\therefore$ Maximum Displacement $q_{max1} = Z_1 g\, C_1/\omega_{n1}^2 = 0.227 \times 32.17 \times 0.665/3.09^2 = 0.507$ ft

$Z_2 = 0.313 \times 0.382/1.382 = 0.087$, and RSA for El Centro $\Rightarrow C_2 = 1.356$

$\therefore$ Maximum Displacement $q_{max2} = Z_2 g\, C_2/\omega_{n2}^2 = 0.087 \times 32.17 \times 1.356/8.09^2 = 0.058$ ft

$\therefore$ Using the square-root-of-sum-of-squares (SRSS) rule,

Maximum Displacement $u_{max1} \cong \sqrt{\{(0.507 \times 1)^2 + (0.058 \times 1)^2\}} = 0.510$ ft

and $u_{max2} \cong \sqrt{\{(0.507 \times 1.618)^2 + (-0.058 \times 0.618)^2\}} = 0.821$ ft

$\therefore$ Maximum story forces are $F_2 = 25 \times (0.821-0.510) = 7.77$ k, $F_1 = 25 \times 0.510 - 7.77 = 4.99$ k

$\therefore$ Maximum Base shear $V_b = 4.99 + 7.77 = 12.76$ k

(ii) For Kobe data, $Z_1 = 0.553 \times 2.618/3.618 = 0.400$, $C_1 = 1.526$

$\therefore$ Maximum Displacement $q_{max1} = Z_1 g\, C_1/\omega_{n1}^2 = 0.400 \times 32.17 \times 1.526/3.09^2 = 2.057$ ft

$Z_2 = 0.553 \times 0.382/1.382 = 0.153$, $C_2 = 1.407$

$\therefore$ Maximum Displacement $q_{max2} = Z_2 g\, C_2/\omega_{n2}^2 = 0.153 \times 32.17 \times 1.407/8.09^2 = 0.106$ ft

Maximum Displacement $u_{max1} \cong \sqrt{\{(2.057 \times 1)^2 + (0.106 \times 1)^2\}} = 2.060$ ft

and $u_{max2} \cong \sqrt{\{(2.057 \times 1.618)^2 + (-0.106 \times 0.618)^2\}} = 3.329$ ft

$\therefore$ Maximum story forces are $F_2 = 25 \times (3.329-2.060) = 31.73$ k, $F_1 = 25 \times 2.060 - 31.73 = 19.76$ k

$\therefore$ Maximum Base shear $V_b = 19.76 + 31.73 = 51.49$ k

(iii) For Dhaka, $Z_1 = 0.15 \times 2.618/3.618 = 0.109$, and BNBC $\Rightarrow C_1 = 0.779$

$\therefore$ Maximum Displacement $q_{max1} = Z_1 g\, C_1/\omega_{n1}^2 = 0.109 \times 32.17 \times 0.779/3.09^2 = 0.285$ ft

$Z_2 = 0.15 \times 0.382/1.382 = 0.042$, $C_2 = 1.479$

$\therefore$ Maximum Displacement $q_{max2} = Z_2 g\, C_2/\omega_{n2}^2 = 0.042 \times 32.17 \times 1.479/8.09^2 = 0.031$ ft

Maximum Displacement $u_{max1} \cong \sqrt{\{(0.285 \times 1)^2 + (0.031 \times 1)^2\}} = 0.287$ ft

and $u_{max2} = \sqrt{\{(0.285 \times 1.618)^2 + (-0.031 \times 0.618)^2\}} = 0.462$ ft

$\therefore$ Maximum story forces are $F_2 = 25 \times (0.462-0.287) = 4.37$ k, $F_1 = 25 \times 0.287 - 4.37 = 2.80$ k

$\therefore$ Maximum Base shear $V_b = 2.80 + 4.37 = 7.17$ k

(iv) Using BNBC for hard soils, $C = 1.25/T_{n1}^{2/3} \leq 2.75$; $\therefore T_{n1} = 2.033$ sec $\Rightarrow C = 0.779$

$\therefore$ For El Centro data, Base shear $V_b = 0.313 \times 0.779 \times (2 \times 32.17) = 15.69$ kips

$\therefore F_t = 0.07 T_n V_b = 0.07 \times 2.033 \times 15.69 = 2.23$ k $\Rightarrow$ Story Forces are 4.49 and 11.20 kips.

For Kobe data, Base shear $V_b = 0.553 \times 0.779 \times (2 \times 32.17) = 27.72$ kips

$\therefore F_t = 0.07 \times 2.033 \times 27.72 = 3.94$ k $\Rightarrow$ Story Forces are 7.93 and 19.79 kips.

For Dhaka, $Z = 0.15 \Rightarrow$ Base shear $V_b = 0.15 \times 0.779 \times (2 \times 32.17) = 7.52$ kips

$\therefore F_t = 0.07 \times 2.033 \times 7.52 = 1.07$ k $\Rightarrow$ Story Forces are 2.15 and 5.37 kips.

The results from both the examples suggest an overestimation of the El Centro base shear and an underestimation of the Kobe base shear by using the equation suggested in BNBC. This is quite natural because the code equation is derived by averaging the results from numerous earthquakes. The examples further show that the base shear for Dhaka is much smaller than the forces suggested by the two major earthquakes, and can at best represent a moderate earthquake.

Results from RHA and RSA

Although the RSA provides a very convenient method for dynamic seismic analysis, it is only an approximation of the RHA, provided by time series analysis. Whereas the two methods provide identical results for SDOF systems, their results can be different for MDOF systems.

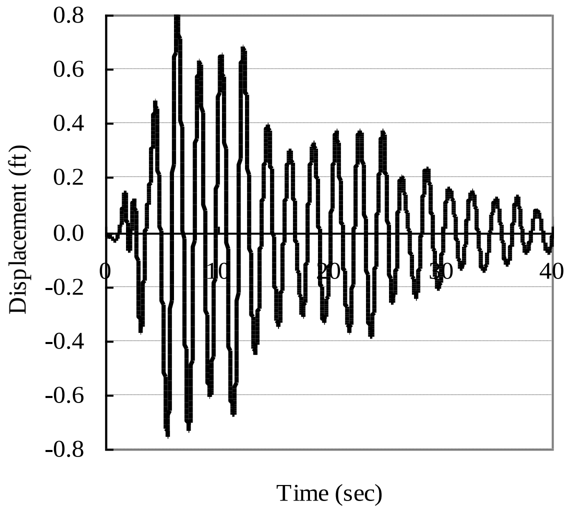


Fig. 19.3: Top Floor displacement (El Centro)

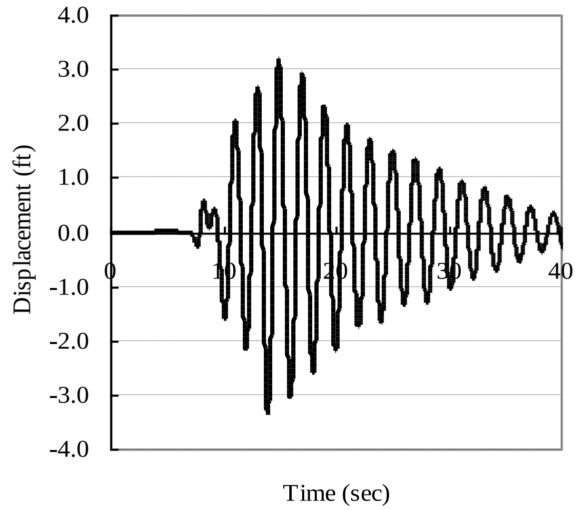


Fig. 19.4: Top Floor displacement (Kobe)

Figs. 19.3 and 19.4 show the temporal variations of the top floor displacements (for the El Cento and Kobe data respectively) of the MDOF system analyzed in Example 19.2, where the maximum values of the displacements come out to be 0.802 and 3.355 ft, which compares quite favorably with the RSA results (0.822 and 3.330 ft) shown in Example 19.2. Although not shown in the figures, the maximum first floor displacements (0.535 and 2.009 ft) are also quite similar to the values obtained in Example 19.2 (i.e., 0.515 and 2.064 ft).

Therefore, the results from RHA and RSA match quite well in this particular case. However, their differences can be quite significant for more complex structures where the natural frequencies are quite close, the displacements can be a combination of deflections and rotations, especially for three-dimensional structures. The limitations of the RSA to deal with structural nonlinearity make it less acceptable for nonlinear systems.

Inelastic Seismic Response, Ductility and Seismic Detailing

Building structures are rarely expected to remain within the elastic limit during major earthquakes, and the inelastic material behavior can result in the reduction of forces acting on them as well as a corresponding increase in deformations [Fig. 16.2(b)]. Thus the design base shear forces calculated from elastic analysis can be reduced significantly, but at the same time the structure should be designed to withstand the corresponding increase in deformations. The concept of *ductility* is introduced to allow for the consideration of inelastic deformations.

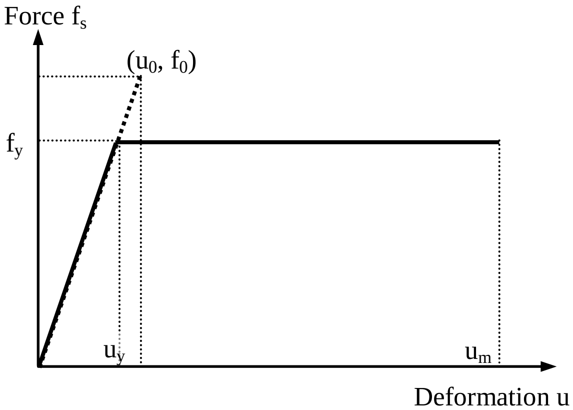


Fig. 20.1: Elastoplastic system and the corresponding linear system

Fig. 20.1 shows the elastoplastic force-deformation response of a nonlinear system. Here the force f_s remains proportional to the deformations up to the yield point (u_y, f_y), beyond which the force remains constant upto the failure deformation u_m.

On the other hand, if the system remained linearly elastic, the corresponding force and deformation would be f_0 and u_0, where $f_0 > f_y$, but $u_0 > u_m$. This again emphasizes that although a nonlinear system has to withstand a smaller force (i.e., upto its yield strength only), the deformation u_m it must withstand before failure is greater. Both these aspects need to be incorporated in the structural analysis and design of members subjected to seismic vibrations.

The ratio between the forces f_0 and f_y is called the *yield reduction factor* (denoted here by R_y), while the ratio between the deformations u_m and u_y is the *ductility factor* (denoted here by μ)

$R_y = f_0/f_y$ ……………………………………………….(20.1)

$\mu = u_m/u_y$ ……………………………………………….(20.2)

Using $R_y = f_0/f_y = u_0/u_y \Rightarrow \mu = u_m/u_y = (u_m/u_0)\, R_y \Rightarrow u_m = \mu\, u_0/R_y$ …………...……………(20.3)

The following simple relationship (also Fig. 20.2) is one of the early attempts to relate R_y and μ, which can be used to construct the *Inelastic Response Spectrum*.

$R_y = 1$, for $T_n < T_a$

$= \sqrt{(2\mu-1)}$, for $T_b < T_n < T_{c1}$

$= \mu$, for $T_n > T_c$ …..…………….(20.4)

The intermediate values (i.e., between T_a and T_b, or between T_{c1} and T_c) can be obtained by interpolation. Here T_a = 0.03 sec, T_b = 0.125 sec, while T_{c1} and T_c depend on the damping ratio of the system. For a damping ratio of 5%, it is reasonable to assume $T_{c1} \cong$ 0.35 sec, and T_c = 0.55 sec.

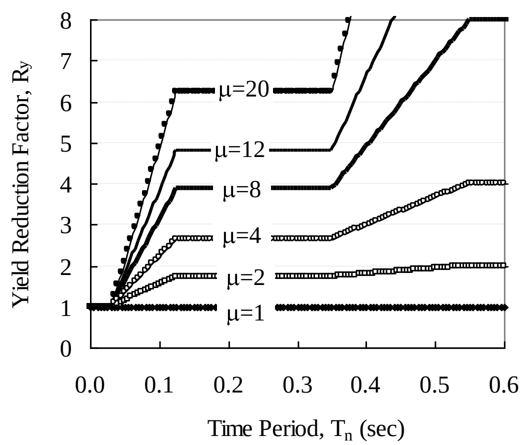


Fig. 20.2: Variation of R_y with T_n

Example 20.1

Calculate the inelastic base shear force V_b and the corresponding relative displacement u_{max} for El Centro data for the SDOF system of (i) Example 19.1 with u_y = 0.1 ft, (ii) # 1 of Problem set 1 with u_y = 0.02 in.

Solution

(i) For this system, Time period T_n = 1.257 sec, Damping ratio ξ = 0.05
For El Centro data, maximum elastic displacement u_0 = 0.376 ft, base shear $V_{b(e)}$ = 9.40 kips
$\therefore R_y = u_0/u_y = 3.76$ and $T_n > T_c \Rightarrow \mu = R_y = 3.76$
$\therefore$ Inelastic $V_b = V_{b(e)}/R_y$ = 9.40/3.76 = 2.50 kips, Inelastic $u_m = \mu\, u_y$ = 0.376 ft

(ii) Mass, m = 0.0259 lb-in/sec^2, k = 100 lb/in, ω_n = 62.16 rad/sec T_n = 0.101 sec, ξ = 0.05
For El Centro data, C = 2.323, $u_0 = ZCg/\omega_n^2$ = 0.0726 in, $V_{b(e)} = k\, u_0$ = 7.26 lb
$\therefore R_y = u_0/u_y$ = 3.63, T_n = 0.101 sec $\Rightarrow \mu$ = 10.17; $\therefore$ Inelastic V_b = 7.26/3.63 = 2 lb, $u_m = \mu\, u_y$ = 0.203 in

In Example 20.1, R_y is used as a reduction factor for the elastic base shear, because it is used in Eq. (20.1) to reduce the 'elastic' force f_0 (for an 'equivalent' linear system) to the 'inelastic' force f_y (for a nonlinear system). Table 20.1 shows the code-recommended values of Response Modification Coefficient R (equivalent to R_y) for various types of structures, which is used to reduce the design base shear.

Table 20.1: Response Modification Coefficient, R for Structural Systems

Basic Structural System	Description Of Lateral Force Resisting System	R
(a) Bearing Wall System	Light framed walls with shear panels	6~8
	Shear walls (Concrete/Masonry)	6
	Light steel framed bearing walls with tension only bracing	4
	Braced frames where bracing carries gravity loads	4~6
(b) Building Frame System	Steel eccentric braced frame (EBF)	10
	Light framed walls with shear panels	7~9
	Shear walls (Concrete/Masonry)	8
	Concentric braced frames (CBF)	8
(c) Moment Resisting Frame System	Special moment resisting frames (SMRF) (Steel/Concrete)	12
	Intermediate moment resisting frames (IMRF) (Concrete)	8
	Ordinary moment resisting frames (OMRF)	
	(i) Steel	6
	(ii) Concrete	5
(d) Dual System	Shear walls	7~12
	Steel EBF (with Steel SMRF or OMRF)	6~12
	Concentric braced frame (CBF)	6~10
(e) Special Structural Systems	According to Sec 1.3.2, 1.3.3, 1.3.5 of BNBC	

[Note: Some of these systems are prohibited in Seismic Zone2 and/or Zone 3]

Ductility and Seismic Detailing

Ductility may be broadly defined as the ability of a structure to undergo inelastic deformations beyond the initial yield deformation with no decrease in the load resistance. While ductility helps in reducing induced forces and in dissipating some of the input energy, it also demands larger deformations to be accommodated by the structure. Modern building codes provide for reduction of seismic forces through provision of special ductility requirements. Many such provisions have been incorporated in the BNBC also [i.e., in Chapters 8 and 10 of Part 6 (Structural Design) of the 1993 edition].

In order to maintain overall ductile behavior of the structure with minimal damage, it becomes necessary to achieve, in relative terms, combinations of

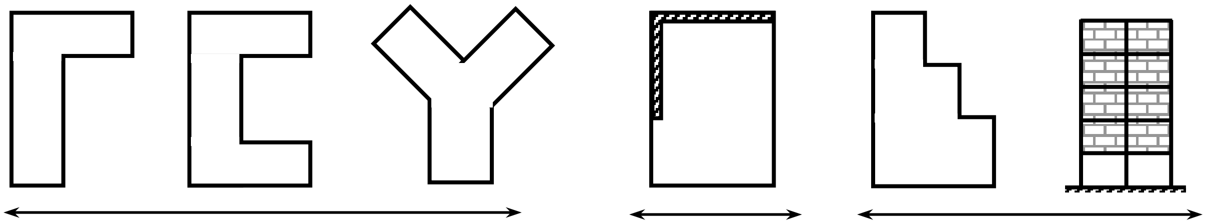
* Continuity in construction (i.e., avoid sudden changes in plan or elevation);
* Strong foundations and weak structure (i.e., the foundations should not fail before the structure);
* Strong columns and weak beams (i.e., the columns should not fail before the beams);
* Members stronger in shear than in flexure (i.e., they should not fail in shear before failing in flexure, because shear failure in much more brittle and sudden).

Since ductility is a major concern for RC structures that are widely used in building construction, seismic detailing of RC structures is a topic of particular interest. The main design considerations in providing ductility of RC structures include

* Using materials of 'medium strength'; i.e., materials strong enough to avoid brittle tensile failure but not too strong to result in brittle tensile/compressive failure;
* Using a low tensile steel ratio and/or using compressive steel in order to avoid concrete crushing before yielding of steel;
* Providing adequate stirrups to ensure that shear failure does not precede flexural failure;
* Confining concrete and compressive steel by closely spaced hoops/spirals;
* Proper detailing with regard to anchorage, splicing, minimum reinforcement, etc, so that the structural members can develop the forces they are designed for.

Seismic Design and Detailing of RC Structures

Discontinuity in Construction Unfavorable for Seismic Design



Complicated Plans Asymmetrical Plan Discontinuous Elevations

Fig. 21.1: Unfavorable Discontinuity in Building Configurations

Code Prescribed Seismic Detailing of RC Structural Elements

1. Materials

	Specification	Possible Explanation
Concrete	$f_c' \geq 20$ Mpa ($\cong 3$ ksi) for 3-storied or taller buildings	Weak concretes have low shear and bong strengths and cannot take full advantage of subsequent design provisions
Steel	$f_y \leq 415$ Mpa ($\cong 60$ ksi), preferably ≤ 250 Mpa ($\cong 36$ ksi)	Lower strength steels have (a) a long yield region, (b) greater ductility, (c) greater f_{ult}/f_y ratio

2. Flexural Members (members whose factored axial stress $\leq f_c'/10$)

	Specification	Possible Explanation
Size	$b/d \geq 0.3$	To ensure lateral stability and improve torsional resistance
	$b \geq 8''$	To (a) decrease geometric error, (b) facilitate rod placement
	$d \leq L_c/4$	Behavior and design of deeper members are significantly different
Longitudinal Reinforcement	$N_{s(top)}$ and $N_{s(bottom)} \geq 2$	Construction requirement
	$\rho \geq 0.1\sqrt{f_c'}/f_y$ (f_c', f_y in ksi) at both top and bottom	To avoid brittle failure upon cracking
	$\rho \leq 0.025$ at top or bottom	To (a) cause steel yielding before concrete crushing and (b) avoid steel congestion
	$A_{s(bottom)} \geq 0.5A_{s(top)}$ at joint and $A_{s(bottom)/(top)} \geq 0.25A_{s(top)\,(max)}$ at any section	To ensure (a) adequate ductility and (b) minimum reinforcement for moment reversal
	Both top and bottom bars at an external joint must be anchored $\geq L_d + 10d_b$ from inner face of column with 90° bends	To ensure (a) adequate bar anchorage, (b) joint ductility
	Lap splices are allowed for $\leq 50\%$ of bars, only where stirrups are provided @$\leq d/4$ or $4''$ c/c	Closely spaced stirrups are necessary within lap lengths because of the possibility of loss of concrete cover
	Lap splice lengths $\geq L_d$ and are not allowed within distance of 2d from joints or near possible plastic hinges	Lap splices are not reliable under cyclic loading into the inelastic range
Web Reinforcement	Web reinforcements must consist of closed vertical stirrups with 135° hooks and $10d_t$ ($\geq 3''$) extensions	To provide lateral support and ensure strength development of longitudinal bars
	Design shear force is the maximum of (a) shear force from analysis, (b) shear force due to vertical loads plus as required for flexural yielding of joints	It is desirable that the beams should yield in flexure before failure in shear

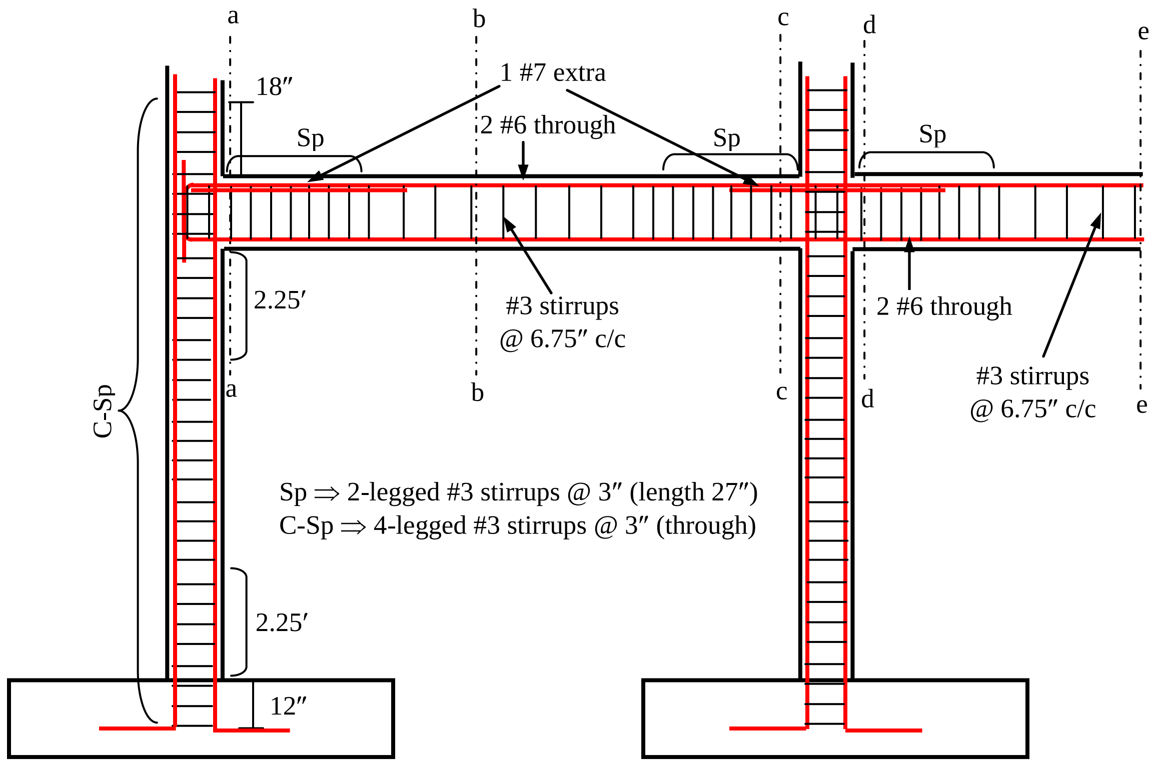
	Spacing of hoops within 2d (beginning at ≤ 2″) at either end of a beam must be ≤ d/4, $8d_b$; elsewhere $S_t \leq d/2$	To (a) provide resistance to shear, (b) confine concrete to improve ductility, (c) prevent buckling of longitudinal compression bars

3. Axial Members (members whose factored axial stress ≥ $f_c'/10$)

	Specification	Possible Explanation
Size	$b_c/h_c \geq 0.4$	To ensure lateral stability and improve torsional resistance
	$b_c \geq 12''$	To avoid (a) slender columns, (b) column failure before beams
Longitudinal Reinforcement	Lap splices are allowed only for ≤ 50% of bars, only where stirrups are provided @$\leq b_c/4$ or 4″	Closely spaced stirrups are necessary within lap lengths because of the possibility of loss of concrete cover
	Lap splice lengths ≥ L_d and only allowed in the center half of columns	Lap splices are not reliable under cyclic loading into the inelastic range
	$0.01 \leq \rho_g \leq 0.06$	To (a) ensure effectiveness and (b) avoid congestion of longitudinal bars
	$\sum M_{c,ult} \geq 1.2 \sum M_{b,ult}$ at joint	To obtain 'strong column weak beam condition' to avoid column failure before beams
Transverse Reinforcement	Transverse reinforcement must consist of closed spirals or rectangular/ circular hoops with 135° hooks with $10d_t$ (≥ 3″) extensions	To provide lateral support and ensure strength development of longitudinal bars
	Parallel legs of rectangular hoops must be spaced @ ≤ 12″ c/c	To provide lateral support and ensure strength development of longitudinal bars
	Spacing of hoops within L_0 (≥ d_c, $h_c/6$, 18″) at each end of column must be ≤ $b_c/4$, 4″; else $S_t \leq b_c/2$	To (a) provide resistance to shear, (b) confine concrete to improve ductility, (c) prevent buckling of longitudinal compression bars
	Design shear force is the maximum of (a) shear force from analysis, (b) shear force required for flexural yielding of joints	It is desirable that the columns should yield in flexure before failure in shear
	Special confining reinforcement (i.e., $S_t \leq b_c/4$, 4″) should extend at least 12″ into any footing	To provide resistance to the very high axial loads and flexural demands at the base
	Special confining reinforcement (i.e., $S_t \leq b_c/4$, 4″) should be provided over the entire height of columns supporting discontinued stiff members and extend L_d into the member	Discontinued stiff members (e.g., shear walls, masonry walls, bracings, mezzanine floors) may develop significant forces and considerable inelastic response
	For special confinement, area of circular spirals ≥ $0.11\ S_t d\ (f_c'/f_y)(A_g/A_c-1)$, rectangular hoops ≥ $0.3\ S_t d\ (f_c'/f_y)(A_g/A_c-1)$	To ensure load carrying capacity upto concrete spalling, taking into consideration the greater effectiveness of circular spirals compared to rectangular hoops. It also ensures toughness and ductility of columns

4. Joints of Frames

	Specification	Possible Explanation
Transverse Reinforcement	Special confining reinforcement (i.e., $S_t \leq b_c/4$, 4″) should extend through the joint	To provide resistance to the shear force transmitted by framing members and improve the bond between steel and concrete within the joint
	$S_t \leq b_c/2$, 6″ through joint with beams of width $b \geq 0.75b_c$	Some confinement is provided by the beams framing into the vertical faces of the joint

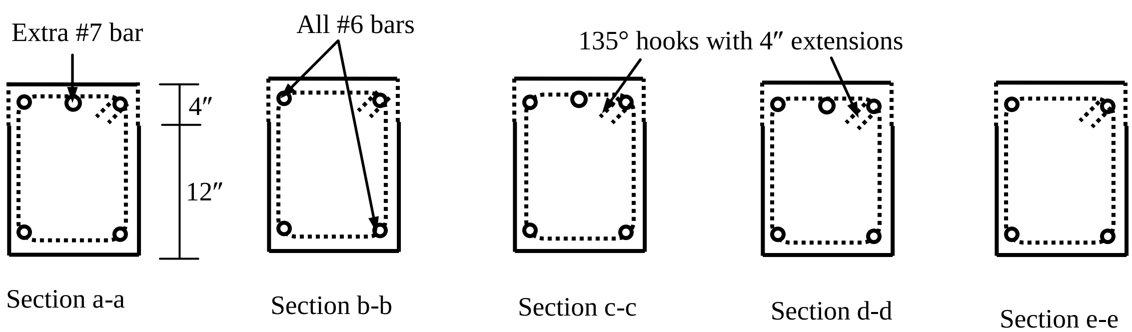


Lap-splices not allowed here
Elsewhere, it is only allowed for 50% bars with special confinement

Anchorage at end joints $L_{anch} = L_d + 10\ d_b$

L_d for #7 bars = $0.04\ A_s\ f_y/\sqrt{f_c'} = 0.04 \times 0.60 \times 40/\sqrt{(3/1000)} \times 1.4 = 24.53''$

$L_{anch} = 24.53 + 10 \times 7/8 = 33.29''$; i.e., 34″



Beam Sections (with reinforcements)

Fig. 21.2: Typical Building Frame satisfying Provisions of Seismic Detailing

Earthquake Repair and Retrofit

Earthquake repairing is to make the existing damaged structure safer for future earthquake so that it can perform better during any future earthquake. It includes renewal of any part of a damaged or deteriorated structure to provide the same level of strength and ductility, which it had prior to the damage.

Seismic retrofitting is to upgrade the earthquake resistance of the structure up to the level of present-day building codes by appropriate techniques. The concepts of retrofitting include repairing and remolding, thereby upgrading of the structural system to improve the performance, function or appearance.

<u>Retrofitting Strategies for RC Structures</u>

1. Global Strategies

    (i) Adding shear wall

    (ii) Adding infill wall

    (iii) Adding bracing

    (iv) Adding wing walls or external buttressing

    (v) Wall thickening

    (vi) Mass reduction

    (vii) Supplemental damping

    (viii) Base isolation

2. Local Strategies

    (i) Jacketing of Beams

    (ii) Jacketing of Columns

    (iii) Jacketing of Beam-Column joints

    (iv) Strengthening of individual footings

<u>Repairing and Retrofitting Strategies for Masonry Structures</u>

(i) Injecting grout or epoxy

(ii) Injecting cement mortar and flat chips

(iii) Wire mesh and cement plaster

(iv) Shotcrete

(v) Adding reinforcements

(vi) Confining with RC or steel

Problems on Earthquake Engineering

1. (i) Use the standard surface-wave formula to calculate the magnitude of an earthquake if it originates at a focal depth of 500 km, the maximum amplitude of ground vibration recorded at an epicentral distance of 5 km is 10 cm and the frequency of surface-wave is 0.05 Hz.
   (ii) For this earthquake, calculate the ground vibration amplitude at an epicentral distance of 50 km.

   Solution
   (i) Using the standard surface-wave formula, with
   $A = 10$ cm $= 10^5$ μm, $T = 1/f = 1/0.05 = 20$ sec, $D = d/h = 5/500 = 0.01$ rad $= 0.573°$
   $M_S = \log_{10}(A/T) + 1.66 \log_{10}(D) + 3.30 = \log_{10}(10^5/20) + 1.66 \log_{10}(0.573) + 3.30 = 6.60$
   (ii) Using $M_S = \log_{10}(A/T) + 1.66 \log_{10}(D) + 3.30$
   $\Rightarrow 6.60 = \log_{10}(A/20) + 1.66 \log_{10}(50/500) + 3.30 \Rightarrow A = 0.219$ cm

2. Use the BNBC response spectrum for Dhaka to calculate the elastic peak deformation and base shear for the (20′ × 20′) floor system described in # 2 of Problem set 2, for k_f equal to
   (i) 2×10^6 lb/in, and (ii) 2×10^4 lb/in [assume $\xi = 5\%$].

   Solution
   For this floor system, $k_1 = 4.02 \times 10^4$ lb/in, $m_1 = 207.25$ lb-sec^2/in
   (i) As calculated earlier, $k_f = 2 \times 10^6$ lb/in $\Rightarrow k_{eff} = k_1k_f/(k_1 + k_f) = 3.94 \times 10^4$ lb/in
   $\Rightarrow \omega_n = 13.79$ rad/sec, $T_n = 2\pi/\omega_n = 0.456$ sec
   $\therefore$ Using BNBC response spectrum, $C = 1.25/T^{2/3} = 1.25/(0.456)^{2/3} = 2.110$
   $\therefore$ Elastic base shear $V_{b(e)} = ZCW = 0.15 \times 2.110 \times (20 \times 20) \times 200/1000 = 25.32$ kips
   $\therefore$ Elastic maximum deformation $u_0 = V_b/k_{eff} = 25.32 \times 1000/(3.94 \times 10^4) = 0.643''$
   (ii) $k_f = 2 \times 10^4$ lb/in $\Rightarrow k_{eff} = k_1k_f/(k_1 + k_f) = 1.34 \times 10^4$ lb/in
   $\Rightarrow \omega_n = 8.03$ rad/sec, $T_n = 2\pi/\omega_n = 0.783$ sec
   $\therefore$ Using BNBC response spectrum, $C = 1.25/T^{2/3} = 1.25/(0.783)^{2/3} = 1.471$
   $\therefore$ Elastic base shear $V_{b(e)} = ZCW = 0.15 \times 1.471 \times (20 \times 20) \times 200/1000 = 17.65$ kips
   $\therefore$ Elastic maximum deformation $u_0 = V_b/k_{eff} = 17.65 \times 1000/(1.34 \times 10^4) = 1.317''$

3. Answer Question 2 using the response spectrum for El Centro earthquake.

   Solution
   For this floor system, $k_1 = 4.02 \times 10^4$ lb/in, $m_1 = 207.25$ lb-sec^2/in
   (i) For $T_n = 0.456$ sec, $C = 2.695$
   $\therefore$ Elastic base shear $V_{b(e)} = ZCW = 0.313 \times 2.695 \times (20 \times 20) \times 200/1000 = 67.48$ kips
   $\therefore$ Elastic maximum deformation $u_0 = V_b/k_{eff} = 67.48 \times 1000/(3.94 \times 10^4) = 1.713''$
   (ii) For $T_n = 0.783$ sec, $C = 1.563$
   $\therefore$ Elastic base shear $V_{b(e)} = ZCW = 0.313 \times 1.563 \times (20 \times 20) \times 200/1000 = 39.14$ kips
   $\therefore$ Elastic maximum deformation $u_0 = V_b/k_{eff} = 39.14 \times 1000/(1.34 \times 10^4) = 2.921''$

4. A 12′ long vertical cantilever pipe (made of steel, with $E = 29 \times 10^6$ psi) supports a 5200 lb weight attached at the tip. Determine the peak deformation and bending stress in the cantilever due to the El Centro data, assuming $\xi = 2\%$, with the properties of the pipe being
   (i) $d_0 = 4.5''$, $d_i = 4.026''$, $t = 0.237''$, (ii) $d_0 = 6.75''$, $d_i = 6.039''$, $t = 0.356''$.

   Solution
   (i) For this system, $I = \pi[(4.5)^4 - (4.026)^4]/64 = 7.23$ in^4
   $\therefore$ Lateral stiffness, $k = 3EI/L^3 = 3(29 \times 10^6) \times 7.23/(12 \times 12)^3 = 211$ lb/in
   $\therefore$ Mass, $m = W/g = 5200/386 = 13.47$ lb-sec^2/in
   $\Rightarrow \omega_n = \sqrt{(211/13.47)} = 3.958$ rad/sec, $T_n = 2\pi/\omega_n = 1.59$ sec
   $\therefore$ From El Centro response spectrum with $\xi = 2\%$, $C = 0.655$
   $\Rightarrow$ Elastic base shear $V_{b(e)} = ZCW = 0.313 \times 0.655 \times 5200/1000 = 1.066$ kips
   $\therefore$ Maximum bending moment $M = 1.066 \times (12 \times 12) = 153.52$ k-in
   $\Rightarrow$ Maximum bending stress $\sigma_{max} = Mc/I = 153.52 \times (4.5/2)/7.23 = 47.80$ ksi
   (ii) For the new system, $I = \pi[(6.75)^4 - (6.039)^4]/64 = 36.62$ in^4
   $\therefore$ Lateral stiffness, $k = 3EI/L^3 = 3(29 \times 10^6) \times 36.62/(12 \times 12)^3 = 1066.82$ lb/in

$\therefore$ Mass, $m = W/g = 5200/386 = 13.47$ lb-sec^2/in

$\Rightarrow \omega_n = \sqrt{(1066.82/13.47)} = 8.900$ rad/sec, $T_n = 2\pi/\omega_n = 0.706$ sec

$\therefore$ From El Centro response spectrum with $\xi = 2\%$, $C = 2.538$

$\Rightarrow$ Elastic base shear $V_{b(e)} = ZCW = 0.313 \times 2.538 \times 5200/1000 = 4.131$ kips

$\therefore$ Maximum bending moment $M = 4.131 \times (12 \times 12) = 594.84$ k-in

$\Rightarrow$ Maximum bending stress $\sigma_{max} = Mc/I = 594.84 \times (6.75/2)/36.62 = 54.83$ ksi

5. Answer Question 3 assuming yield deformation $u_y = 0.50''$.

Solution

(i) Using $u_0 = 1.713''$ and $u_y = 0.50''$, $R_y = 1.713/0.50 = 3.426$

$\therefore$ Inelastic base shear $V_b = V_{b(e)}/R_y = 67.48/3.426 = 19.70$ kips (also $= k_{eff}\, u_y$)

$\therefore$ For $T_n = 0.456$ sec, Ductility ratio $\mu = 4.13$, Maximum deformation $u_m = \mu\, u_y = 2.065''$

(ii) Using $u_0 = 2.921''$ and $u_y = 0.50''$, $R_y = 2.921/0.50 = 5.842$

$\therefore$ Inelastic base shear $V_b = V_{b(e)}/R_y = 39.14/5.842 = 6.70$ kips (also $= k_{eff}\, u_y$)

$\therefore$ For $T_n = 0.783$ sec, Ductility ratio $\mu = 5.842$, Maximum deformation $u_m = \mu\, u_y = 2.921''$

6. Answer Question 4 assuming yield strength $f_y = 36$ ksi.

Solution

(i) Using $f_0 = 47.80$ ksi and $f_y = 36$ ksi, $R_y = 47.80/36 = 1.33$

$\therefore$ Inelastic base shear $V_b = V_{b(e)}/R_y = 1.066/1.33 = 0.803$ kips

$\therefore$ Maximum bending moment $M = 0.803 \times (12 \times 12) = 115.61$ k-in

$\Rightarrow$ Maximum bending stress $\sigma_{max} = Mc/I = 115.61 \times (4.5/2)/7.23 = 36$ ksi

$\therefore$ For $T_n = 1.59$ sec, Ductility ratio $\mu = R_y = 1.33$

Also, $u_y = V_b/k = 803/211 = 3.805''$, Maximum deformation $u_m = \mu\, u_y = 5.053''$

(ii) Using $f_0 = 54.83$ ksi and $f_y = 36$ ksi, $R_y = 54.83/36 = 1.52$

$\therefore$ Inelastic base shear $V_b = V_{b(e)}/R_y = 4.131/1.52 = 2.712$ kips

$\therefore$ Maximum bending moment $M = 2.712 \times (12 \times 12) = 390.57$ k-in

$\Rightarrow$ Maximum bending stress $\sigma_{max} = Mc/I = 390.57 \times (6.75/2)/36.62 = 36$ ksi

$\therefore$ For $T_n = 0.706$ sec, Ductility ratio $\mu = R_y = 1.52$

Also, $u_y = V_b/k = 2712/1066.82 = 2.542''$, Maximum deformation $u_m = \mu\, u_y = 3.872''$