Diagrammatic Convention for Supports

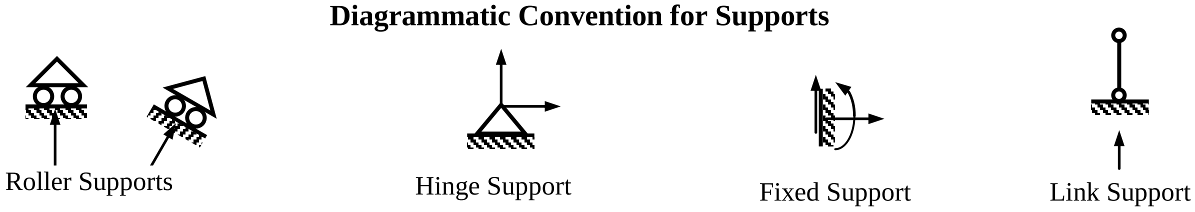


Fig. 1.1: Support Conditions

Axial Force, Shear Force and Bending Moment

Three forces are required to maintain internal equilibrium at every section of a beam.

The force acting along the geometric axis of the beam (i.e., perpendicular to the cross-sectional plane) is called *Axial Force*, the force acting parallel to its cross-sectional plane (i.e., perpendicular to the axis) is *Shear Force* while the moment or couple acting at the section is *Bending Moment*. They maintain the horizontal equilibrium ($\sum F_x = 0$), vertical equilibrium ($\sum F_y = 0$) and moment equilibrium ($\sum M_z = 0$) of the beam segment at the relevant cross-section and are therefore calculated by considering the three equilibrium equations at the section.

For design purposes, it may be necessary to calculate the axial force, shear force and bending moment at several sections of a beam, resulting in diagrams of the forces versus the distance along beam length.

Axial Force, Shear Force and Bending Moment Diagram

Draw the *Axial Force Diagram* (*AFD*), *Shear Force Diagram* (*SFD*) and *Bending Moment Diagram* (*BMD*) of the beam loaded as shown below.

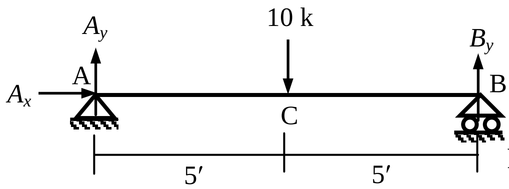


Fig. 1.2: Beam Free Body

The support reactions can be calculated by considering the equilibrium equations of the overall structure; i.e., $\sum F_x = 0$, $\sum M_z = 0$ (at B) and $\sum M_z = 0$ (at A), giving $A_x = 0$, $A_y = 5$ k and $B_y = 5$k.

<u>Example 1.1: Free Body Diagrams for various Segments of the Beam</u>

Free body diagrams are drawn for various beam segments considering the points D, C(left), C(right) and E at distances 3′, 5′ (left), 5′ (right) and 7′ from the support A of the beam.

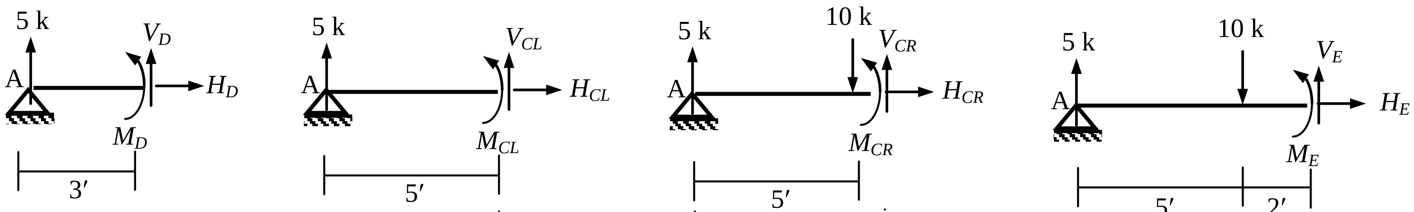


Table 1.1: Results of Example 1.1

x (ft)	*H* (k)	V (k)	*M* (k-ft)
3	0	−5	5 × 3 = 15
5(−)	0	−5	5 × 5 = 25
5(+)	0	+10−5 = +5	5 × 5 − 10 × 0 = 25
7	0	+10−5 = +5	5 × 7 − 10 × 2 = 15

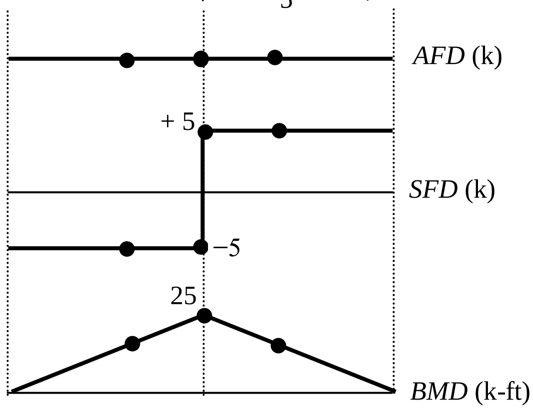


Fig. 1.3: Introduction to *AFD*, *SFD* and *BMD*

Sign Convention for Axial Force, Shear Force and Bending Moment

For this course, the following sign conventions have been chosen for axial force, shear force and bending moment. In short, tension is assumed positive for axial force, shear forces forming clockwise couple are considered positive while a bending moment causing sagging shape is taken as positive.

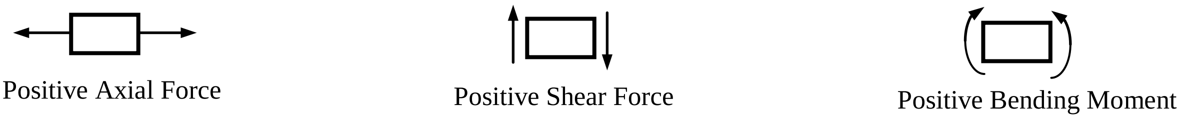


Fig. 1.4: Sign Convention for Axial Force, Shear Force and Bending Moment

Relationship between Applied Load, Shear Force and Bending Moment

Instead of the direct approach of cutting a beam and determining the shear force and bending moment at a section by statics, an efficient alternative approach can be used. Certain fundamental differential relations need to be derived for this purpose. These can be used for the construction of shear force and bending moment diagrams of beams.

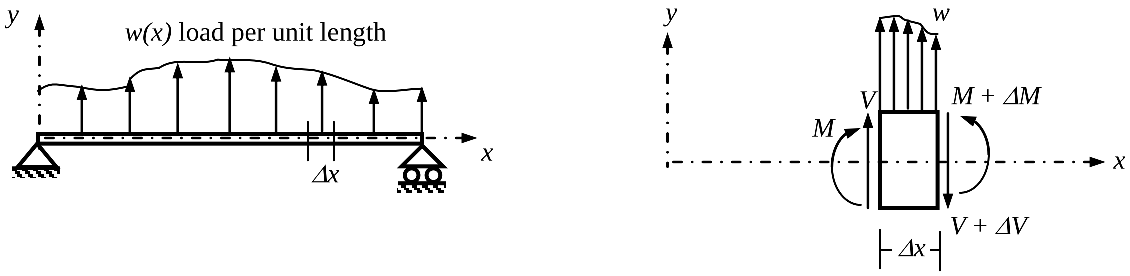


Fig. 1.5: Beam and Beam elements between adjoining sections

Consider an element Δx long, isolated by two adjoining sections taken perpendicular to its axis in the beam loaded as shown above. As the shear force and bending moment may vary from one section to the next, they are assumed to be V and M on the left section and ($V + \Delta V$) and ($M + \Delta M$) on the right section of the beam element. However, if the length Δx is infinitesimal (i.e., $\Delta x \to 0$), no variation of the applied load $w(x)$ is considered; i.e., $w(x)$ is assumed to be constant within the length Δx.

From the equilibrium of vertical forces; i.e., $\Sigma F_y = 0$

$$\Rightarrow V + w\,\Delta x - (V + \Delta V) = 0 \Rightarrow \Delta V/\Delta x = w \qquad \text{………………..(1.1)}$$

From the equilibrium of moments about the right sections; i.e., $\Sigma M_z = 0$

$$\Rightarrow M + V\,\Delta x + (w\,\Delta x)\,\Delta x/2 - (M + \Delta M) = 0 \Rightarrow \Delta M/\Delta x = V + w\,\Delta x/2 \qquad \text{………………..(1.2)}$$

In the limit $\Delta x \to 0$, the above equations take the following forms,

$$dV/dx = w \qquad \text{………………..(1.3)}$$

$$dM/dx = V \qquad \text{………………..(1.4)}$$

Combining Eq. (1.3) and (1.4), $d^2M/dx^2 = w$ ………………..(1.5)

Eq. (1.5) can be used for determining support reactions of the beams, while Eqs. (1.3) and (1.4) are very convenient for the construction of shear force and bending moment diagrams, using the equations

$$V = \int w\,dx + C_1 \qquad \text{………………..(1.6)}$$

and $M = \int V\,dx + C_2$ ………………..(1.7)

Integration Method for SFD and BMD

Example 1.2

Derive equations for the shear force and bending moment of the beams loaded as shown below. Also draw the corresponding SFD and BMD.

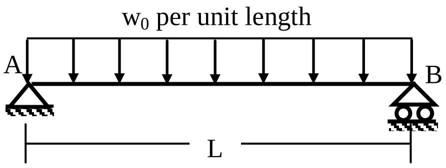


Fig. 1.6(a): Simply Supported Beam under UDL

$w(x) = -w_0$ ………..(1.8)
$V(x) = -w_0\ x + C_1$ ………..(1.9)
$M(x) = -w_0\ x^2/2 + C_1\ x + C_2$ ………(1.10)

Boundary conditions
$M(0) = 0$ and $M(L) = 0$

$M(0) = 0$ in Eq. (1.10)
$\Rightarrow 0 = 0 + 0 + C_2$
$\Rightarrow C_2 = 0$ ………(1.11)

$M(L) = 0$ in Eq. (1.10)
$\Rightarrow 0 = -w_0\ L^2/2 + C_1\ L + 0$
$\Rightarrow C_1 = w_0\ L/2$ ………(1.12)

$\therefore$ Eq.(1.9) $\Rightarrow V(x) = -w_0\ x + w_0\ L/2$ ……(1.13)

(1.10) $\Rightarrow M(x) = -w_0\ x^2/2 + w_0 L/2\ x$ ……(1.14)

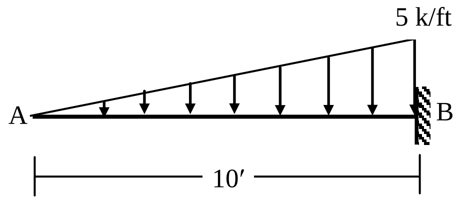


Fig. 1.6(b): Cantilever Beam under triangular load

$w(x) = -0.5\ x$ ………(1.15)
$V(x) = -0.5\ x^2/2 + C_1$ ……..…(1.16)
$M(x) = -0.5\ x^3/6 + C_1\ x + C_2$ ………(1.17)

Boundary conditions
$V(0) = 0$ and $M(0) = 0$

$V(0) = 0$ in Eq. (1.16)
$\Rightarrow 0 = 0 + C_1$
$\Rightarrow C_1 = 0$ ………(1.18)

$M(0) = 0$ in Eq. (1.17)
$\Rightarrow 0 = 0 + 0 + C_2$
$\Rightarrow C_2 = 0$ ………(1.19)

$\therefore$ Eq.(1.16) $\Rightarrow V(x) = -0.5\ x^2/2$ ………(1.20)

(1.17) $\Rightarrow M(x) = -0.5\,x^3/6$ ……..…(1.21)

The following SFD and BMD are drawn for (b) using EXCEL

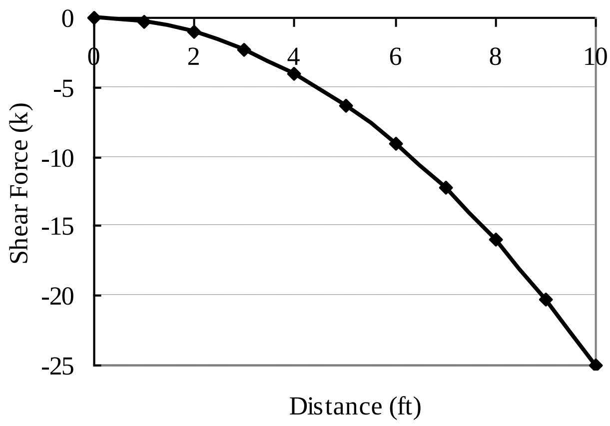


Fig. 1.7(a): SFD for Example 1.2(b)

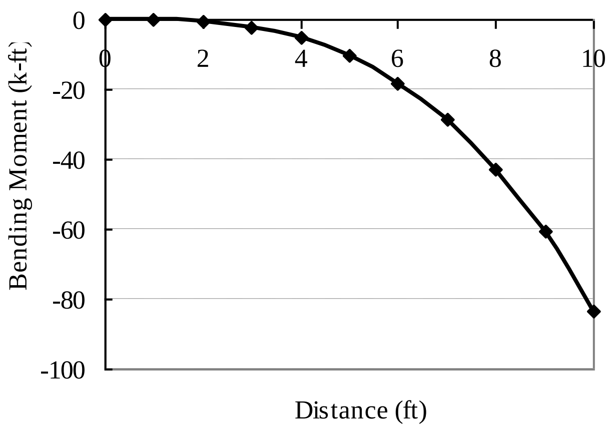


Fig. 1.7(b): BMD for Example 1.2(b)

Representing Different Loadings by Singularity Functions

Singularity Functions:

$f(x) = \langle x-a\rangle^n \Rightarrow f(x) = 0$, when $x \le a$ and $f(x) = \langle x-a\rangle^n$, when $x > a$ [where $n \ge 0$] …………(1.22)

$\therefore f(x) = \langle x-a\rangle^0 \Rightarrow f(x) = 0$, when $x \le a$ and $f(x) = 1$, when $x > a$

However $\langle x-a\rangle^n$ has no physical significance if $n < 0$, and is written only as a notation with an asterisk ($*$) as subscript; e.g., $f(x) = \langle x-a\rangle^{-1}{}_*$

The integration and differentiation of singularity functions follow the rules for ordinary polynomial functions;i.e., $\int\langle x-a\rangle^n\,dx = \langle x-a\rangle^{n+1}/(n+1) + C_1$ and $d(\langle x-a\rangle^n)/dx = n\langle x-a\rangle^{n-1}$ ………….(1.23)

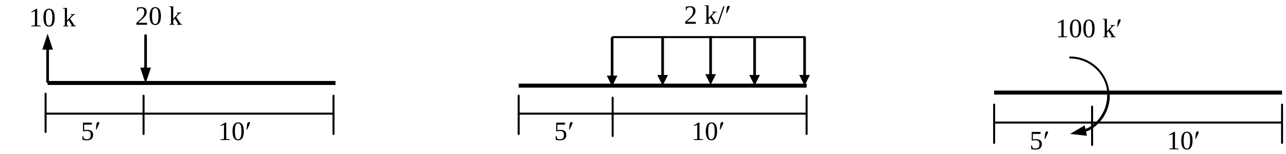
e.g., $\int\langle x-a\rangle^2\,dx = \langle x-a\rangle^3/3 + C_1$ and $d(\langle x-a\rangle^2)/dx = 2\langle x-a\rangle^1$

By definition, $\int\langle x-a\rangle^n{}_*\,dx = \langle x-a\rangle^{n+1}{}_* + C_1$ if $n < 0$ ……..…………(1.24)

e.g., $\int\langle x-a\rangle^{-2}{}_*\,dx = \langle x-a\rangle^{-1}{}_* + C_1$ and $\int\langle x-a\rangle^{-1}{}_*\,dx = \langle x-a\rangle^0 + C_1$

Singularity Functions for Common Loadings:

Common loadings are expressed in terms of the following singularity functions



w(x) = $10\langle x-0\rangle^{-1}{}_* - 20\langle x-5\rangle^{-1}{}_*$ w(x) = $-2\langle x-5\rangle^0$ w(x) = $100\langle x-5\rangle^{-2}{}_*$

Fig. 1.8: Singularity Functions for Common Loadings

Example 1.3

Derive the equations for the shear force and bending moment of the beams loaded as shown below.

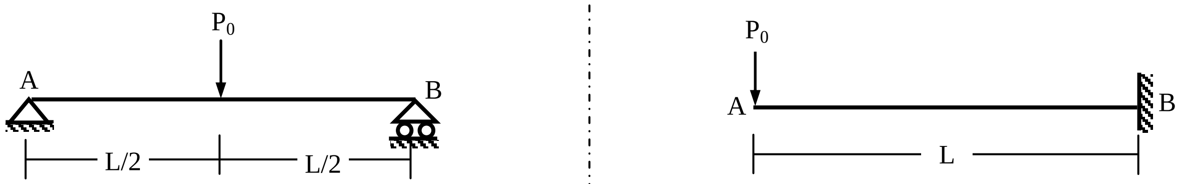


Fig. 1.9: (a) Simply Supported Beam and (b) Cantilever Beam under concentrated load

$w(x) = -P_0\langle x-L/2\rangle^{-1}{}_*$ ……..(1.25)

$V(x) = -P_0\langle x-L/2\rangle^0 + C_1$ ……..(1.26)

$M(x) = -P_0\langle x-L/2\rangle^1 + C_1 x + C_2$ …….(1.27)

Boundary conditions: M(0) = 0 and M(L) = 0

M(0) = 0 in Eq. (1.27)

$\Rightarrow 0 = 0 + 0 + C_2 \Rightarrow C_2 = 0$ …….(1.28)

M(L) = 0 in Eq. (1.27)

$\Rightarrow 0 = -P_0 L/2 + C_1 L + 0 \Rightarrow C_1 = P_0/2$ …....(1.29)

$\therefore$Eq.(1.26) $\Rightarrow V(x) = -P_0\langle x-L/2\rangle^0 + P_0/2$ …...(1.30)

(1.27) $\Rightarrow M(x) = -P_0\langle x-L/2\rangle^1 + P_0/2\, x$ ……(1.31)

$w(x) = -P_0\langle x-0\rangle^{-1}{}_*$ ……..(1.32)

$V(x) = -P_0\langle x-0\rangle^0 + C_1$ ……..(1.33)

$M(x) = -P_0\langle x-0\rangle^1 + C_1 x + C_2$ …….(1.34)

Boundary conditions: V(0) = 0 and M(0) = 0

V(0) = 0 in Eq. (1.33)

$\Rightarrow 0 = 0 + C_1 \Rightarrow C_1 = 0$ …….(1.35)

M(0) = 0 in Eq. (1.34)

$\Rightarrow 0 = 0 + 0 + C_2 \Rightarrow C_2 = 0$ …....(1.36)

$\therefore$Eq.(1.33) $\Rightarrow V(x) = -P_0\langle x-0\rangle^0$ …...(1.37)

(1.34) $\Rightarrow M(x) = -P_0\langle x-0\rangle^1$ ……(1.38)

Example 1.4: Derive the equations for the SF and BM of the beam loaded as shown below (in Fig. 1.10).

$w(x) = 10\langle x-0\rangle^{-1}{}_* + R_1\langle x-5\rangle^{-1}{}_* - 1\langle x-5\rangle^0 + 1\langle x-15\rangle^0 + 0.15\langle x-30\rangle^1$

BCs: V(0) = 0, M(0) = 0, M(40) = 0

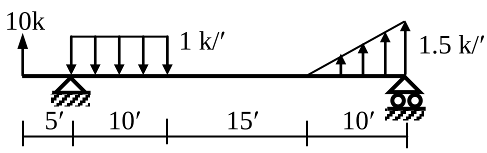


Fig. 1.10

Summation Method for SFD and BMD

The earlier formulations derived the following relations between applied load, shear force and bending moment,

$dV/dx = w$ ………………….(1.3)

$dM/dx = V$ ……………….(1.4)

$V = \int w\,dx + C_1$ ……………….(1.6)

and $M = \int V\,dx + C_2$ ……………….(1.7)

While these equations have been used in the integration method of SFD and BMD, for hand calculations it is more convenient to use their physical meanings; i.e.,
(i) Eqs. (1.3) and (1.4) $\Rightarrow$ Slope of the SFD is the applied load, while slope of the BMD is the shear force
(ii) Eq. (1.6) $\Rightarrow$ *Shear Force* is the *summation of the vertical forces along the beam (*C_1 is the SF at left end*)*
and (1.7) $\Rightarrow$ *Bending Moment* is equal to the *area under the SFD (*C_2 is the BM at left end*)*

The conclusions (ii) are particularly useful and form the basic concepts of the *Summation Method* of drawing the SFD and BMD by hand calculation.

Based of that, the nature of SFD and BMD for common loading conditions is shown in Table 1.2.

Table 1.2: Nature of Load, SF and BM

w(x)	Concentrated Load	UDL	Triangular
V(x)	Straight Line, No Slope	Sloped Straight Line	Parabolic
M(x)	Sloped Straight Line	Parabolic	Cubic

Example 1.5
Use the Summation Method to draw the SFD and BMD of the beams loaded as shown below.

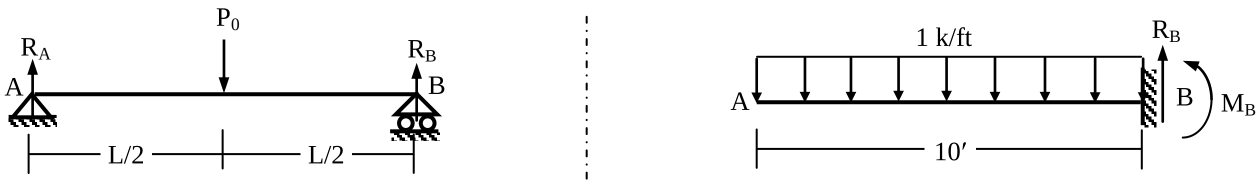
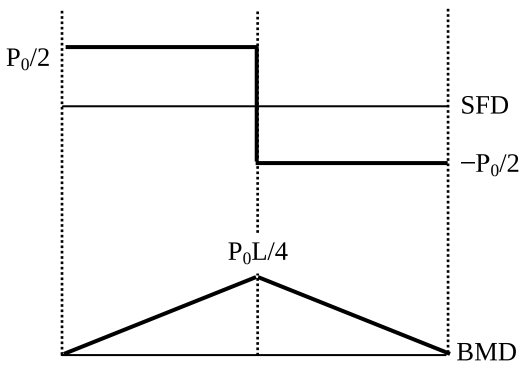
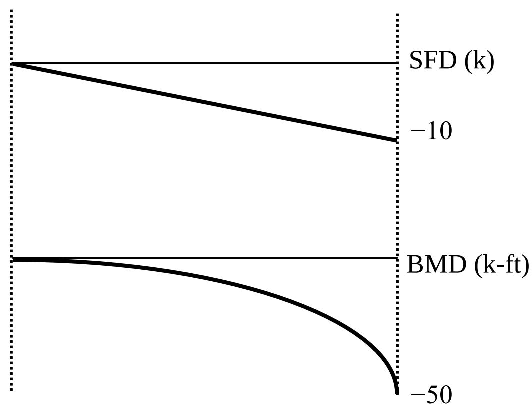


Fig. 1.11: (a) Simply Supported Beam under concentrated load and (b) Cantilever Beam under UDL

$\sum M_A = 0 \Rightarrow P_0\,L/2 - R_B\,L = 0 \Rightarrow R_B = P_0/2$
$\sum F_y = 0 \Rightarrow R_A - P_0 + R_B = 0 \Rightarrow R_A = P_0/2$



$\sum F_y = 0 \Rightarrow R_B - 1 \times 10 = 0 \Rightarrow R_B = 10$ k
$\sum M_B = 0 \Rightarrow -M_B - 1 \times 10 \times 5 = 0 \Rightarrow M_B = -50$ k-ft

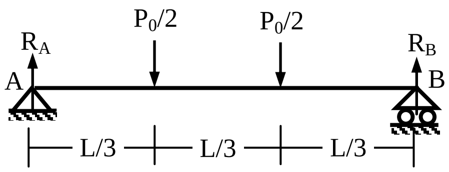


Example 1.6
Use the Summation Method to draw the SFD and BMD of the beams shown in Example 1.2(a), (b), 1.3(b).

Typical SFD and BMD for Beams

Example 1.6

Use the Summation Method to draw the SFD and BMD of the beams loaded as shown below.



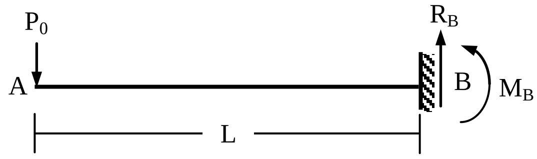
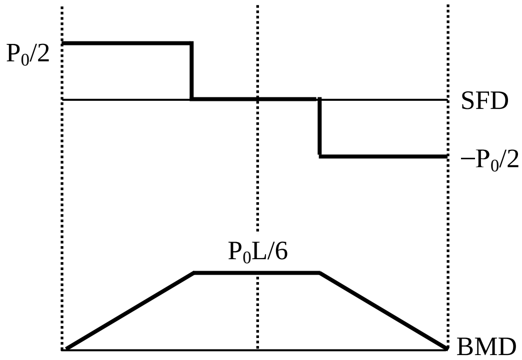


Fig. 1.12: (a) Simply Supported Beam under 2-Point load and (b) Cantilever Beam under concentrated load

$\sum M_A = 0 \Rightarrow (P_0/2)\,L/3 + (P_0/2)\,2L/3 - R_B\,L = 0$

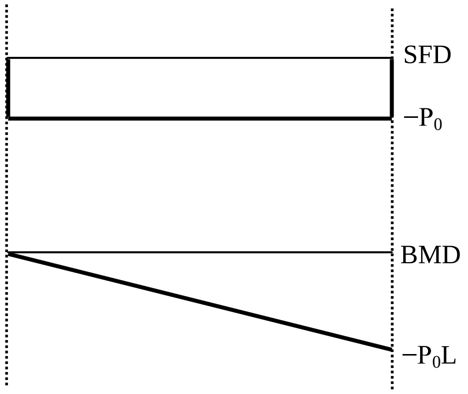
$\Rightarrow R_B = P_0/2$

$\sum F_y = 0 \Rightarrow R_A - P_0/2 - P_0/2 + R_B = 0 \Rightarrow R_A = P_0/2$



$\sum F_y = 0 \Rightarrow R_B - P_0 = 0 \Rightarrow R_B = P_0$

$\sum M_B = 0 \Rightarrow -M_B - P_0\,L = 0 \Rightarrow M_B = -P_0\,L$



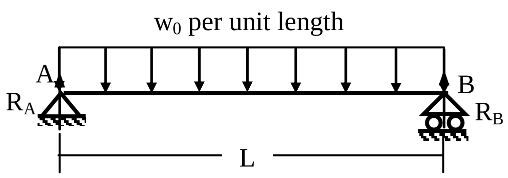


Fig. 1.6(a): Simply Supported Beam under UDL

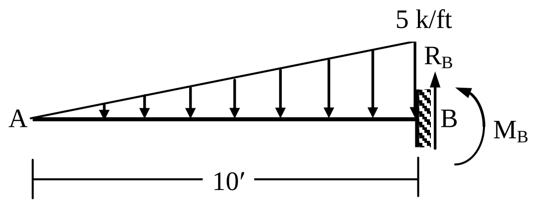
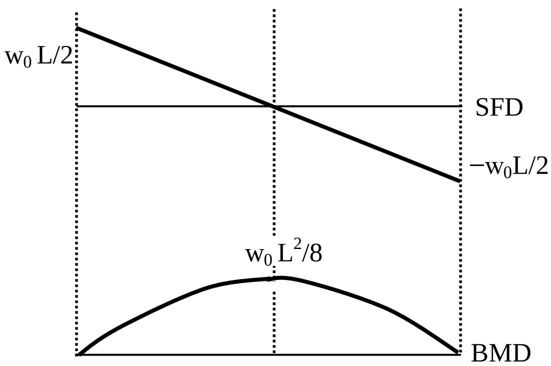


Fig. 1.6(b): Cantilever Beam under triangular load

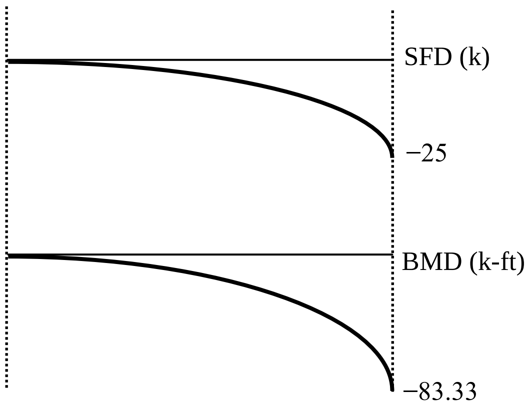
$\sum M_A = 0 \Rightarrow (w_0\,L)\,L/2 - R_B\,L = 0 \Rightarrow R_B = w_0L/2$

$\sum F_y = 0 \Rightarrow R_A - w_0\,L + R_B = 0 \Rightarrow R_A = w_0L/2$



$\sum F_y = 0 \Rightarrow R_B - 5 \times 10/2 = 0 \Rightarrow R_B = 25$ k

$\sum M_B = 0 \Rightarrow -M_B - 25 \times 10/3 = 0 \Rightarrow M_B = -83.33$ k-ft



Further Topics on SFD and BMD

SFD and BMD for Beams with Internal Hinge

Internal hinges are often used in beams (particularly bridges) to make them statically determinate. Since no bending moment can develop in an internal hinge, it provides an additional equation for calculating the support reactions, which would otherwise be impossible using statics only.

A beam is not continuous at an internal hinge, i.e., its slopes on both sides of it are different. Details of this are discussed in studies on beam deflection.

Example 1.7

Use the Integration Method and Summation Method to draw the SFD and BMD of the beam shown below.

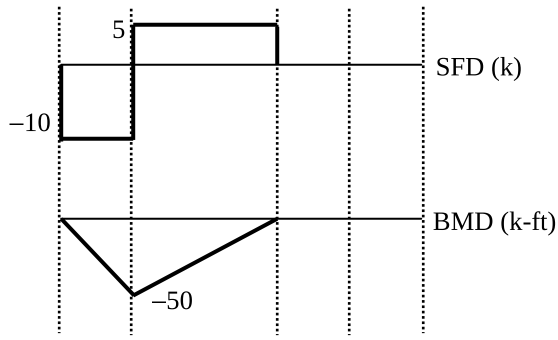
Fig. 1.7: Beam with Internal Hinge

$w(x) = -10\langle x-0\rangle^{-1}_{*} + R_B\langle x-5\rangle^{-1}_{*} + R_C\langle x-15\rangle^{-1}_{*}$
$V(x) = -10\langle x-0\rangle^{0} + R_B\langle x-5\rangle^{0} + R_C\langle x-15\rangle^{0} + C_1$
$M(x) = -10\langle x-0\rangle^{1} + R_B\langle x-5\rangle^{1} + R_C\langle x-15\rangle^{1} + C_1\, x + C_2$

Boundary conditions: $V(0) = 0$, $M(0) = 0$, $M(20) = 0$, $M(25) = 0$
$V(0) = 0 \Rightarrow C_1 = 0$, $M(0) = 0 \Rightarrow C_2 = 0$
$M(20) = 0 \Rightarrow 0 = -200 + 15\, R_B + 5\, R_C = 0$
and $M(25) = 0 \Rightarrow 0 = -250 + 20\, R_B + 10\, R_C = 0$
Solving $\Rightarrow R_B = 15$, $R_C = -5$

$\therefore V(x) = -10\langle x-0\rangle^{0} + 15\langle x-5\rangle^{0} - 5\langle x-15\rangle^{0}$
and $M(x) = -10\langle x-0\rangle^{1} + 15\langle x-5\rangle^{1} - 5\langle x-15\rangle^{1}$

$BM_D = 0 \Rightarrow -R_E \times 5 = 0 \Rightarrow R_E = 0$
$\sum M_C = 0 \Rightarrow R_B \times 10 - 10 \times 15 = 0 \Rightarrow R_B = 15$ k
$\sum F_y = 0 \Rightarrow R_B + R_C + R_E - 10 = 0 \Rightarrow R_C = -5$ k



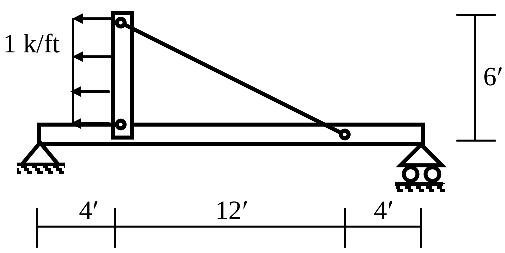
SFD and BMD for Frames

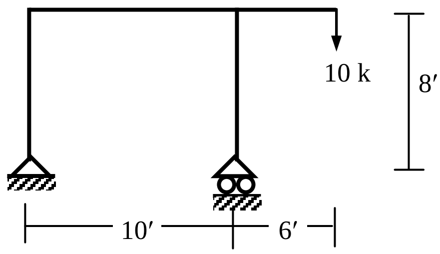
Since frames are composed of beams (and columns), their SFD and BMD follow the same basic process as for individual beams. The best process would be to draw the freebody diagram of each structural element (i.e., each beam and column) and draw their SFD and BMD just like for beams.

The final diagrams in such cases will be an assembly of SFD and BMD of all structural elements of the frame.

Example 1.8

Draw the SFD and BMD of the frames shown below.

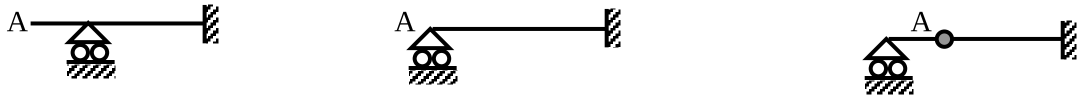




Summary Notes on BMD

1. BM = 0 at points A

   (i) Free End (ii) Hinge/Roller Supported End (iii) Internal Hinge

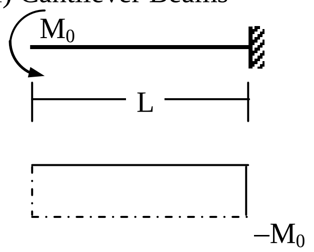


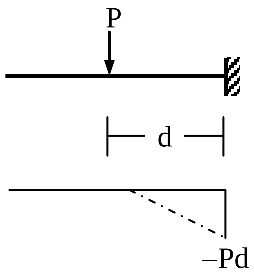
2. BM ≠ 0 at points B (in general, but can be = 0 only for special loading cases)

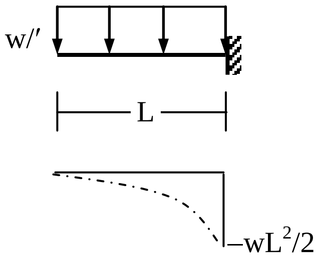
   (i) Fixed End (ii) Internal Roller/Hinge support

3. Verify and memorize the following BMDs

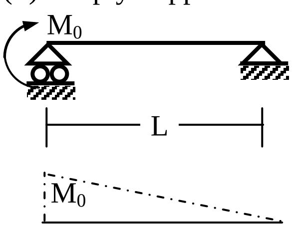
   (i) Cantilever Beams

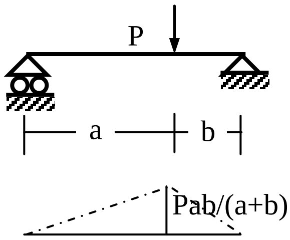


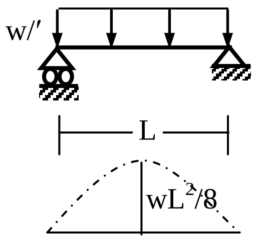




   (ii) Simply Supported Beams

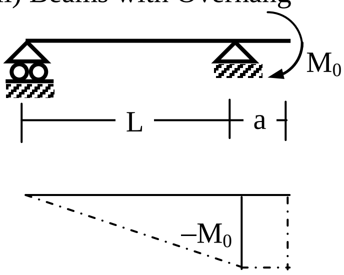


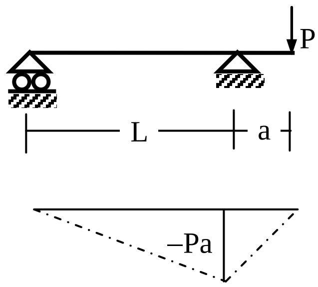


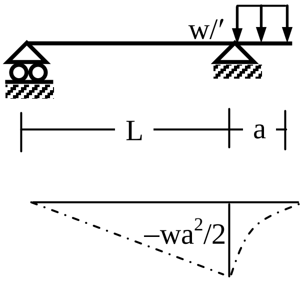


(= PL/4, if a = b = L/2)

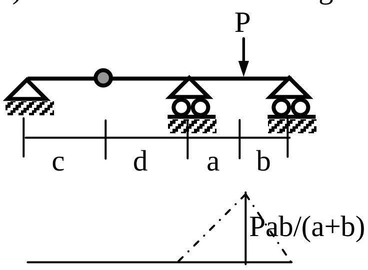
   (iii) Beams with Overhang

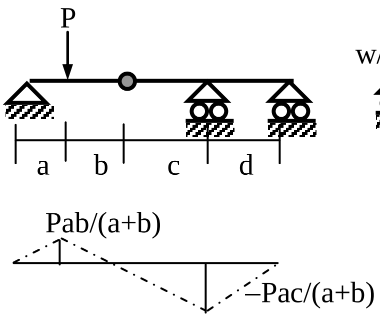


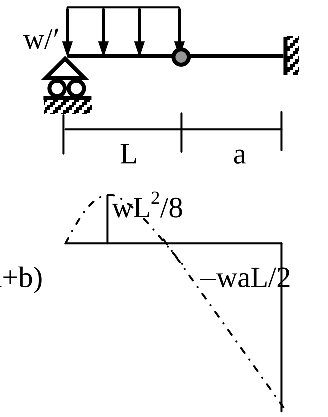




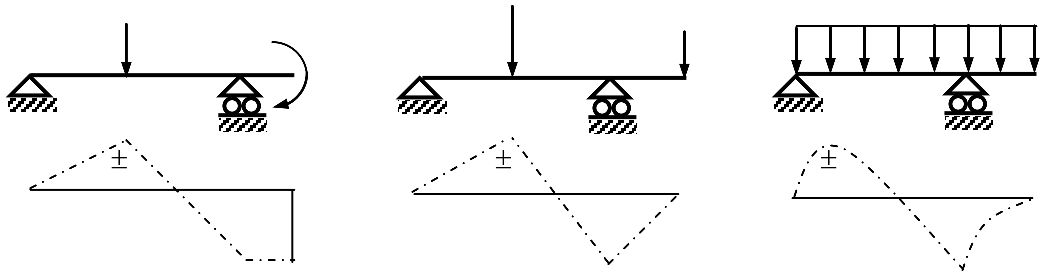
   (iv) Beams with Internal Hinge



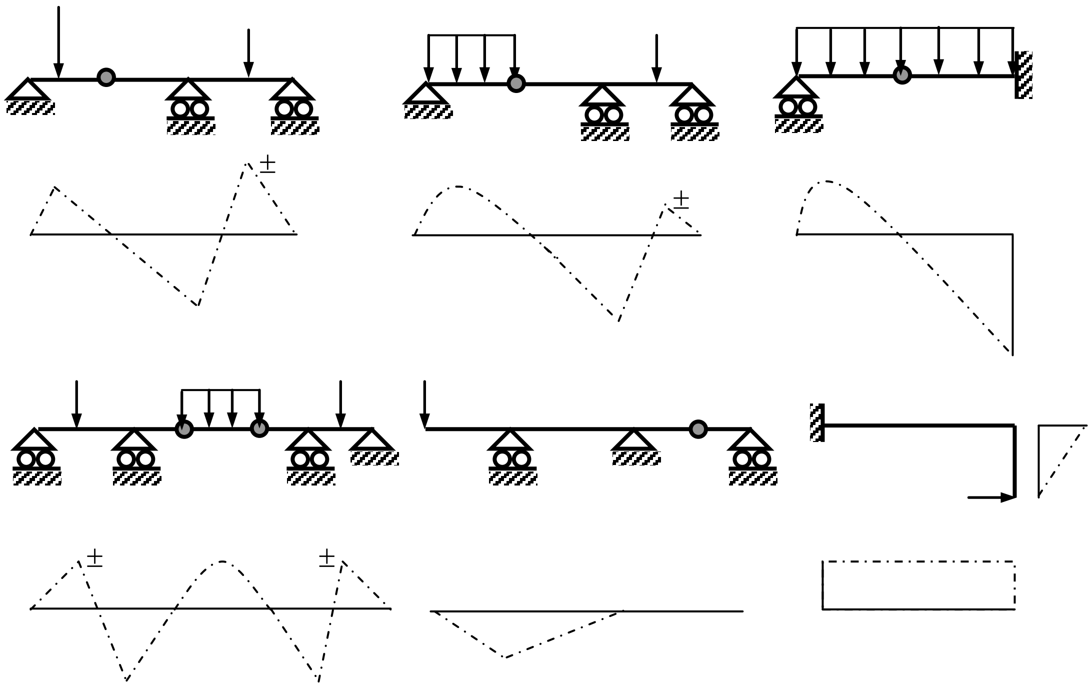




4. Qualitative BMDs

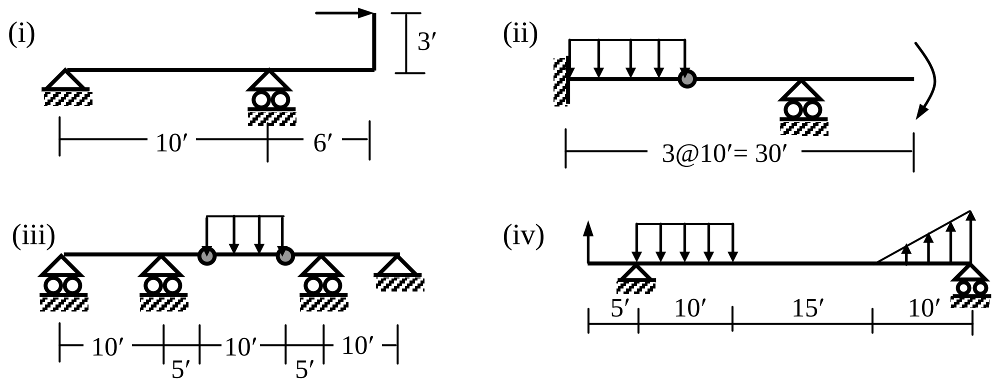


[± implies that the ordinate can be positive or negative depending on the loads and spans]

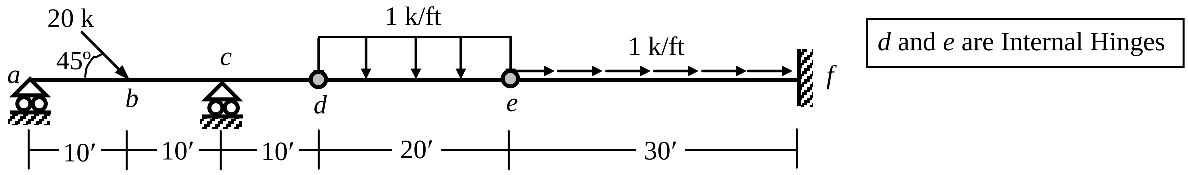


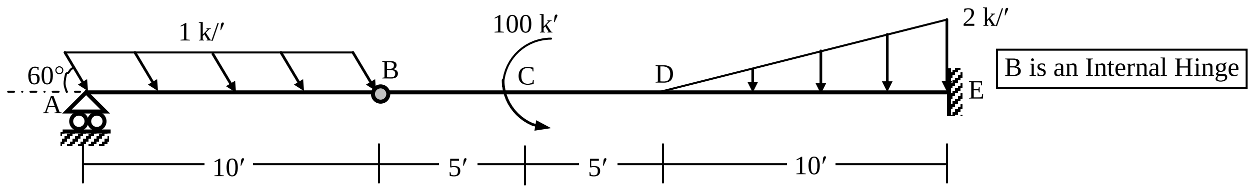
Practice Problems on SFD and BMD

1. Draw the SFD and BMD of the beams loaded as shown below
   [Assume concentrated moment = 100 k′, load = 10 k, UDL = 1 k/′, peak of triangular load = 1.5 k/′].

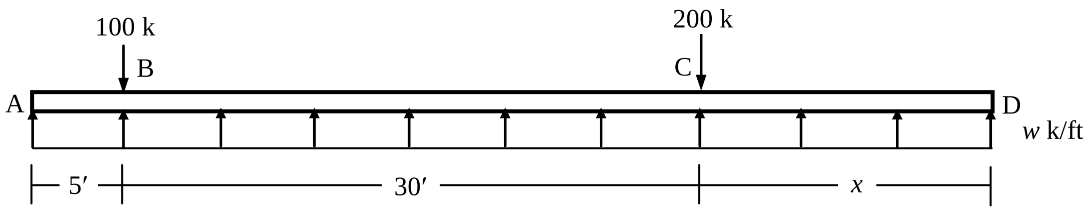


2. Draw the AFD, SFD and BMD of the beams loaded as shown below.

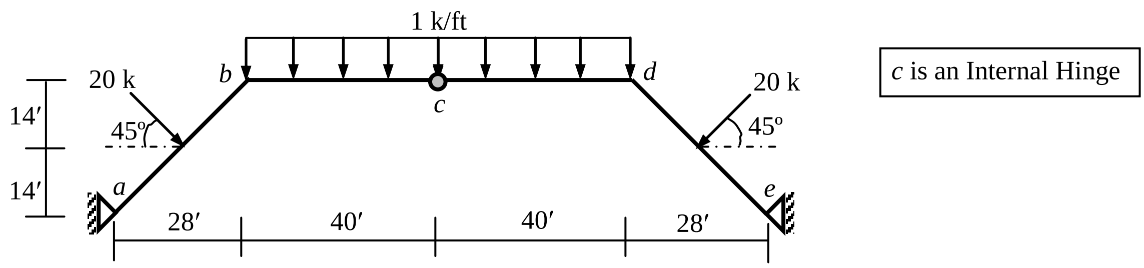




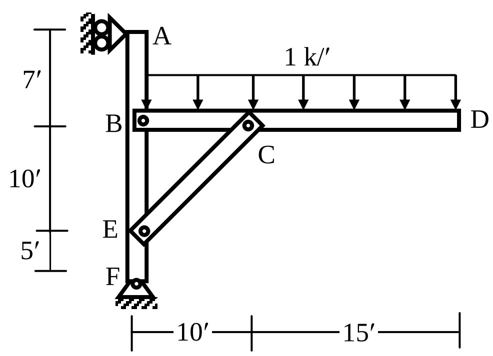
3. For the load distribution over the length of footing ABCD shown in the figure below, calculate
   (i) the length x and uniformly distributed load w k/ft required to maintain equilibrium,
   (ii) the shear force at the left and right of B and bending moment at C using Singularity Functions.



4. Draw the AFD, SFD and BMD of the beam *bcd* in the frame *abcde* loaded as shown below.



5. Draw the AFD, SFD and BMD of column ABEF in the frame shown below [B, C, E, F are pin joints].



Stress, Strength and Factor of Safety

Stress

Stress is defined as the internal force on a body per unit area. Thus if an internal axial force *P* acts on a cross-sectional area *A*, the axial stress on the area is

$$\sigma = P/A \qquad \text{…………………..(2.1)}$$

Fig. 2.1 shows a body being subjected to an external axial load of *P*, which causes an internal force *P* at every cross-section of the body. Therefore, the axial stress σ at each cross-section of the body is equal to *P/A*.

The commonly used units of stress are lb/in^2 (psi), kip/in^2 (ksi), kg/cm^2, N/m^2 (Pascal or Pa), kN/m^2 (kilo-Pascal or kPa), MN/m^2 (mega-Pascal or MPa, also given by N/mm^2) etc.

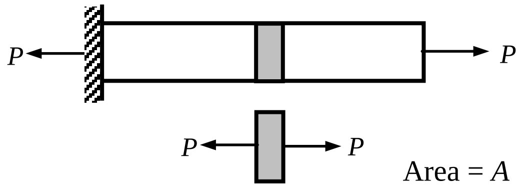


Fig. 2.1: Concept of Normal Stress

In general, stresses can be classified as *Normal Stress* (acting perpendicular to the area) and *Shear Stress* (acting parallel to the area). Under this broad classification, several types of normal and shear stress may act on structures due to various types of loading. Other than the axial loading mentioned, normal stress may also be caused by bending moments. Shear stresses are also caused by direct or flexural shear forces as well as torsional moments. In general, if a shear force *V* acts parallel to an area *A*, the *average* shear stress on the area is

$$\tau = V/A \qquad \text{………………..(2.2)}$$

Strength

Strength is the ability of a body to resist stress. The purpose of calculating stresses in structural members is to compare them with the experimentally determined material strengths in order to assure desired performance.

Physical testing in a laboratory can provide information regarding a material's resistance to stress. In a laboratory, specimens of known material, manufacturing process and heat treatment are subjected to successively increasing known forces until they finally rupture. The force necessary to cause rupture is called the *Ultimate Load*. The *Ultimate Strength* of the material is obtained by dividing this ultimate load by the original cross-sectional area of the specimen. Therefore, the units of stress and strength are the same.

The common types of strength tests performed in the laboratory on Civil Engineering materials are

* Tension Test (of steel, concrete, cement)
* Compression Test (of concrete, brick, timber, soil)
* Direct Shear Test (of metals, concrete, soil)
* Torsional Shear Test (of metals, concrete)
* Static Bending Test (of steel, concrete, timber)
* Impact Test (of metals, aggregate)
* Fatigue/Abrasion Test (of metals, concrete, aggregate)
* Non-Destructive Test (of metals, concrete)

Besides laboratory tests, field tests are also performed (e.g., on soil) to determine material strengths in a more realistic scenario.

Factor of Safety

Although a material can be stressed up to its ultimate strength before it ruptures, practical application and serviceability do not often allow it to be so highly stressed. Some safety is required to allow for the uncertainties and variations in applied loads, field conditions and material strengths. The permissible limit up to which a material can be stressed is called its *Allowable Stress*. Therefore the ratio of ultimate strength and allowable stress is a 'safety ratio' of the material (i.e., indicates how safe it is) and is often called the *Factor of Safety*; i.e.,

Factor of Safety = Ultimate Strength/Allowable Stress ……………..(2.3)

Axial Force Diagrams

<u>Concentrated Axial Loads</u>

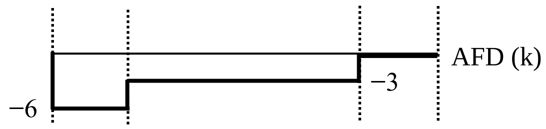
The concentrated loads on axially loaded members are predominant in many cases, e.g., in columns of multi-storied structures where the distributed loads (e.g., self-weights) are negligible compared to the loads from various floors.

<u>Example 2.1</u>

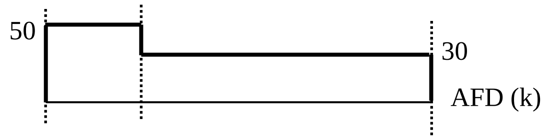
Draw the AFD of the beams loaded axially by the concentrated loads shown below.

Fig. 2.2: Concentrated axial forces on (a) Simply Supported Beam, (b) Cantilever Beam

$\sum F_x = 0 \Rightarrow X_A - 3 - 3 = 0 \Rightarrow X_A = 6$ k



$\sum F_x = 0 \Rightarrow -50 + 20 + X_B = 0 \Rightarrow X_B = 30$ k



<u>Distributed Axial Loads</u>

Significant examples where distributed axial loads are important include vertically suspended members subjected to self-weight or pile foundations where the loads from super-structure are resisted by skin friction; i.e., resistance of soil distributed along the pile-length.

<u>Example 2.2</u>

Draw the AFD of the piles loaded axially by the distributed loads shown below.

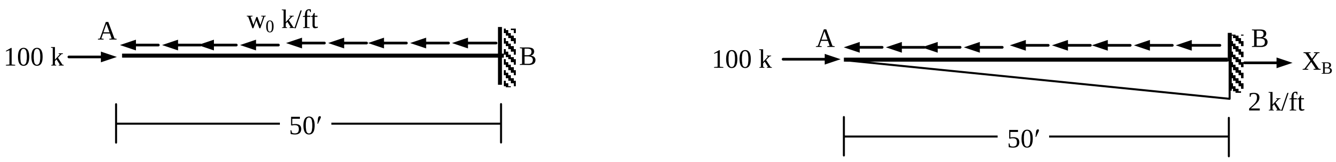
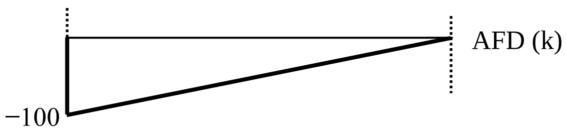


Fig. 2.3: Piles subjected to (a) Uniformly distributed axial forces, (b) Linearly distributed axial forces

$\sum F_x = 0 \Rightarrow 100 - 50w_0 = 0 \Rightarrow w_0 = 2$ k/ft



$\sum F_x = 0 \Rightarrow 100 - (50 \times 2)/2 + X_B = 0 \Rightarrow X_B = -50$ k

Design of Bolted (Riveted) Joints

Bolted (or riveted) joints are widely used in connections between various structural members. Its versatility makes it useful for connections between members made of different materials; e.g., steel, timber.

There are two types of bolted joints; i.e., *Lap Joints* and *Butt Joints*. In a lap joint, the plates to be connected are lapped over one another and fastened together by one or more rows of connectors (Fig. 2.4). In butt joint, the plates are butted together and joined by two cover plates connected to each of the main plates (occasionally, only one cover plate is used). The joints are identified here (as single-row, double-row and so on) by the number of rows of connectors that fasten the cover plate to each main plate (Fig. 2.5).

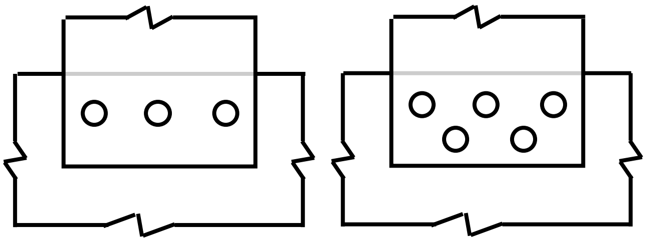


Fig. 2.4: Lap Joints (a) Single Row, (b) Double Row

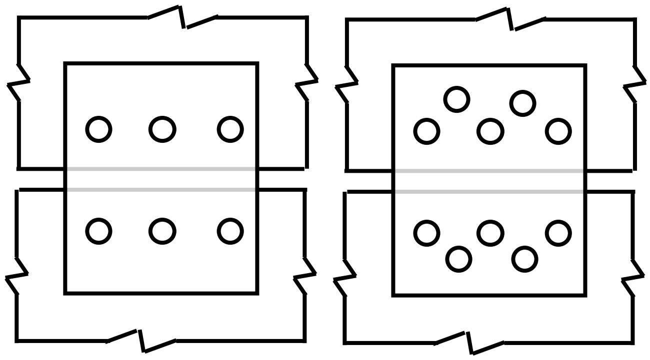


Fig. 2.5: Butt Joints (a) Single Row, (b) Double Row

Stresses in Bolted Joints

There are three major types of stresses that may work on bolted joints; i.e., (a) shear stress in bolts, (b) tensile (tearing) stress in plates, (c) bearing stress between bolts and plates. Other types of failure are possible if the joint is not designed properly; e.g., tearing or shearing of plate behind a connector hole that is too close to edge.

Example 2.3

Calculate the maximum shearing, tearing and bearing stresses in the riveted joint shown below (in Fig. 2.6) when subjected to a force P = 25 k. Also comment on the adequacy of the joint if the allowable shearing, tearing and bearing stresses are 15, 20 and 25 ksi respectively.

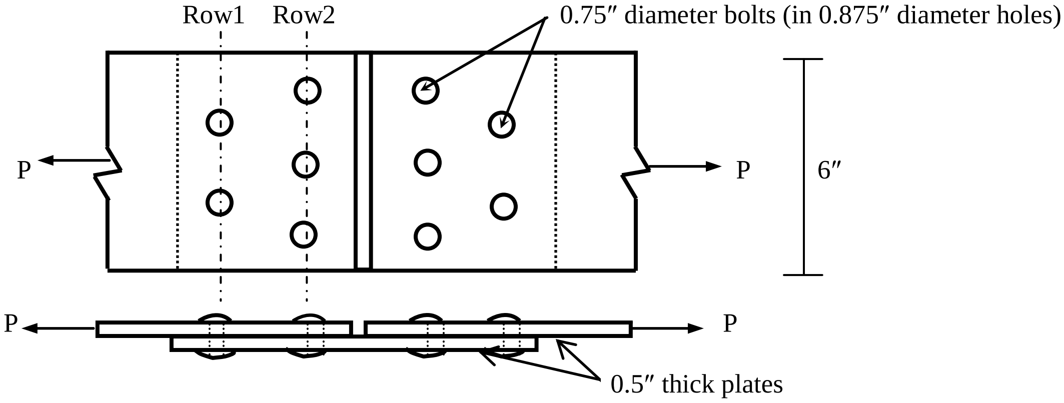


Fig. 2.6: Bolted joint subjected to force P

The force P can be assumed to be equally distributed among the bolts; i.e., each bolt taking (25/5 =) 5 k force parallel to its cross-sectional area.

$\therefore$ Shear stress in each bolt = $5/[\pi/4 \times (0.75)^2] = 11.32$ ksi

Maximum tensile force in Row1 = 25 k $\Rightarrow$ Maximum Tensile stress = $25/[0.5 \times (6 - 2 \times 0.875)] = 11.76$ ksi
Maximum tensile force in Row2 = 25 − 10 = 15k $\Rightarrow$ Tensile stress = $15/[0.5 \times (6 - 3 \times 0.875)] = 8.89$ ksi

Bearing stress between each bolt and plate (main plate and cover plate) = $5/[(0.75) \times (0.5)] = 13.33$ ksi

$\therefore$ Joint is adequate for bolt shear (11.32 < 15 ksi), plate tension (11.76 < 20 ksi) and bearing (13.33 < 25 ksi).

Design of Welded Connections

Welding is a method of joining metals by fusion. With heat from either an electric arc or an oxyacetylene torch, the metal at the joint is melted and fused with additional metal from a welding rod. When cool, the weld material and the base metal form a continuous and homogenous joint. The reliability of welded connections has increased to the point where they are used extensively to supplement or replace riveted or bolted connections in structural and machine design. It is frequently more economical to fabricate a member by welding simple component parts together than to use a complicated casting.

The two principal types of welds are butt welds [Fig. 2.7 (a)] and fillet welds [Fig. 2.7 (b)]. The strength of a butt weld is equal to the allowable stress multiplied by the product of the length of the weld times the thickness of the thinner plate of the joint.

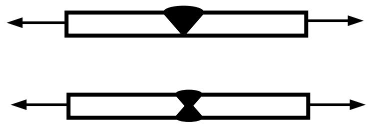


Fig. 2.7: (a) Single and Double V-Butt Weld

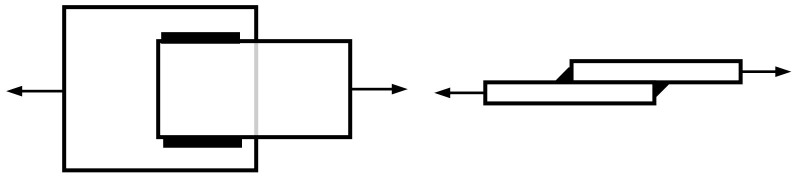


Fig. 2.7: (b) Side Fillet and End Fillet Weld

The strength of transverse fillet welds is determined by the shearing resistance of the throat of the weld regardless of the direction of the applied load. In the *45°* fillet weld in Fig. 2.8, with the leg equal to *t*, the shearing area through the length of weld *L* times the throat depth, or $A = L\,(t \sin 45°) = 0.707\,Lt$.

$\therefore$ Shear stress = Shear force/Shearing area; i.e., $\tau = V/A$

$\Rightarrow \tau = V/(0.707\,Lt) \Rightarrow L = V/(0.707t\,\tau)$ ……..………..(2.4)

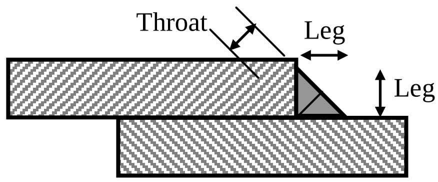


Fig. 2.8: Weld Leg and Throat

<u>Example 2.4</u>

In Fig. 2.9 shown below, calculate the length of 3/8-inch weld joints required on sides

(i) AB and CD only, (ii) AB, AD and CD to connect the 0.5″ thick channel section ABCD to the 0.625″ thick plate EFGH. Axial force of 50 kips passes through centroid of ABCD [Given: Allowable shear stress = 16 ksi].

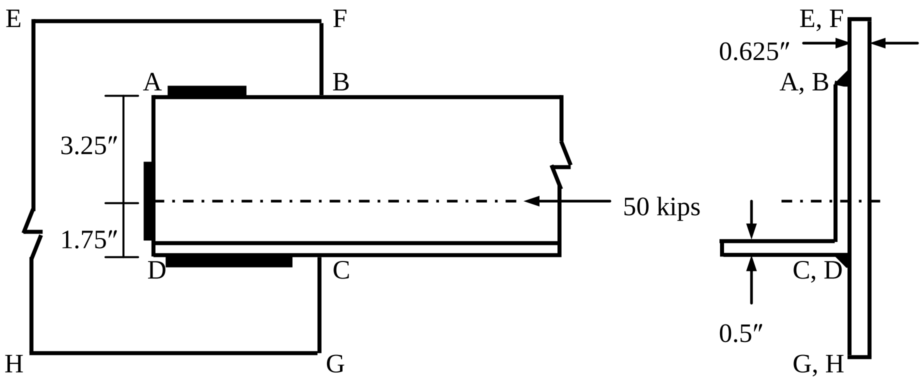


Fig. 2.9: Welded joint subjected to 50 kip force

(i) For welds on AB and CD, the axial force is resisted by $F_{AB} = 50 \times 1.75/5 = 17.5$ k, $F_{CD} = 50 \times 3.25/5 = 32.5$ k

$\therefore$ Weld lengths are $L_{AB} = F_{AB}/(0.707\ t\ \tau_{all}) = 17.5/(0.707 \times 3/8 \times 16) = 4.13″$, $L_{CD} = F_{CD}/(0.707\ t\ \tau_{all}) = 7.66″$; i.e., use welds of length 4.25″ and 7.75″ respectively.

(ii) The weld lengths on AB and CD can be reduced if the connection is welded on AD also. For the resultant of weld forces to pass through the centroid, welds length on AD can be $L_{AD} = 1.50 \times 2 = 3.00″$

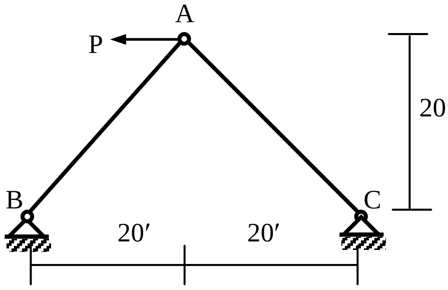
This leaves a weld length of (4.13″ + 7.66″ – 3.00″ =) 8.79″ to be divided on sides AB and CD

$\therefore$ Weld length on AB is = $L_{AB} = 8.79 \times 1.75/5 = 3.08″$, and on CD is = $L_{CD} = 8.79 \times 3.25/5 = 5.71″$;

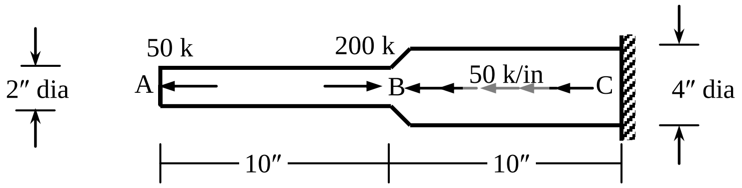
i.e., use welds of length 3.25″ and 5.75″ respectively.

Practice Problems on Stresses

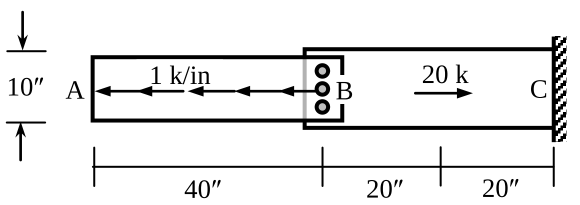
1. Calculate the allowable value of force P for the truss shown below [Given: Cross-sectional area of both AB and AC = 2 in^2, ultimate strength in compression = 30 ksi, tension = 36 ksi, factor of safety = 2.0].



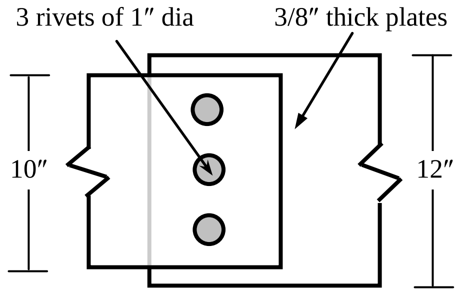
2. Draw the AFD and calculate the maximum tensile and compressive stress in the bar ABC shown below.



3. Calculate the shearing stress in the rivets and the maximum tearing and bearing stresses in the plates at joint B of the structural member ABC loaded as shown below.

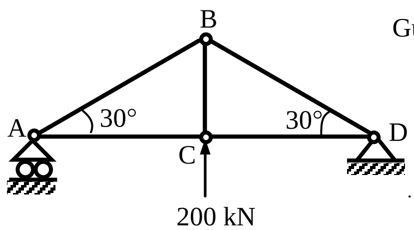


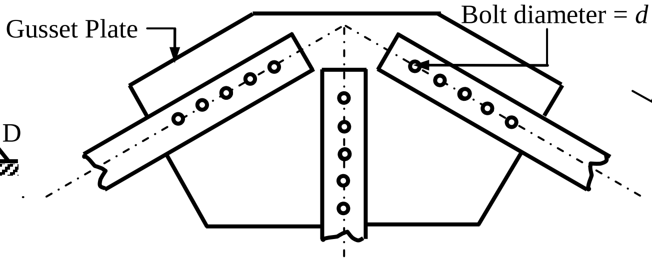
Member ABC



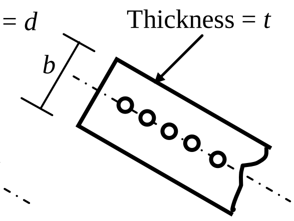
Joint B

4. In the truss ABCD shown below, use factor of safety = 1.5 to calculate the required
(i) Bolt diameter (d), (ii) Thickness (t) of member BD, (iii) Width (b) of member BD
[Given: Shear strength = 150 MPa, bearing strength = 250 MPa, axial strength = 200 MPa].



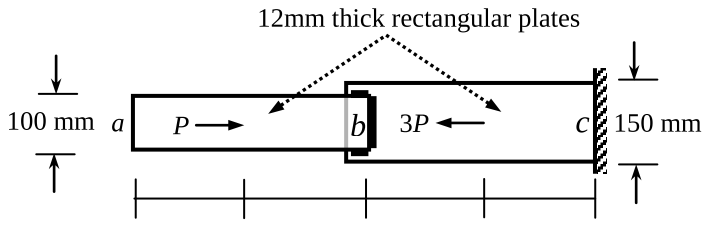


Joint B (Enlarged)

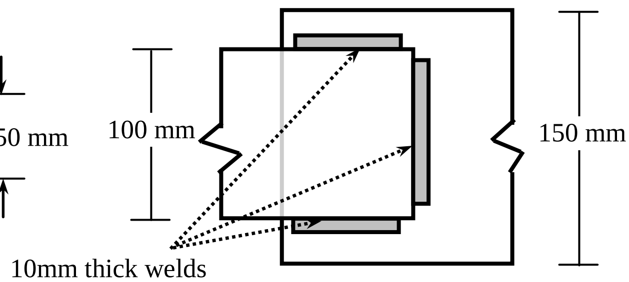


Member BD

5. (i) Calculate the maximum allowable value of P for the axially loaded member *abc* shown below.
(ii) For the force P calculated in (i), determine the lengths of 10 mm welds to connect the members *ab* and *bc* at joint *b*
[Given: Allowable stress in shear = 180 MPa, tension = 200 MPa, compression = 150 MPa].

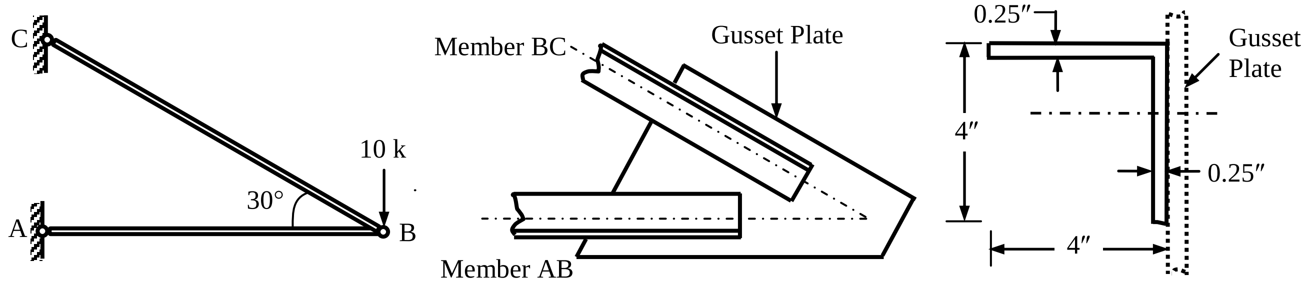


<u>Member *abc*</u>



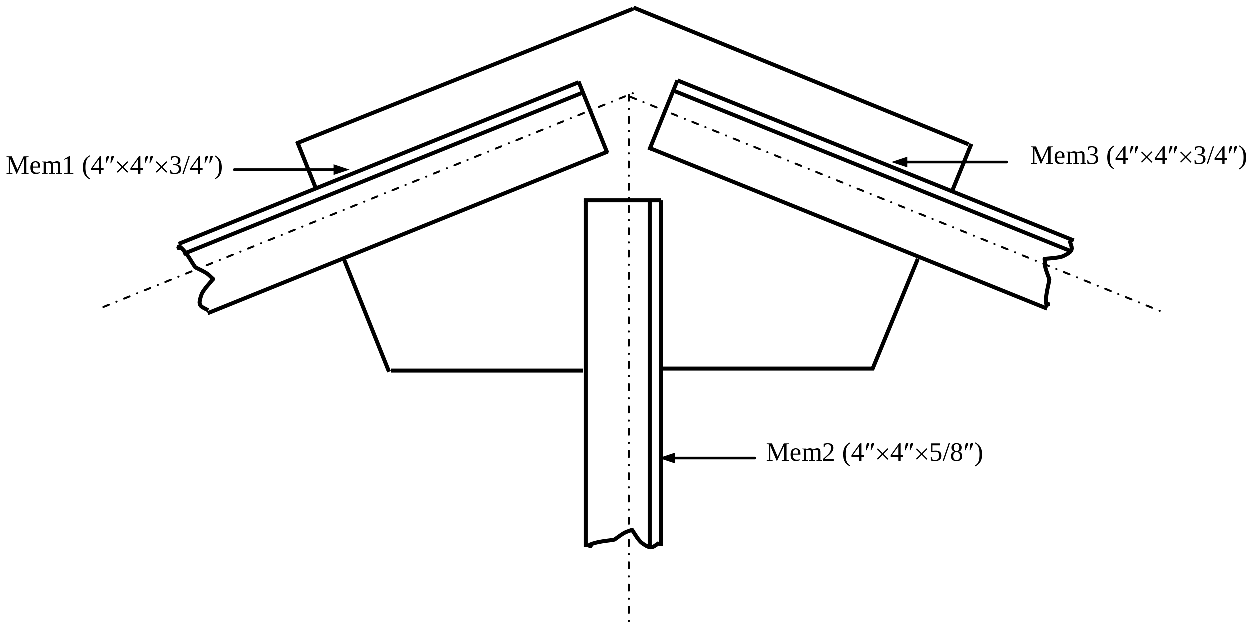
<u>Joint *b*</u>

6. In the structure ABC loaded as shown in the figure below,
   (i) Check the adequacy of the member BC,
   (ii) Design the welds for member BC with Gusset Plate at joint B
   [Given: Allowable axial stress in member BC = 18 ksi, Allowable shear stress in welds = 15 ksi].



Joint B (Enlarged) Cross-section of BC

7. Design the welds at the Gusset Plate of a truss (connecting members 1, 2 and 3) in the figure shown below [Given: F1 = F3 = 45 kips, F2 = 25 kips, Allowable shear stress in the welds = 16 ksi].



Stress, Strain and Stress-Strain Diagram

Stress is defined as the internal force on a body per unit area. Thus if an internal axial force P acts on a cross-sectional area A, the axial stress on the area is

$$\sigma = P/A \qquad \text{…………………..(3.1)}$$

Fig. 3.1(a) shows a body being subjected to an external axial load of P, which causes an internal force P as a reaction at every cross-section of the body. Therefore, the axial stress σ on the body is equal to P/A. The commonly used units of stress are lb/in^2 (psi), kip/in^2 (ksi), kg/cm^2, N/m^2 (Pascal or Pa), kN/m^2 (kilo-Pascal or kPa) etc.

Several types of stress may act on structures under various types of load. Other than the axial loading mentioned, stresses are caused by direct or flexural shear forces as well as flexural and torsional moments. In general, stresses can be classified as normal stress (acting perpendicular to the area) and shear stress (acting parallel to the area). This chapter deals with normal stresses.

An obvious effect of stress is the deformation it causes in the body. *Strain* is the deformation caused in a body per unit length. If a body of length L [Fig. 3.1(b)] undergoes an axial deformation of Δ, the axial strain caused in the body is

$$\varepsilon = \Delta/L \qquad \text{…………………..(3.2)}$$

Strain is a non-dimensional quantity but often units like in/in, cm/cm etc are used for strain. Like stress, strain can be broadly classified as normal strain and shear strain.

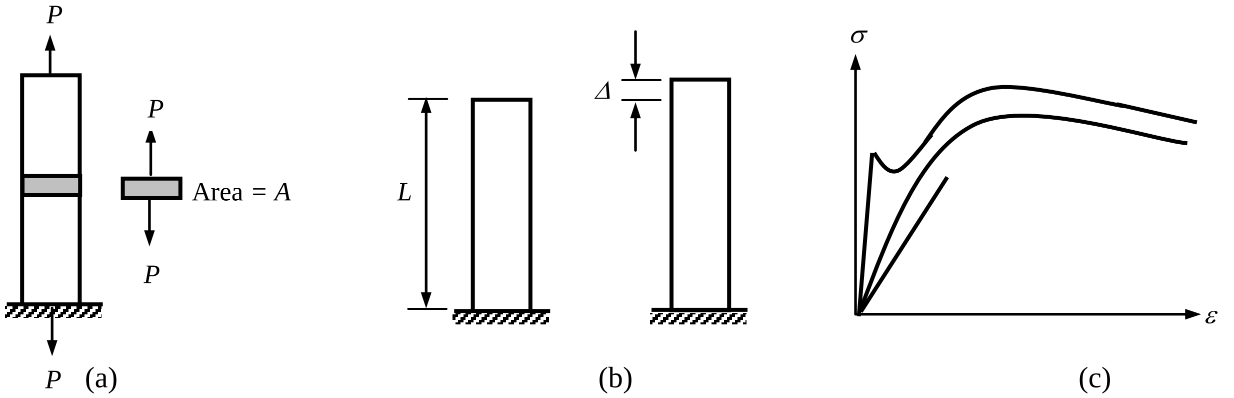


Fig. 3.1: (a) Concept of stress, (b) Concept of strain, (c) Different stress-strain diagrams

The diagram showing the stress (along y-axis) and strain (along x-axis) on a body is called its *Stress-Strain (σ-ε) Diagram*. Usually it is typical of the material, but also depends on the size of the specimen, the rate of loading, etc. Fig. 3.1(c) shows typical σ-ε diagrams for different materials.

Essential Elements of Stress-Strain Diagram

The stress vs. strain (σ-ε) diagrams discussed in the previous section are often used in studying various mechanical properties of materials under the action of loads. Depending on the type of materials, the σ-ε diagrams are drawn for specimens subjected to tension (typically for mild steel, aluminum and several other metals, less often for granular materials) or compression (more often for concrete, timber, soil and other granular materials).

Several elements of the σ-ε diagrams are used in Strength of Materials as well as structural analysis and design. Figs. 3.2 (a) and 3.2 (b) show two typical stress-strain diagrams often encountered in material testing. The first of them represents a material with an initial linear σ-ε relationship followed by a pronounced yield region, which is often followed by a strain hardening and failure region (typical of Mild Steel). The second curve represents a material with nonlinear σ-ε relationship almost from the beginning and no distinct yield region (typical of concrete and timber).

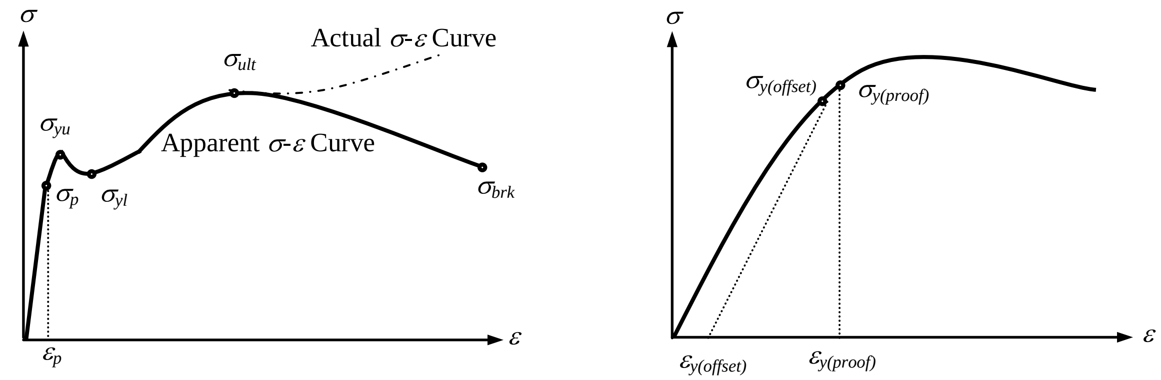


Fig. 3.2: Typical stress-strain diagrams for (a) Yielding materials, (b) Non-yielding materials

Up to a certain limit of stress and strain, the σ-ε diagram for most materials remain linear (or nearly so); i.e., the stress remains proportional to the strain initially. Up to this limit, the material follows the Hooke's Law, which states that deformation is proportional to applied load. The corresponding stress is called the *Proportional Limit (*or *Elastic Limit)*, which is denoted by σ_p in Fig. 3.2(a), and the strain is denoted by ε_p. The ratio of σ_p and ε_p (or any stress and strain below these) is called the *Modulus of Elasticity or Young's Modulus* and is denoted by E.

$$E = \sigma_p/\varepsilon_p \qquad \text{.....................(3.3)}$$

The area under the σ-ε diagram indicates the energy dissipated per unit volume in straining the material under study. The corresponding area up to the proportional limit is called *Modulus of Resilience* and is given by the following equation

$$\text{Modulus of Resilience} = \sigma_p\, \varepsilon_p/2 = \sigma_p^2/2E \qquad \text{.....................(3.4)}$$

Longitudinal strain is accompanied by lateral strain as well, of different magnitude and opposite sign. If the longitudinal strain is ε_{long} and the corresponding lateral strain is $-\varepsilon_{lat}$, the ratio between the two is called the *Poisson's Ratio*, often denoted by ν.

$$\nu = -\varepsilon_{lat}/\varepsilon_{lomg} \qquad \text{.....................(3.5)}$$

The Modulus of Elasticity and the Poisson's Ratio are two basic material constants used universally for the linear elastic analysis and design.

Within proportional limit, the σ-ε diagram passes through the origin. Therefore, the strain sustained within the proportional limit can be fully recovered upon withdrawal of the load; i.e., without any permanent deformation of the material. However, this is not applicable if the material is stressed beyond σ_p. If load is withdrawn after stressing the material beyond yield point, the σ-ε diagram follows the initial straight path during the process of unloading and therefore does not pass through the origin; i.e., the strain does not return to zero even when stress becomes zero.

In many materials, the proportional limit is followed by a stress (or small range of stresses) where the material is elongated (i.e., strained) without any significant change in stress. This is called the *Yield Strength* for the material and is often denoted by σ_y. As shown in Fig. 3.2 (a), yielding occurs within a range of stress rather than any particular stress. The upper limit of the region is called *Upper Yield Strength* while the lower limit is called *Lower Yield Strength* of the material. In Fig. 3.2 (a), they are denoted by σ_{yu} and σ_{yl} respectively.

However materials with σ-ε diagrams similar to Fig. 3.2 (b) do not have any particular yield point or region. In order to indicate the stress where the material is strained within the range of typical yield strains or undergoes permanent deformation typical of yield points, two methods have been suggested to locate the 'yield point' of non-yielding materials. One of the is the *Proof Strength*, which takes the stress $\sigma_{y(proof)}$ as the yield point of the material corresponding to a pre-assigned strain indicated by $\varepsilon_{y(proof)}$. The other is the *Offset Method*, which takes as yield point a point corresponding to a permanent strain of $\varepsilon_{y(offset)}$. Therefore the *Yield Strength by Offset Method* is obtained by drawing a straight line from $\varepsilon_{y(offset)}$ parallel to the initial tangent of the σ-ε diagram and taking as yield strength the point where this line intersects the σ-ε diagram.

Beyond yield point, the strains increase at a much faster rate with nominal increase in stress and the material moves towards failure. However, the material can usually take stresses higher than its yield strength. The maximum stress a material can sustain without failure is called the *Ultimate Strength*, which is denoted by σ_{ult} in Fig. 3.2 (a). For most materials, the stress decreases as the material is strained beyond σ_{ult} until failure occurs at a stress called *Breaking Strength* of the material, denoted by σ_{brk} in Fig. 3.2 (a).

Here it may be mentioned that the stress does not actually decrease beyond ultimate strength in the true sense. If the *Actual σ-ε Diagram* [indicated by dotted lines in Fig. 3.2 (a)] of the material is drawn using the instantaneous area (which is smaller than the actual area due to Poisson's effect) and length of the specimen instead of the original area and length, the stress will keep increasing until failure. All the other σ-ε diagrams shown in Figs. 3.1 and 3.2 (and used for most Civil Engineering applications) are therefore called *Apparent σ-ε Diagrams*.

The total area under the σ-ε diagram is called the *Modulus of Toughness*. Physically, this is the energy required to break a specimen of unit volume.

Ductility is another property of vital importance to structural design. This is the ability of the material to sustain strain beyond elastic limit. Quantitatively it is taken as the final strain in the material (the strain at failure) expressed in percentage.

Table 3.1 shows some useful mechanical properties of typical engineering materials (quoted from available literature). However, these properties may vary significantly depending on the ingredients used and the manufacturing process. For example, although the yield strength of Mild Steel is shown to be 40 ksi, other varieties with yield strengths of 60 ksi and 75 ksi are readily available. The properties of concrete are even more unpredictable. Here, the ultimate strength is mentioned to be 3 ksi, but concretes with much lower and higher ultimate strengths (1~7 ksi) are used in different construction works. The typical properties of timber also vary significantly depending on the type of timber (e.g., Gurjan, Jarul), seasoning (e.g., green, air-dry) or type of loading (compression or bending, parallel or perpendicular to grain).

Table 3.1: Useful Mechanical Properties of Typical Engineering Materials

Material	σ_p (ksi)	σ_y (ksi)	σ_{ult} (ksi)	E (ksi)	ν	Modulus of Resilience (ksi)	Modulus of Toughness (ksi)	Ductility (%)	Reduction of Area (%)
Mild Steel	35	40	60	29000	0.25	0.02	15	35	60
Aluminum	60	70	80	10000	0.33	0.18	7	10	30
Concrete	1.4	2.0	3.0	3000	0.30	0.0003	0.006	0.30	0.60
Timber	3.4	4.5	5.3	2000	0.30	0.003	0.007	0.80	1.60

[Note: Properties may vary significantly depending on ingredients and manufacturing process]

Three-Dimensional Stresses

Uniaxial normal stress (discussed in the previous section) is only an idealized stress scenario. In practice, an element can be subjected to a combination of normal and shear stresses acting in different directions.

Figs. 3.3(a) and 3.3(b) show the three-dimensional stresses acting on an element along the *x*, *y* and *z* axes as well as the two-dimensional stresses on the *x*-*y* plane.

The sign convention followed here is

σ_{ij} = Normal stress on plane *i* along *j*-axis ....(3.6)

τ_{ij} = Shear stress on plane *i* along *j*-axis ....(3.7)

For example

σ_{xx} = Normal stress on plane *x* (i.e., *yz*) along *x*-axis

τ_{yx} = Shear stress on plane *y* (i.e., *xz*) along *x*-axis

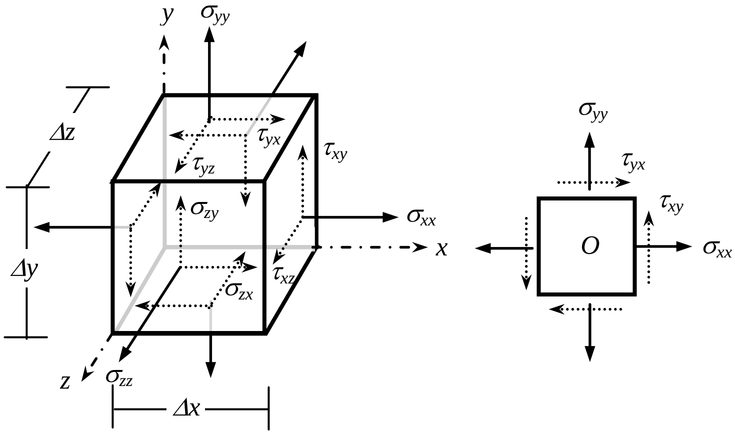


Fig. 3.3: Stresses (a) Three-dimensional, (b) On *x*-*y* plane

<u>Shear Stress and Strain</u>

As mentioned before, shear stress acts parallel to a surface whereas normal stress acts perpendicular to it. And as shown in Fig. 3.3, an element can be subjected to shear stresses in addition to normal stresses. For the equilibrium of the element in Fig. 3.3(b), summation of moments about point O ⇒

$$-\{\tau_{xy}(\Delta y)(\Delta z)\}\times(\Delta x) + \{\tau_{yx}(\Delta x)(\Delta z)\}\times(\Delta y) = 0 \Rightarrow \tau_{yx} = \tau_{xy} \qquad \text{..........................(3.8)}$$

It can be shown similarly that, $\tau_{zy} = \tau_{yz}$ and $\tau_{xz} = \tau_{zx}$ ....(3.9), (3.10)

Therefore the *shear stresses in mutually perpendicular planes are equal*.

Shear stresses would naturally cause shear strains in an element, as shown in the *x*-*y* plane in Fig. 3.4. This is the change of angle between the planes *x* and *y*; i.e., shear strain (denoted by γ_{xy}) inclines the sides of the deformed element in relation to the original axes.

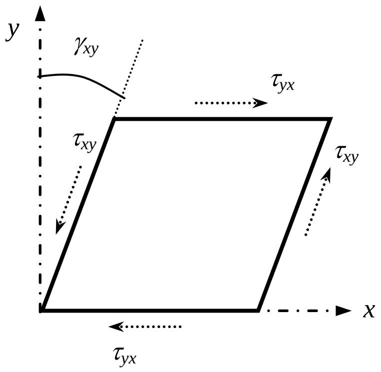


Fig. 3.4: Concept of Shear Strain

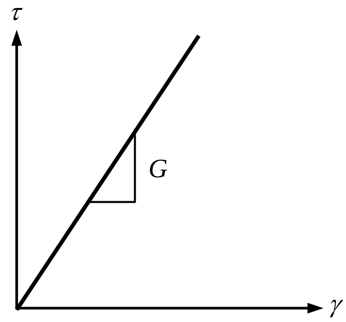


Fig. 3.5: Shear Stress vs. Shear Strain

Within the elastic range, shear stress is related to shear strain by

$$\tau = G\,\gamma \qquad \text{.........................(3.11)}$$

where *G* is the *Modulus of Rigidity* or *Shear Modulus* of the material and is related to elastic modulus by

$$G = E/(2\,(1+\nu)) \qquad \text{.........................(3.12)}$$

Constitutive Relations for Isotropic Materials: Generalized Hooke's Law

In this section, six basic relations between a general state of stress and strain are derived using the principle of superposition from the previously established simpler stress-strain equations. This set of equations is referred to as the *Generalized Hooke's Law* and is valid for *Isotropic Materials*; i.e., materials having the same properties in all directions (as against anisotropic materials, with significantly different properties in different directions).

According to the basic concept of Hooke's Law, a linear relationship exists between the applied stress and the resulting strain (e.g., $\varepsilon_{xx} = \sigma_{xx}/E$). During this process, a lateral contraction or expansion of a body also takes place (i.e., ε_{yy} and ε_{zz}), depending on whether it is being stretched or compressed. The extent of the lateral deformation is obtained using the Poisson's Ratio (i.e., $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$). Therefore, a uniaxial normal stress σ_{xx} causes normal strains in three directions; i.e., σ_{xx}/E, $-\nu\sigma_{xx}/E$ and $-\nu\sigma_{xx}/E$ along the *x*, *y* and *z* axis respectively. Similarly, the normal stress σ_{yy} causes normal strains, $-\nu\sigma_{yy}/E$, σ_{yy}/E and $-\nu\sigma_{yy}/E$ and the normal stress σ_{zz} causes normal strains $-\nu\sigma_{zz}/E$, $-\nu\sigma_{zz}/E$ and σ_{zz}/E.

Based on the above, the three normal strains according to generalized Hooke's Law for *Isotropic Linearly Elastic Materials* can be written as

$$\varepsilon_{xx} = \sigma_{xx}/E - \nu\sigma_{yy}/E - \nu\sigma_{zz}/E \quad \text{………………………(3.13a)}$$

$$\varepsilon_{yy} = -\nu\sigma_{xx}/E + \sigma_{yy}/E - \nu\sigma_{zz}/E \quad \text{………………………(3.13b)}$$

$$\varepsilon_{zz} = -\nu\sigma_{xx}/E - \nu\sigma_{yy}/E + \sigma_{zz}/E \quad \text{………………………(3.13c)}$$

The relationships between the shear stresses and strains are more direct and follow Eq. (3.11); i.e.,

$$\gamma_{xy} = \tau_{xy}/G \quad \text{………………………(3.14a)}$$

$$\gamma_{yz} = \tau_{yz}/G \quad \text{………………………(3.14b)}$$

$$\gamma_{zx} = \tau_{zx}/G \quad \text{………………………(3.14c)}$$

<u>Example 3.1</u>

The rectangular prism shown below is subjected to normal force in the *x* direction and is restrained in the *y* and *z* directions (i.e., $\varepsilon_{yy} = 0$, $\varepsilon_{zz} = 0$). Calculate the normal stresses (σ_{xx}, σ_{yy}, σ_{zz}) and strain (ε_{xx}) that develop in the prism [Given: Modulus of Elasticity = 2000 ksi, Poisson's ratio = 0.30].

The normal stress in *x*-direction is

$\sigma_{xx} = 60/(5 \times 3) = 4$ ksi

$\therefore \varepsilon_{yy} = (-0.30 \times 4 + \sigma_{yy} - 0.30\ \sigma_{zz})/2000 = 0$

$\Rightarrow \sigma_{yy} - 0.3\ \sigma_{zz} = 1.20$

$\therefore \varepsilon_{zz} = (-0.30 \times 4 - 0.30\ \sigma_{yy} + \sigma_{zz})/2000 = 0$

$\Rightarrow -0.3\ \sigma_{yy} + \sigma_{zz} = 1.20$

Solving, $\sigma_{yy} = 1.71$ ksi, $\sigma_{zz} = 1.71$ ksi

$\therefore \varepsilon_{xx} = (4 - 0.30 \times 1.71 - 0.30 \times 1.71)/2000$

$= 1.49 \times 10^{-3}$

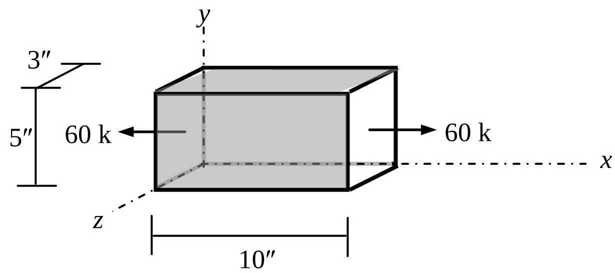


Fig. 3.6: Force on Rectangular Prism

$\therefore$ Elongation of the prism in the x-direction, $\Delta_{xx} = (1.49 \times 10^{-3}) \times 10 = 0.0149$ in

Tensile force in y-direction, $P_{yy} = 1.71 \times (10 \times 3) = 71.4$ kips

and in z-direction, $P_{zz} = 1.71 \times (10 \times 5) = 85.7$ kips

<u>Example 3.2</u>

Calculate all the shear forces, stresses and strains if the prism is also subjected to a uniformly distributed shear force of $V_{xy} = 45$ k.

Axial Deformations

Calculation of structural deformations is one of the main objectives of the study of axial strains. Excessive deformation can render a structure useless even if it is safe in terms of stress induced. It may also cause excessive stresses adjacent structural elements. Axial deformation in a structure is caused by axial strains. The figure below (Fig. 3.7) shows a differential element of length Δx in an axially loaded bar.

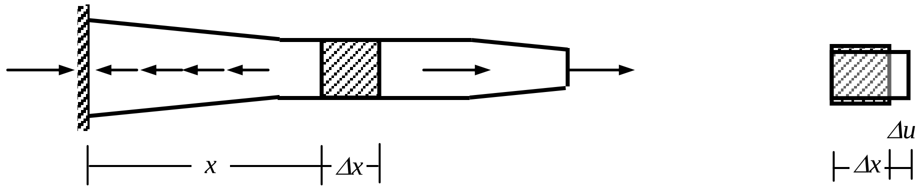


Fig. 3.7: Axial Deformation of Differential Element

If the applied loads cause the differential element to extend by a differential amount Δu, the axial strain

$\varepsilon_{xx} = \Delta u/\Delta x$, and in the limit $\Delta x \rightarrow 0,\ \varepsilon_{xx} = du/dx$ …………………(3.15)

$\Rightarrow du = \varepsilon_{xx}\, dx = \sigma_{xx}/E\, dx = P_{xx}/AE\, dx \Rightarrow \int du = \int P_{xx}/AE\, dx$

where P_{xx} = Axial force, A = Cross-sectional area, E = Modulus of elasticity at distance x

$\therefore$ The axial deformation between points A and B is given by

$u_B - u_A = \int (P_{xx}/AE)\, dx$ …………………(3.16)

Therefore, using Eq. (3.15), the axial deformation between sections A and B can be calculated if the axial force, cross-sectional area and modulus of elasticity are known at any section. For the special case when they are all constant, the axial deformation takes the special form

$u_B - u_A = P_{xx}L/AE$ …………………(3.17)

Example 3.3

Calculate the axial deformations at point B and C of the axially loaded bars shown below in Fig. 3.8
[Given: E = 2000 ksi, Members AB and BC are (6″ × 3″/8) and (4″ × 3″/8) sections respectively].

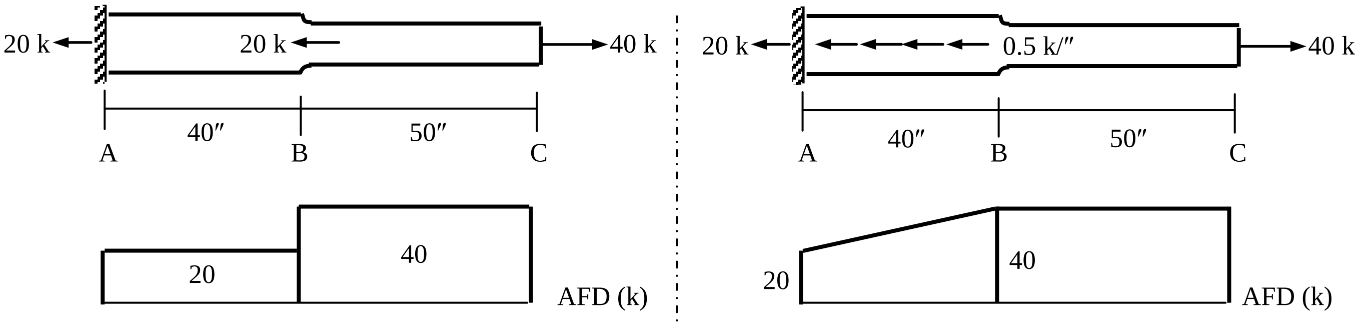


Fig. 3.8: Axially Loaded Bars with AFD

$u_B - u_A = P_{AB}L_{AB}/A_{AB}E$
$\Rightarrow u_B - 0 = 20 \times 40/\{(6 \times 3/8) \times (2000)\}$
$\Rightarrow u_B = 0.178''$

$u_C - u_B = P_{BC}L_{BC}/A_{BC}E$
$\Rightarrow u_C - u_B = 40 \times 50/\{(4 \times 3/8) \times (2000)\}$
$\Rightarrow u_C = 0.844''$

$u_B - u_A = \int P_{AB}\, dx/A_{AB}E$
$\Rightarrow u_B - 0 = (20 + 40)/2 \times 40/\{(6 \times 3/8) \times (2000)\}$
$\Rightarrow u_B = 0.267''$

$u_C - u_B = P_{BC}L_{BC}/A_{BC}E$
$\Rightarrow u_C - u_B = 40 \times 50/\{(4 \times 3/8) \times (2000)\} = 0.667''$
$\Rightarrow u_C = 0.933''$

Example 3.4

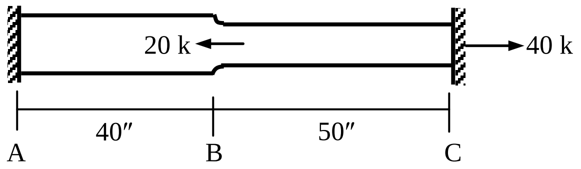
Calculate u_B and u_C if the depth of member AB varies linearly between A and B from 6″ to 4″.

Analysis of Statically Indeterminate Bars

A large portion of civil engineering structures is statically indeterminate; i.e., they cannot be analyzed by statics alone. Analysis of such structures require knowledge of displacements; i.e., axial deformation. Solution of statically indeterminate structures is one of the main objectives of the study of axial deformations.

Example 3.5

Draw the axial force diagram of the statically indeterminate axially loaded bar shown below in Fig. 3.9
[Given: E = 2000 ksi, Members AB and BC are (6″ × 3″/8) and (4″ × 3″/8) sections respectively].



The structure is solved by first withdrawing the support at C and then ensuring that the elongation at that point is zero. Fig. 3.10 shows the two components of the problem.

Fig. 3.9: Statically Indeterminate Axially Loaded Bar

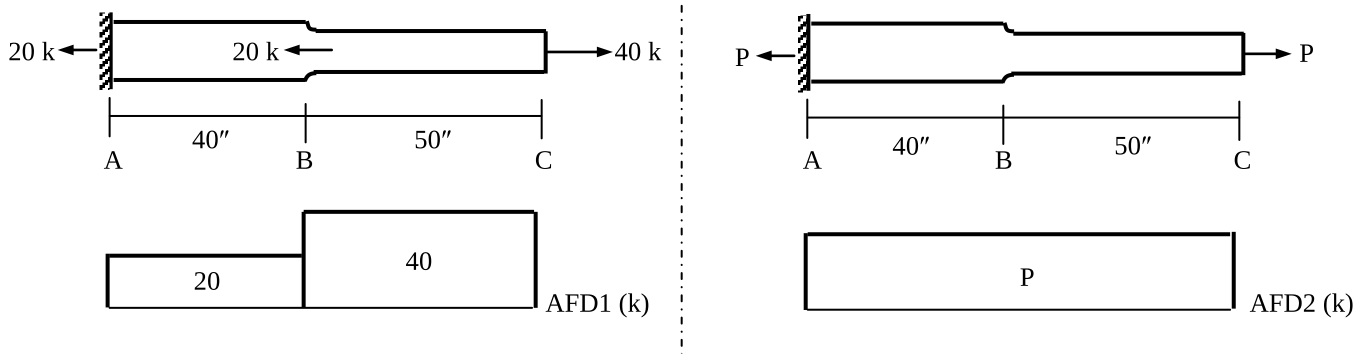


Fig. 3.10: Statically Determinate Components of the Problem

$u_{C1} - u_{A1} = P_{AB}L_{AB}/A_{AB}E + P_{BC}L_{BC}/A_{BC}E$

$\Rightarrow u_{C1} - 0 = 20 \times 40/\{(6 \times 3/8) \times (2000)\} + 40 \times 50/\{(4 \times 3/8) \times (2000)\}$

$\Rightarrow u_{C1} = 0.178 + 0.667 = 0.844''$

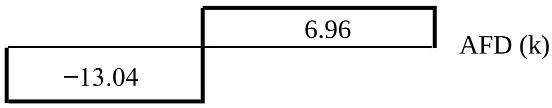
$u_{C2} - u_{A2} = P\, L_{AB}/A_{AB}E + PL_{BC}/A_{BC}E$

$\Rightarrow u_{C2} = P\,\{40/(6 \times 3/8) + 50/(4 \times 3/8)\}/(2000)$

$= P\,(17.78 + 33.33)/2000 = P/39.13$

$\therefore u_C = u_{C1} + u_{C2} = 0.844 + P/39.13 = 0$

$\Rightarrow P = -33.04$ k



Example 3.6

Calculate the forces in wires A, B and C supporting the rigid bar ABC loaded as shown in Fig. 3.11
[Given: E = 10,000 ksi and A = 0.20 in^2 for wire A and C, while E = 30,000 ksi, A = 0.30 in^2 for B].

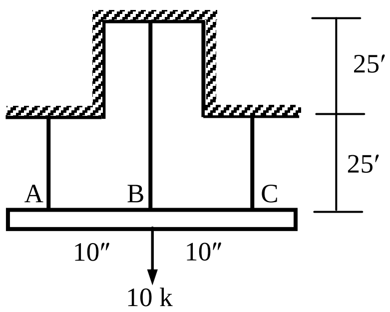


Fig. 3.11: Rigid Bar ABC Supported by A, B, C

$\sum M_B = 0 \Rightarrow P_A \times 10 - P_C \times 10 = 0 \Rightarrow P_A = P_C = P$ (let)

$\therefore \sum F_y = 0 \Rightarrow P_B + P + P - 10 = 0 \Rightarrow P_B = 10 - 2P$

Rigid bar ABC $\Rightarrow \Delta_A\ (= \Delta_C) = \Delta_B$

$\Rightarrow P \times (25 \times 12)/(10000 \times 0.2) = (10 - 2P) \times (50 \times 12)/(30000 \times 0.3)$

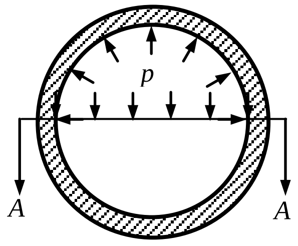
$\Rightarrow P/(0.2) = (10 - 2P) \times (2)/(3 \times 0.3) \Rightarrow 4.5\,P = 20 - 4P$

$\Rightarrow P = 2.35$ k

$\therefore P_B = 10 - 2P = 5.3$ k

Stresses and Strains in Thin-Walled Pressure Vessels

A cylindrical tank carrying gas or fluid under a pressure p is subjected to tensile forces that resist the bursting forces developed across longitudinal and transverse sections. Consider a typical section *A-A* through the pressure loaded cylinder shown in Fig. 3.12 (a). A free body diagram of the lower half of the cylinder isolated by the cutting plane is shown in Fig. 3.12 (b).



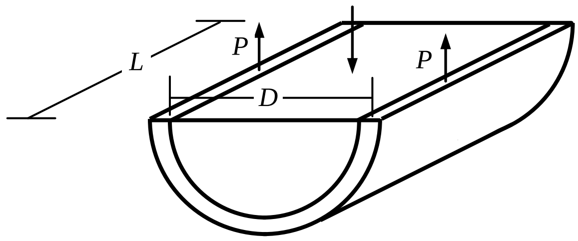


Fig. 3.12: Calculation of Bursting Force *F*

As shown in Fig. 3.12, the cylinder is occupied by a fluid, which transmits pressure equally in all directions. From the accompanying free-body diagram, the bursting force F, acting over the flat surface of the fluid, equals the pressure p multiplied by the area DL over which it acts, i.e.,

$$F = pDL \qquad \text{…..………..(3.18)}$$

The stress in the longitudinal section that resists the bursting force F is obtained by dividing it by the area of the two cut surfaces. This gives $\sigma_t = F/A = pDL/2tL$

$$\Rightarrow \sigma_t = pD/2t \qquad \text{..……………(3.19)}$$

This stress is called the *Tangential stress* because it acts tangent to the surface of the cylinder. Other common names are *Circumferential Stress*, *Hoop Stress* and *Girth Stress*.

Considering next the free-body diagram of a transverse system (Fig. 3.13), where the bursting force acts over the end of the cylinder is resisted by the resultant P of the tensile forces acting over the transverse section; i.e.,

$P = F \Rightarrow \pi(D + t)t\ \sigma_l = p(\pi D^2/4)$

$\Rightarrow \sigma_l \cong pD/4t$ ……………………………(3.20)

where σ_l is called the *Longitudinal Stress* because it acts parallel to the longitudinal axis of the cylinder.

Eqs. (3.19) and (3.20) show that the longitudinal stress is one-half the tangential stress.

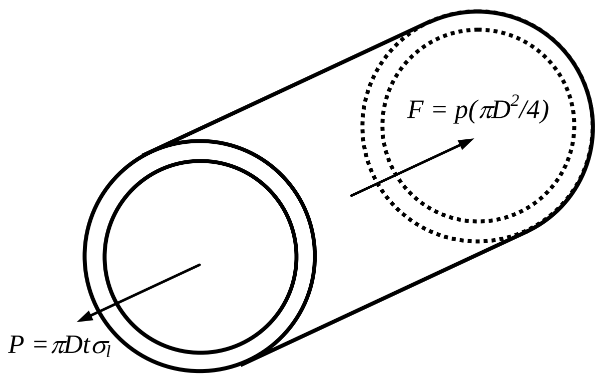


Fig. 3.13: Bursting Force on Transverse Section

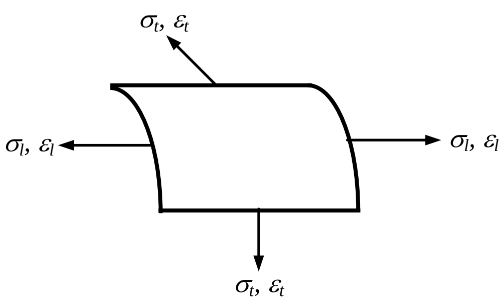


Fig. 3.14: Longitudinal and Hoop Stress and Strain

Since the longitudinal and transverse stresses act in perpendicular planes, the corresponding strains obtained by

$$\varepsilon_t = \sigma_t/E - \nu\sigma_l/E = (pD/2t)\ (1-\nu/2) \qquad \text{…………………….(3.21a)}$$

$$\varepsilon_l = \sigma_l/E - \nu\sigma_t/E = (pD/2t)\ (0.5-\nu) \qquad \text{…………………….(3.21b)}$$

Connections in Thin-Walled Pressure Vessels

Instead of a single thin-walled piece of metal, pressure vessels are often composed of several metal sheets bolted or welded together. And since Eqs. (3.19) and (3.20) show that the longitudinal stress is one-half the tangential stress, the longitudinal joint (Fig. 3.15) should be twice as strong as the hoop joint, otherwise the permissible internal pressure will depend on the strength of the longitudinal joint.

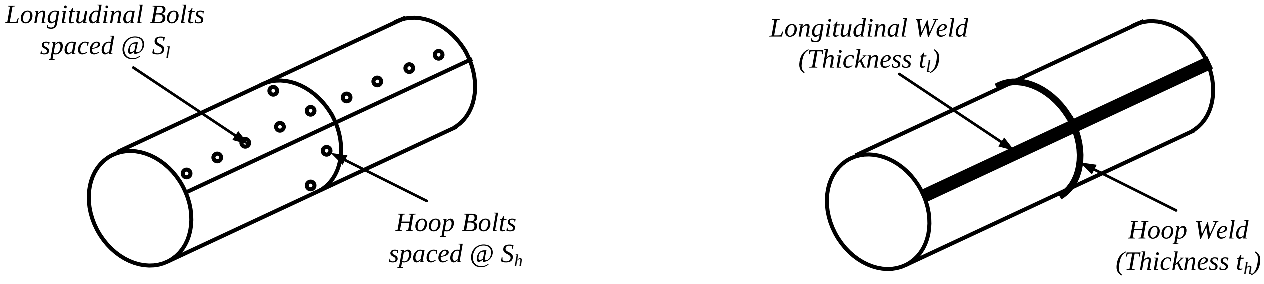


Fig. 3.15: Longitudinal and Hoop Joints (a) Bolted, (b) Welded

Bolt Connections

For connections joined together by bolts spaced at S_l along the length of the vessel, the force $(pD/2)\,S_l$ is to be resisted by the shear force acting on each bolt.

$$\therefore (pD/2)\, S_l = \tau_{all}\,(\pi d^2/4) \Rightarrow S_l = (\tau_{all}/p)\,(\pi d^2/2D) \qquad \text{………………….. (3.22 a)}$$

It is reasonable to conclude from previous analysis that the bolts would be distributed at twice this spacing along the perimeter of the vessel, to be acted on by longitudinal pressure and resisted by bolt shear stress. Here, the longitudinal force $p(\pi D^2/4)$ is resisted by all the bolts [number = Perimeter/Spacing = $\pi(D + t)/S_h$] around the perimeter of the vessel.

$$p\,(\pi D^2/4) = [\pi(D + t)/S_h]\,[\tau_{all}\,(\pi d^2/4)] \Rightarrow S_h \cong (\tau_{all}/p)\,(\pi d^2/D) \qquad \text{………………….. (3.22 b)}$$

which is indeed twice the spacing of the longitudinal bolts, shown in Eq. (3.22 a).

Weld Connections

For connections joined together by welds of thickness t_l along the length of the vessel, the force $(pD/2)L$ is to be resisted by the shear force acting on each weld.

$$pDL/2 = \tau_{all}\,(0.707 t_l\, L) \Rightarrow t_l = (p/\tau_{all})\,(0.707D) \qquad \text{………………….. (3.23 a)}$$

On the other hand, the longitudinal force [$p\,(\pi D^2/4)$] will be resisted by circumferential welds of thickness t_h and length $\pi(D + t)$; i.e.,

$$p\,(\pi D^2/4) = \tau_{all}\,[0.707\, t_h \pi(D + t)] \Rightarrow t_h \cong (p/\tau_{all})\,(0.353D) \qquad \text{………………….. (3.23 b)}$$

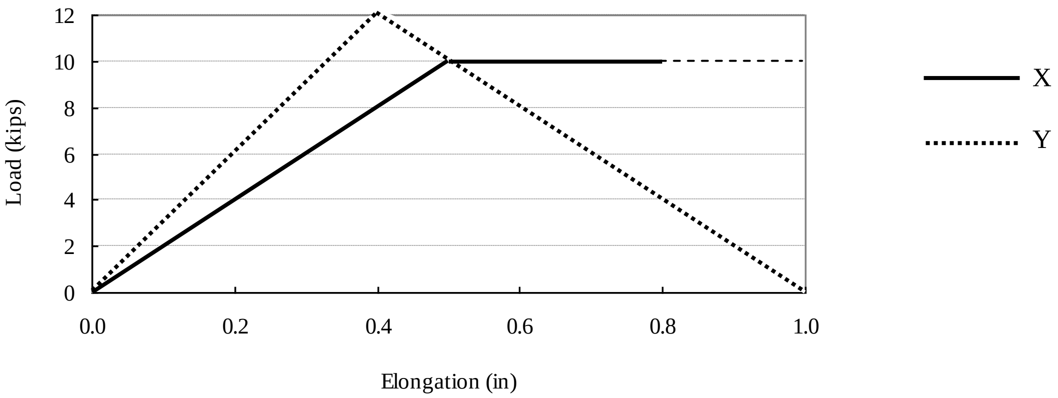
which is half the thickness required for longitudinal welds, as expected.

Possible reasons for failure of Pressure Vessels

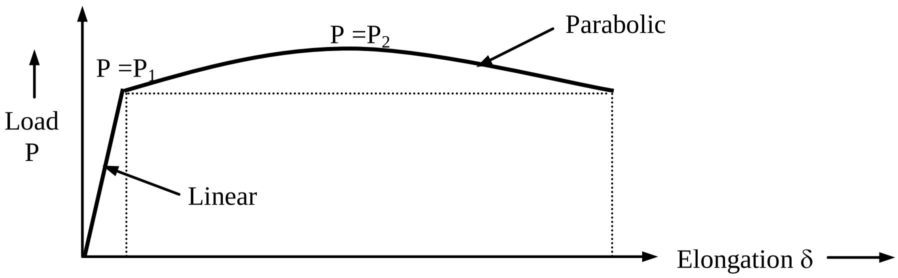
* Increased temperature (due to fire/heat) can
  - Increase gas pressure p
  - Melt the wall-metal
  - Reduce the metal-strength
* Decreased temperature can reduce the strength and ductility of the metal
* Careless handling/storage or accidents can lead to impact loading that may damage the wall
* Poor design may result in inadequate welds or bolts
* Poor workmanship can lead to porous welding/reduced weld thickness than designed
* Corrosion can reduce the thickness of the wall, weld, bolt or may create holes within the wall causing stress concentration

Practice Problems on Strain

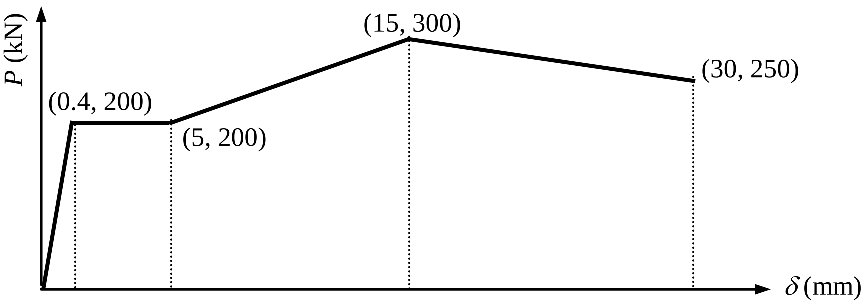
1. The figure below shows the load vs. elongation diagrams of specimens X and Y. If their cross-sectional areas are $A_X = 0.20$ in^2, $A_Y = 0.25$ in^2 and gage-lengths are $L_X = L_Y = 5''$ respectively, determine which specimen is made of (i) stronger, (ii) stiffer, (iii) more resilient, (iv) more ductile, (v) tougher material.



2. The figure below shows the idealized load (P) vs. elongation (δ) diagram of a 2″ long mild steel specimen with X-sectional area = 0.20 in^2. If $P_1 = 40$ kN, $P_2 = 50$ kN, $\delta_1 = 0.075$ mm, $\delta_2 = 12.5$ mm, calculate (i) ultimate strength, (ii) modulus of elasticity, (iii) % elongation, (vi) modulus of toughness.

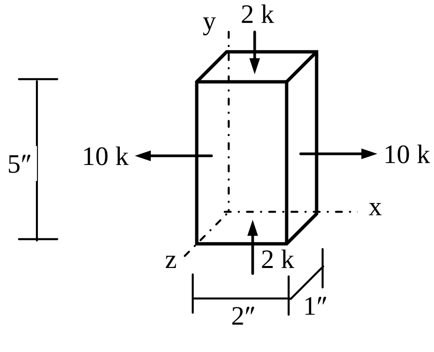


3. The figure below shows the axial force (P) vs. elongation (δ) diagram of a 200 mm long mild steel specimen of 25 mm diameter. Calculate the (i) Young's modulus, (ii) apparent and actual breaking strength and (iii) energy needed to break the specimen.

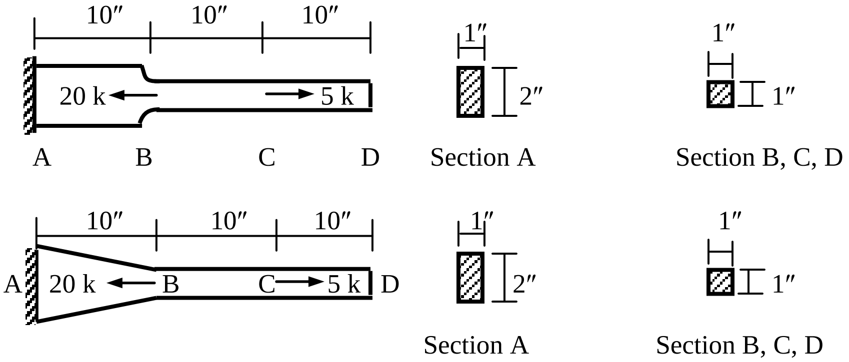


4. The Proof Strength, yield strength from Offset Method and ultimate strength of a (2″ ×2″) timber specimen (6″ long) are all equal to 6 ksi, while its proportional limit is 3 ksi. Calculate the
(i) modulus of elasticity, (ii) modulus of resilience, (iii) ultimate deformation of the specimen.
Also draw its load (P) vs. deformation (δ) diagram indicating appropriate values of P and δ.

5. The rectangular prism shown below is subjected to uniformly distributed normal forces in the x and y directions and is restrained in the z direction (i.e., $\varepsilon_{zz} = 0$). Calculate the normal stresses (σ_{xx}, σ_{yy}, σ_{zz}) and strains (ε_{xx}, ε_{yy}, ε_{zz}) that develop in the prism [Given: E = 2000 ksi, Poisson's ratio = 0.20].

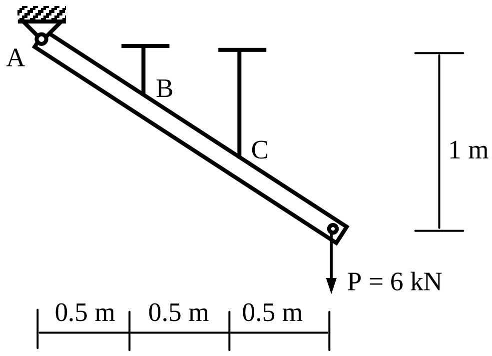


6. For members ABCD shown below, draw the axial stress diagram and calculate the elongations at point A, B, C and D. Neglect the effects of stress concentration [Given: Modulus of Elasticity = 30,000 ksi].

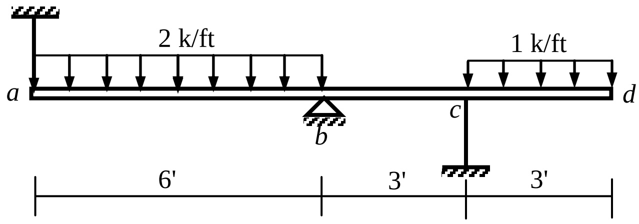


7. Rework Question 6 if both ends A and D are restrained.

8. A rigid bar is supported by a pin at A and two linearly elastic wires at B and, as shown in the figure below. The area of the wire B is 60 mm^2 and wire C is 120 mm^2. Determine the reactions at A, B and C caused by the applied force P = 6 kN.



9. Calculate the forces and axial stresses in wires *a* and *c* supporting the rigid beam *abcd* shown below. Also calculate the deflection of the beam at *a*, *c* and *d*
[Given: E = 30,000 ksi, A = 0.20 in^2, L = 15″ for wire *a*, E = 20,000 ksi, A = 0.10 in^2, L = 15″ for *c*]



10. A cylindrical steel pressure vessel 400 mm in diameter with a wall thickness of 20 mm is subjected to an internal pressure of 4.5 MPa.
    (i) Calculate the tangential and longitudinal stresses and strains in the steel.
    (ii) To what value may the internal pressure be increased in the allowable stress in steel is 120 MPa
    (iii) If the internal pressure is increased until the vessel burst, sketch the type of fracture that would occur

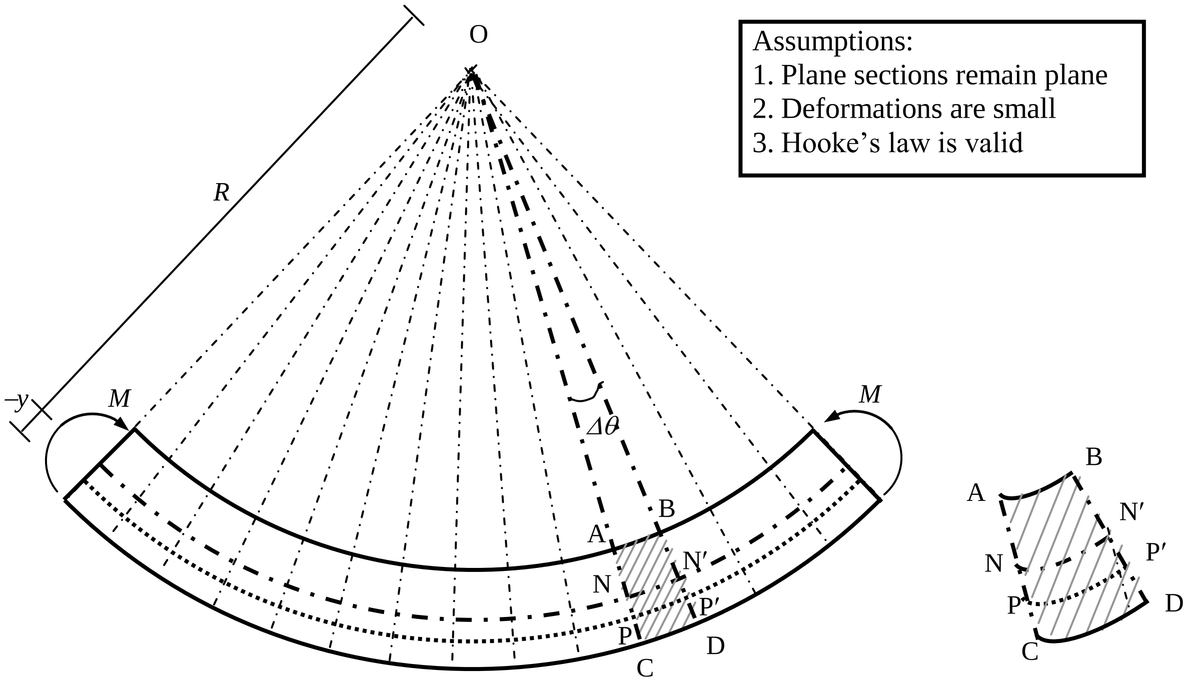
    [Given: Modulus of elasticity of steel = 200 GPa, Poisson's ratio = 0.25].

11. For a gas cylinder of 6′ diameter and 0.25″ wall thickness, calculate the
    (i) maximum internal pressure that the cylinder can be subjected to,
    (ii) corresponding tangential and longitudinal stresses and strains in the wall of the cylinder,
    (iii) required spacing of 1″ diameter bolts and thickness of welds (both longitudinal and circumferential) if the cylinder is subjected to an internal pressure one-third the maximum pressure calculated in (i).

    [Given: Allowable tensile stress in the wall = 20 ksi, Allowable shear stress in bolts and welds = 16 ksi, Modulus of elasticity of steel = 30×10^3 ksi, Poisson's ratio = 0.25].

12. A large pipe, 1.5 m in diameter, is composed of wooden pieces and bound together by steel hoops 300 mm^2 in cross-sectional area. If the permissible tensile stress in the hoops is 130 MPa, what is the maximum spacing between hoops under a head water of 30 m [Given: Unit weight of water = 1 ton/m^3].

Bending Stress and Neutral Axis



Assumptions:
1. Plane sections remain plane
2. Deformations are small
3. Hooke's law is valid

Fig. 4.1: Pure Bending of Beam

If NN′ = Δs and PP′ = $\Delta s + \Delta u$

$\therefore$ If $\Delta s \to 0$, Normal Strain, $\varepsilon_x = \Delta u/\Delta s \to du/ds$ ……………...…….(4.1)

Also, $\Delta u = -y\ \Delta\theta \Rightarrow \Delta u/\Delta s = -y\ \Delta\theta/\Delta s$ …………………(4.2)

$\therefore$ If $\Delta s \to 0,\ du/ds = -y\ d\theta/ds$ ……………..…..(4.3)

Also, Curvature $\kappa = 1/R = d\theta/ds$

$\therefore$ Eq. (4.1) $\Rightarrow \varepsilon_x = -y\ \kappa$ …………………(4.4)

Using stress-strain relationship, $\sigma_x = E\varepsilon_x \Rightarrow$ Normal Stress, $\sigma_x = -E\ y\ \kappa$ …………………(4.5)

$\therefore$ Total Normal force on the section, $\sum dF_x = \int \sigma_x\ dA = -E\kappa \int y\ dA$ …………………(4.6)

Since $\sum dF_x = 0 \Rightarrow -E\kappa \int y\ dA = 0 \quad \Rightarrow \int y\ dA = 0$ …………………(4.7)

Eq. (4.7) $\Rightarrow$ The beam bends about its Centroidal Axis, which is also called its *Neutral Axis*

$\therefore$ Total moment on the section, $\sum dM_z = \sum -y\ dF_x = \int -y\ \sigma_x\ dA = E\kappa \int y^2\ dA$

$\therefore M_z = E\kappa I_z \quad \Rightarrow \kappa = M_z/EI_z$ …………………(4.8)

Combining Eqs. (4.4) and (4.8) $\Rightarrow \sigma_x = -M_z\ y/I_z$ …………………(4.9)

$\therefore$ Maximum normal stress,

$\sigma_{x(max)} = \pm M_z\ y_{max}/I_z$, commonly denoted by $\sigma_{x(max)} = \pm M_z\ c/I_z$ ………..………(4.10)

Example 4.1: Calculate the maximum bending stress in the simply supported beam shown below.

Maximum bending moment, $M_{max} = wL^2/8 = 12.5$ k′ = 150 k″
Moment of inertia $I_z = 10 \times 10^3/12 = 833.33$ in^4
$y_{max} = c = 10/2 = 5''$

Maximum normal stress
$\sigma_{x(max)} = \pm 150 \times 5/833.33 = \pm 0.9$ ksi

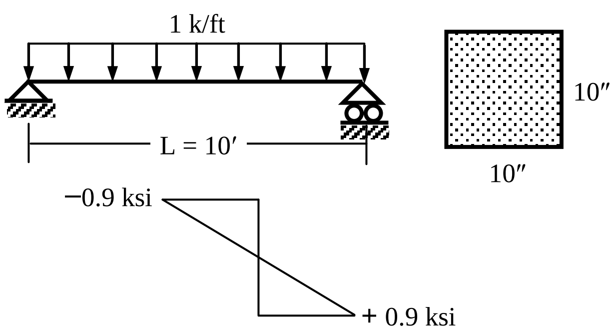


Fig. 4.2: Bending Stress in Simply Supported Beam

Bending Stress in Composite Sections

Instead of homogeneous materials, engineering structures are quite often made of composite sections. For example, concrete and wooden beams are sometimes reinforced by steel and metal straps, plastics reinforced with fibers. The structural analysis of these sections is somewhat different from the analysis of homogeneous sections. Although the basic assumption '*plane sections remain plane after bending*' is still valid, the resulting stresses are quite different, depending (for linearly elastic materials) on the modulus of elasticity of the different materials of the section.

To use the basic equations of pure bending, the structural analysis of these sections assumes them to be made of a homogeneous material. Instead of changing the modulus of elasticity over the section, the width of various parts is modified proportionately. The stress analysis is made of an *Equivalent* or *Transformed Section* derived.

Example 4.2

For a simply supported beam loaded as shown below, draw the flexural stress and strain diagrams over the composite cross-sectional area at section B.

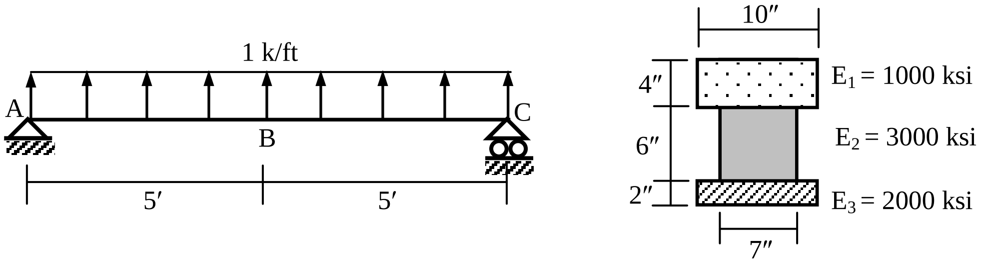


Fig. 4.3: Simply Supported Composite Beam

Maximum bending moment, $M_{max} = -1\times10^2/8 = -12.5$ k-ft $= -150$ k-in

For the *Equivalent Section* (Fig. 4.4), assumed to be made of material1

$\bar{y} = (40 \times 1 + 126 \times 5 + 40 \times 10)/(40 + 126 + 40) = 5.19''$

and $\bar{I} = 20 \times 2^3/12 + 40 \times (5.19-1)^2 + 21 \times 6^3/12 + 126 \times (5.19-5)^2$

$+ 10 \times 4^3/12 + 40 \times (5.19-10)^2 = 2076.90 \text{ in}^4$

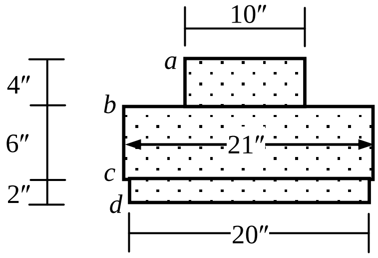


Fig. 4.4: The Equivalent Section

Bending strains

$\varepsilon_a = -\{(-150) \times 6.81/2076.90\}/1000 = 4.92 \times 10^{-4}$

$\varepsilon_b = -\{(-150) \times 2.81/2076.90\}/1000 = 2.03 \times 10^{-4}$

$\varepsilon_c = -\{(-150) \times (-3.19)/2076.90\}/1000 = -2.31 \times 10^{-4}$

$\varepsilon_d = -\{(-150) \times (-5.19)/2076.90\}/1000 = -3.75 \times 10^{-4}$

The corresponding stresses are

$\sigma_a = \varepsilon_a E_1 = 4.92 \times 10^{-4} \times 1000 = 0.492$ ksi

$\sigma_{b1} = \varepsilon_b E_1 = 0.58 \times 10^{-5} \times 1000 = 0.203$ ksi, $\sigma_{b2} = \varepsilon_b E_2 = 0.609$ ksi

$\sigma_{c2} = \varepsilon_c E_2 = -0.692$ ksi, $\sigma_{c3} = \varepsilon_c E_3 = -0.461$ ksi

$\sigma_{d3} = \varepsilon_d E_3 = -0.750$ ksi

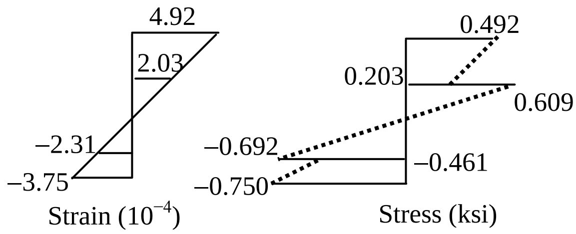


Fig. 4.5: Variation of Flexural Strain and Stress

Example 4.3

If the cross-sectional area of the simply supported beam loaded as in Example 4.2 is made of Reinforced Concrete as shown below, draw the flexural stress diagram over the section B.

Assume that the section is 'uncracked' [Given: $E_{steel} = 30000$ ksi, $E_{concrete} = 3000$ ksi].

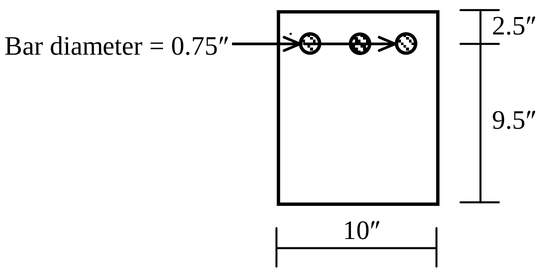


Fig. 4.6: Reinforced Concrete Section

Bending Stress in Reinforced Concrete

Reinforced Concrete is one of the prime examples of a composite section subjected to flexural stress. It is a combination of concrete and reinforcing steel working together in a wide range of structural applications. As a building material, it has unique characteristics, because the very low tensile strength of plain concrete is offset by the high tensile efficiency of the encased steel bars. The following examples illustrate the linearly elastic material behavior of Reinforced Concrete subjected to flexural stress.

The first illustration is Example 4.3 shown before, when concrete does not crack due to tensile stress.

<u>Solution of Example 4.3</u>

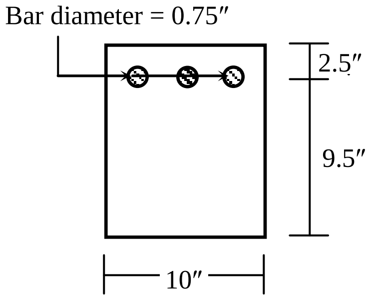


Fig. 4.6 (Repeated):
Reinforced Concrete Section

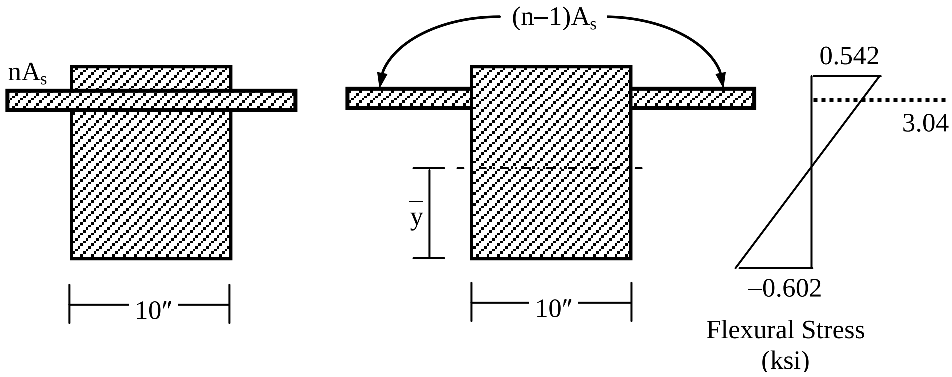


Fig. 4.7: Equivalent Uncracked Section

The modular ratio, $n = E_{steel}/E_{concrete} = 30000/3000 = 10$

$\therefore$ Transformed extra steel area in the Equivalent Cracked section, $(n-1)A_s = 9 \times [3 \times \pi(0.75)^2/4] = 11.93$ in^2

$\bar{y} = (120 \times 6 + 11.93 \times 9.5)/(120 + 11.93) = 6.32''$

$\therefore$ Moment of Inertia, $\bar{I} = 10 \times 12^3/12 + 120 \times (6 - 6.32)^2 + 11.93 \times (9.5 - 6.32)^2 = 1572.91$ in^4

$\therefore$ Maximum compressive stress in concrete, $f_c = M\bar{y}/\bar{I} = 150 \times 6.32/1572.91 = 0.602$ ksi

Maximum tensile stress in concrete, $f_t = 150 \times (12 - 6.32)/1572.91 = 0.542$ ksi

Maximum tensile stress in steel, $f_s = 150 \times (9.5 - 6.32)/1572.91 \times 10 = 3.04$ ksi

As the tensile stress induced in concrete is quite substantial (expected to be greater than its tensile strength), it is more reasonable to assume concrete to have cracked in tension.

<u>Example 4.4</u>

Rework Example 4.3 if the Reinforced Concrete section at B (shown below) is assumed 'cracked'.

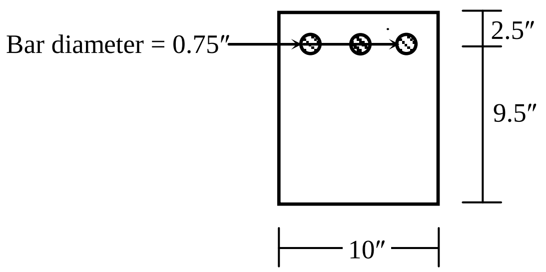


Fig. 4.6 (Repeated):
Reinforced Concrete Section

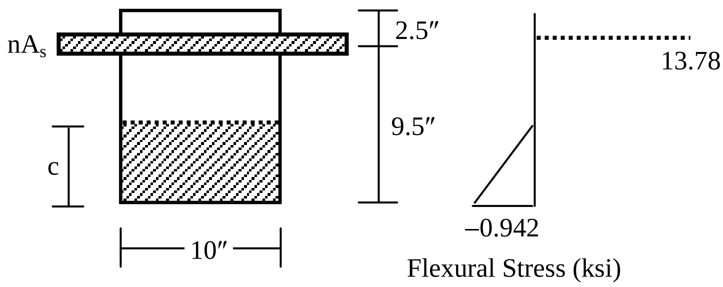


Fig. 4.8: Equivalent Cracked Section

Modular ratio, $n = 10$

$\therefore$ Transformed steel area in the Equivalent Cracked section, $nA_s = 10 \times [3 \times \pi(0.75)^2/4] = 13.25$ in^2

Taking moments of the cracked area about neutral axis $\Rightarrow$

$10\, c \times c/2 = 13.25\,(9.5 - c) \Rightarrow 5c^2 + 13.25\,c - 125.91 = 0 \Rightarrow c = 3.86''$

$\therefore$ Moment of Inertia, $\bar{I} = 10 \times 3.86^3/3 + 13.25 \times (9.5 - 3.86)^2 = 613.30$ in^4

$\therefore$ Maximum compressive stress in concrete, $f_c = Mc/\bar{I} = (150 \times 3.86)/613.30 = 0.945$ ksi

Maximum tensile stress in steel, $f_s = 150 \times (9.5 - 3.86)/613.30 \times 10 = 13.78$ ksi

Plastic Bending of Beams

For reasons of economy, it is important to determine member strengths beyond the elastic limit. The elastic bending theory for beams can be readily extended to inelastic bending by introducing a uniaxial nonlinear stress-strain relationship for the material. The basic requirement for statics and kinematics of deformation will remain the same as for the elastic case.

The elastic perfectly plastic idealization, for reasons of simplicity, is often used, for beams of ductile materials in determining their behavior in bending. Fig. 4.9 shows the elastic-perfectly plastic uniaxial stress-strain (σ-ε) relationship of a material, while Fig. 4.10 shows the development of strain and stress when a beam section made of this material is subjected to pure bending.

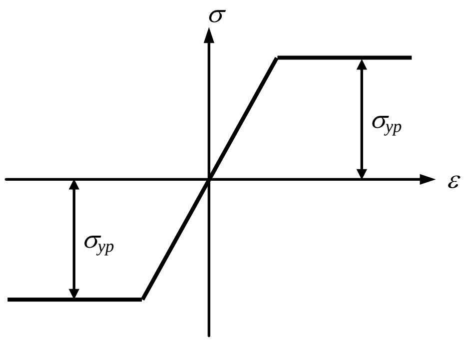


Fig. 4.9: Elastic-Perfectly Plastic σ-ε Relation

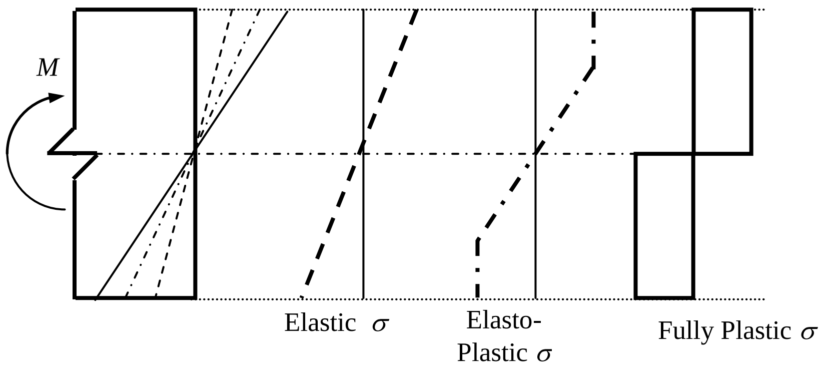


Fig. 4.10: Development of Bending Strain and Stress

As shown in Fig. 4.10, it is reasonable to assume linear variation of strain across the section (i.e., plane sections remain plane after bending), but as the strain increases the corresponding variation of stress is no longer linear. At the extreme case, the stress over almost the entire section may reach the yield point (σ_{yp}); i.e., the section becomes fully plastic and the corresponding bending moment M_p (the *Plastic Moment* of the section) is given by

$$M_p = \sigma_{yp} Z \qquad \text{……….………(4.11)}$$

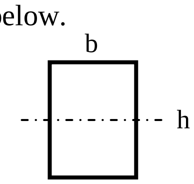
where Z is called the plastic Section Modulus. The ratio of the plastic and elastic section modulus (i.e., Z and S) is called the aspect ratio (α) of the section; i.e.,

$$\alpha = Z/S \qquad \text{……….………(4.12)}$$

If the material has identical σ-ε relation in tension and compression, the neutral axis of the fully plastic section corresponds to two equal areas in tension and compression.

Example 4.5

Calculate the Section Modulus, Plastic Section Modulus and Shape Factor of the sections shown below.



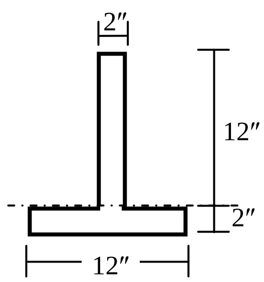
For the rectangular section, the neutral axis divides the area into two segments of (b × h/2).

$\therefore$ Compressive force = Tensile force = σ_{yp} (bh/2)

$\therefore$ Plastic moment M_p = Tensile (or compressive) force × Moment arm = σ_{yp} (bh/2) × h/2

$\therefore M_p = \sigma_{yp} (bh^2/4) \Rightarrow Z = bh^2/4$

Since $S = bh^2/6$, $\alpha = Z/S = 1.5$



For the T-section, the neutral axis divides the area along the flange line.

$\therefore$ Compressive force = Tensile force = $\sigma_{yp} (12 \times 2) = 24\sigma_{yp}$

$\therefore$ Plastic moment M_p = Tensile (or compressive) force × Moment arm = $24\sigma_{yp} \times (1 + 6)$

$\therefore M_p = \sigma_{yp} (168) \Rightarrow Z = 168$ in^3

Also, $\bar{y} = (24 \times 1 + 24 \times 8)/48 = 4.5''$; $c = 14 - 4.5 = 9.5''$

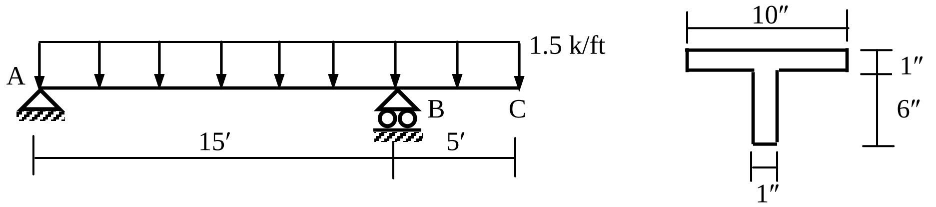
$\bar{I} = 12 \times 2^3/12 + 24 (1 - 4.5)^2 + 2 \times 12^3/12 + 24 (8 - 4.5)^2 = 884$ in^4

$S = 884/9.5 = 93.05$ in^3, $\alpha = Z/S = 1.81$

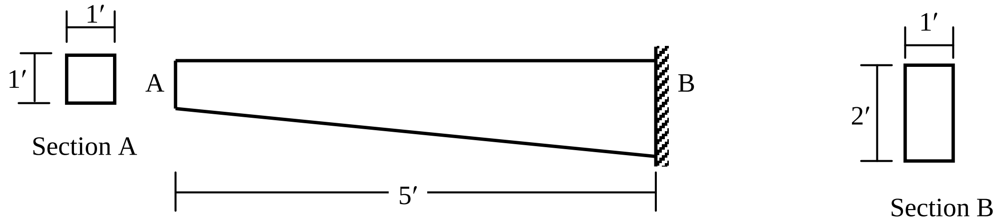
Fig. 4.11

Practice Problems on Bending Stress

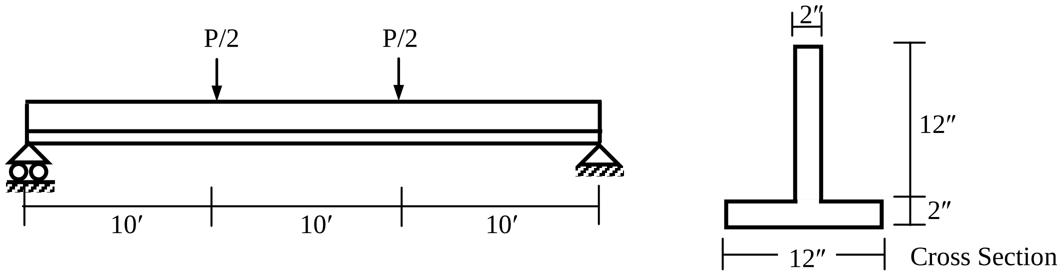
1. Calculate (i) the maximum positive and negative bending moments, (ii) the maximum tensile and compressive flexural stresses in the beam ABC loaded as shown below.



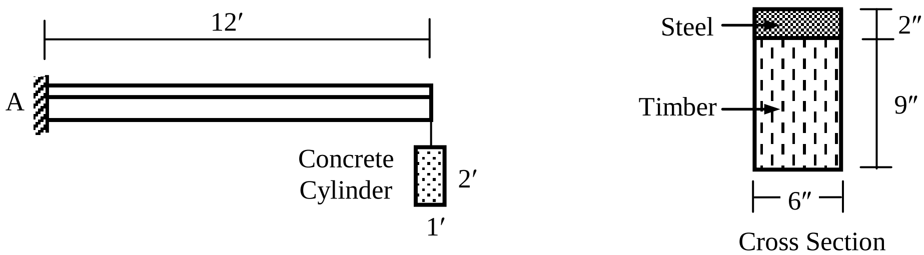
2. The figure below shows a 5′ long cantilever beam of uniformly varying cross-section. The beam is 1′ wide and its depth increases linearly from 1′ at the free end A to 2′ at the fixed end B. If the beam weighs 150 lb per ft^3, calculate the maximum bending stress at B due to the self-weight of the beam.



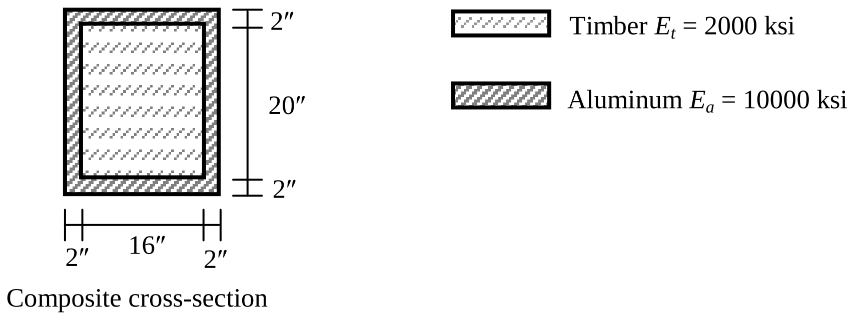
3. Calculate the maximum allowable load P in the simply supported beam loaded as shown below, if the allowable compressive stress in the cross-section is 20 ksi and allowable tensile stress 15 ksi. For this value of P, (i) draw the bending stress and strain diagrams over the section, (ii) calculate the compressive and tensile forces acting on the section [Given: Modulus of elasticity E = 30,000 ksi].



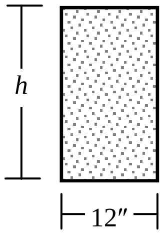
4. A concrete cylinder of 1′ diameter and 2′ height is suspended at the free end of a 12′ cantilever beam as shown in the figure below, which also shows the composite cross-section of the beam, made of steel and timber. Calculate the maximum flexural stresses in the section [Given: Unit weight of concrete = 0.15 k/ft^3, Modulus of elasticity of steel = 30,000 ksi, Modulus of elasticity of timber = 1500 ksi].



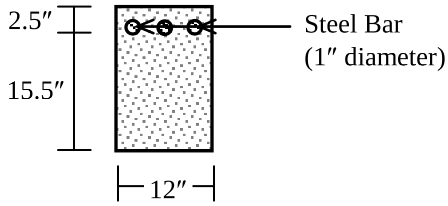
5. For the beam described in Question 4, draw the flexural strain diagram over the composite cross-sectional area shown below at Section A. Also calculate the maximum stress in timber and aluminum.



6. (i) Calculate the required depth 'h' if the Section 1 shown below (made of concrete) is subjected to a negative bending moment of 50 k-ft.
   (ii) Calculate the maximum flexural stress in concrete if the section is made of Reinforced Concrete as shown below in Section 2. Assume the section is 'cracked' due to concrete tension
   [Given: Allowable concrete stress in tension = 200 psi, compression = 1500 psi, n = 10].

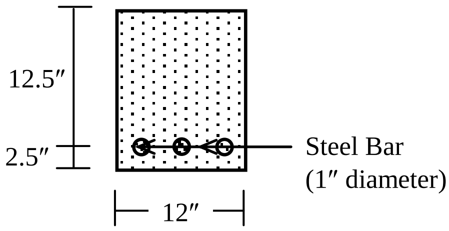


Section 1

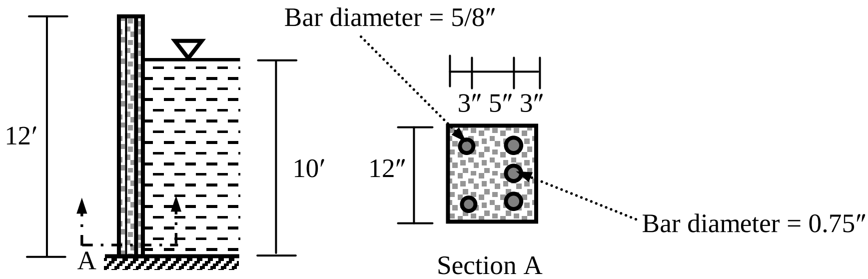


Section 2 (Reinforced Concrete)

7. Calculate the bending moment required to cause the Reinforced Concrete section shown below to crack in concrete tension. Also calculate the corresponding tensile stress in steel bars and maximum compressive stress in concrete [Given: n = 10, Allowable tensile stress in concrete = 300 psi].



8. The side elevation and 1-ft wide cross-section 'A' of the Reinforced Concrete wall of a 12-ft high water tank is shown in the figure below. Calculate the maximum bending stresses in concrete and steel assuming that the section is (i) 'uncracked', (ii) 'cracked' due to concrete tension
   [Given: E_{steel} = 30000 ksi, $E_{concrete}$ = 3000 ksi].



Section A

9. Calculate the (i) Section Modulus, (ii) Plastic Section Modulus and Shape Factor of the inverted T-section shown in Question 3.

10. Calculate the ultimate moment capacity of the RC section shown in Question 7.

Shear Flow and Flexural Shear Stress

The theory of simple bending analyzes a section under pure bending, neglecting the effect of shear force acting on it. But in actual practice when a beam is loaded, the shear force at a section always comes into play along with the bending moment. And sometimes the shear stress at a section assumes much importance in design.

Shear Force at a Section

Consider a small beam segment ABCD of length dx shown in Fig. 5.1(a), in equilibrium under the action of shear forces V and $(V + dV)$ and bending moments M and $(M + dM)$ at sections AB and CD respectively.

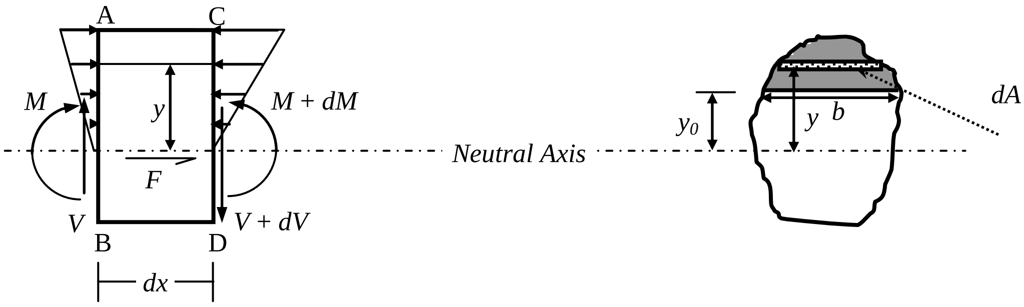


Fig. 5.1: (a) Shear and Moment on a Differential Beam Segment, (b) Cross-Sectional Area

If $\bar{I}$ = Centroidal moment of inertia of the cross-sectional area [shown in Fig. 5.1(b)],

Bending stress on the differential area dA at height y on section AB, $\sigma = -My/\bar{I}$ ……………….(5.1)

and on the differential area on section CD, $\sigma + d\sigma = -(M + dM)\,y/\bar{I}$ ……..………...(5.2)

$\therefore$ The net unbalanced normal force acting on the differential area dA

$$dF = (M + dM)\,y/\bar{I} \times dA - (M)\,y/\bar{I} \times dA = (dM)(dA)\,y\,\bar{I} \qquad \text{……..………...(5.3)}$$

$\therefore$ Total unbalanced normal force on the dark-shaded area

$$F = \int dF = \int (dM)\,(dA)\,y/\bar{I} = (dM)\,A\,\bar{y}/\bar{I} \qquad \text{……..………...(5.4)}$$

This force has to be balanced by a shear force F acting along the length of the beam.

$\therefore$ Shear force per unit length of beam (at height y_0 from the Neutral Axis) = $F/dx = (dM/dx)\,A\,\bar{y}/\bar{I} = VQ/\bar{I}$

$$\text{i.e., Shear Flow, } q = VQ/\bar{I} \qquad \text{……...………...(5.5)}$$

$\therefore$ If b is the width of the section at height y_0 from Neutral Axis, shear stress along horizontal surface $b\,dx$

$$\tau_{yx} = VQ/\bar{I}b \qquad \text{……...………...(5.6)}$$

Since $\tau_{xy} = \tau_{yx}$ (from past formulation of shear stress), shear stress along the vertical surface AB is also

$$\tau = VQ/\bar{I}b \qquad \text{……...………...(5.7)}$$

Example 5.1: Calculate the flexural shear stress at levels 1-1, 2-2 and 3-3 at the support sections of the simply supported beam shown below.

Shear force at support, V = wL/2 = 5 k = 5000 lb

Moment of inertia $\bar{I}_z = 10 \times 10^3/12 = 833.33$ in^4

At level 1-1, $Q_{11} = (10 \times 5) \times 5/2 = 125$ in^3

$\therefore \tau_{11} = VQ_{11}/\bar{I}b = 5000 \times 125/(833.33 \times 10) = 75$ psi

At level 2-2, $Q_{22} = (10 \times 2) \times (3 + 2/2) = 80$ in^3

$\therefore \tau_{22} = VQ_{22}/\bar{I}b = 5000 \times 80/(833.33 \times 10) = 48$ psi

At level 3-3, $Q_{33} = 0$

$\therefore \tau_{33} = VQ_{33}/\bar{I}b = 5000 \times 0/(833.33 \times 10) = 0$

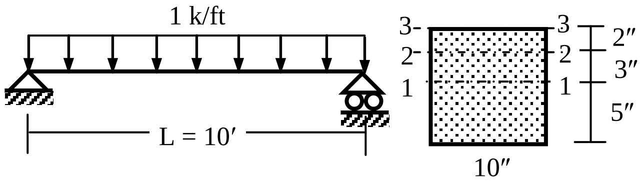


Fig. 5.2: Levels at Section of Simply Supported Beam

Flexural Shear Stress in Typical Sections

The following examples demonstrate the shear stress distribution over several cross-sectional areas.

Example 5.2: Show the variation of flexural shear stress over a rectangular and circular cross-section.

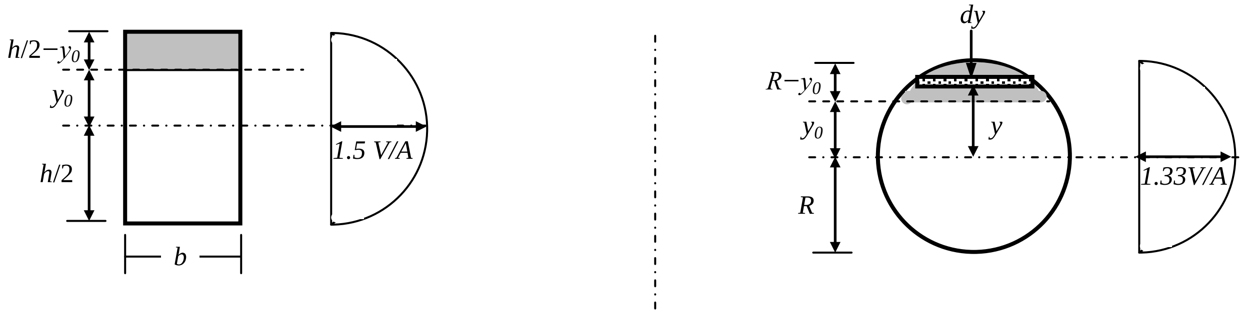


Fig. 5.3: Flexural Shear Stress Distribution for (a) Rectangular Section, (b) Circular Section

For the rectangular section,

Area $A = bh$, Moment of inertia $\bar{I}_z = bh^3/12$

At level y_0, $Q_0 = b \times (h/2 - y_0) \times (h/2 + y_0)/2$

$= b \times \{(h/2)^2 - (y_0)^2\}/2$

$\therefore \tau_0 = VQ_0/\ \bar{I}\, b$

$= V \times [b \times \{(h/2)^2 - (y_0)^2\}/2]/(bh^3/12 \times b)$

$= 1.5\ (V/bh) \times \{1 - (2y_0/h)^2\}$

$\therefore \tau_{max} = 1.5\ (V/bh) = 1.5\ V/A$ [when $y_0 = 0$]

For the circular section,

Area $A = \pi R^2$, Moment of inertia $\bar{I}_z = \pi R^4/4$

At level y, $dQ = \{2\sqrt{(R^2 - y^2)}\ dy\}\ y = 2y\sqrt{(R^2 - y^2)}\ dy$

$\therefore$ At level y_0, $Q_0 = \int 2y\sqrt{(R^2 - y^2)}\ dy = 2/3\ (R^2 - y_0^2)^{3/2}$

$\therefore \tau_0 = VQ_0/\ \bar{I}\, b$

$= V \times [2/3\ (R^2 - y_0^2)^{3/2}]/\{\pi R^4/4 \times 2\sqrt{(R^2 - y_0^2)}\}$

$= (4/3)\ (V/\pi R^2) \times \{1 - (y_0/R)^2\}$

$\therefore \tau_{max} = (4/3)\ (V/\pi R^2) = 1.33\ V/A$ [when $y_0 = 0$]

Example 5.3: Calculate the flexural shear stress at Level 1-1 and 2-2 for the T- and I-sections shown below and loaded as a simply supported beam shown in Example 5.1.

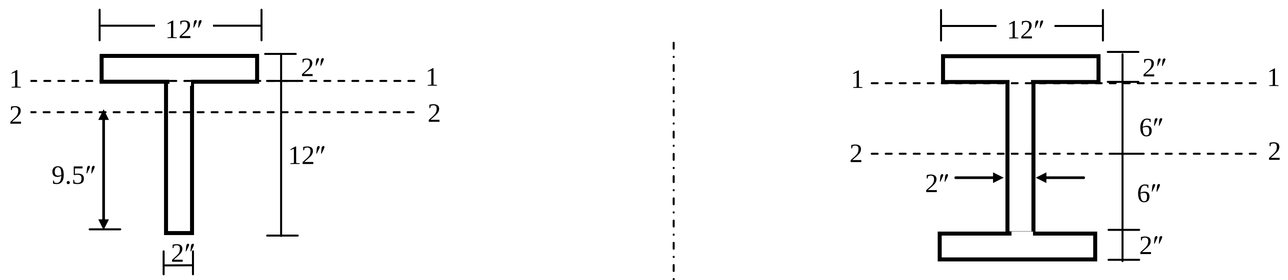


Fig. 5.4: Flexural Shear Stress Distribution for a (a) T-Section, (b) I-Section

For the T-section,

Area A = 24+24 = 48 in^2, $\bar{y}$ = (24×6 + 24×13)/48 = 9.5″

$\therefore$ Moment of inertia

$\bar{I}_z = 24\ (12^2/12 + 3.5^2) + 24\ (2^2/12 + 3.5^2) = 884$ in^4

At level 1-1, $Q_{11} = 24 \times (2.5 + 2/2) = 84$ in^3

At level 2-2, $Q_{22} = Q_{11} + 2 \times 2.5 \times (2.5/2) = 90.25$ in^3

$\therefore \tau_{11}$ (flange) $= VQ_{11}/\ \bar{I}_z\, b_{11} = 5 \times 84/[884 \times 12]$

$= 0.040$ ksi

$\therefore \tau_{11}$ (web) $= 5 \times 84/[884 \times 2] = 0.238$ ksi

$\therefore \tau_{22} = 5 \times 90.25/[884 \times 2] = 0.255$ ksi

$\therefore \tau_{max} = 0.255$ ksi, while $\tau_{avg} = 5/48 = 0.104$ ksi

Neglecting the flange, $\tau_{avg} \cong 5/(28) = 0.179$ ksi

For the I-section,

Area A = 24 + 24 + 24 = 72 in^2, $\bar{y}$ = 8″

$\therefore$ Moment of inertia

$\bar{I}_z = 12 \times 16^3/12 - 10 \times 12^3/12 = 2656$ in^4

At level 1-1, $Q_{11} = 24 \times (6 + 2/2) = 168$ in^3

At level 2-2, $Q_{22} = Q_{11} + 2 \times 6 \times (6/2) = 204$ in^3

$\therefore \tau_{11}$ (flange) $= VQ_{11}/\ \bar{I}_z\, b_{11} = 5 \times 168/[2656 \times 12]$

$= 0.026$ ksi

$\therefore \tau_{11}$ (web) $= 5 \times 168/[2656 \times 2] = 0.158$ ksi

$\therefore \tau_{22} = 5 \times 204/[2656 \times 2] = 0.192$ ksi

$\therefore \tau_{max} = 0.192$ ksi, while $\tau_{avg} = 5/72 = 0.069$ ksi

Neglecting the flange width, $\tau_{avg} \cong 5/(32) = 0.156$ ksi

Flexural Shear and Connections

Structural sections are often built by combining together several simpler sections. The main purpose is to enhance the load-carrying capacity of the sections, which may otherwise be inadequate to withstand the design loads. These sections can be either glued together or joined by bolts/welds to withstand the shear flow working at the joints to separate the sections.

Example 5.4:

(i) Calculate the flexural shear flow at Level 1-1 of the T-sections joined as shown below in (a) and (b) if loaded as the simply supported beam shown in Example 5.1.

(ii) Calculate the spacing of 3/8″ bolts required at the joints to withstand shear flow.

(iii) Calculate the size of welds required at the joints [Given: Allowable shear stress = 15 ksi].

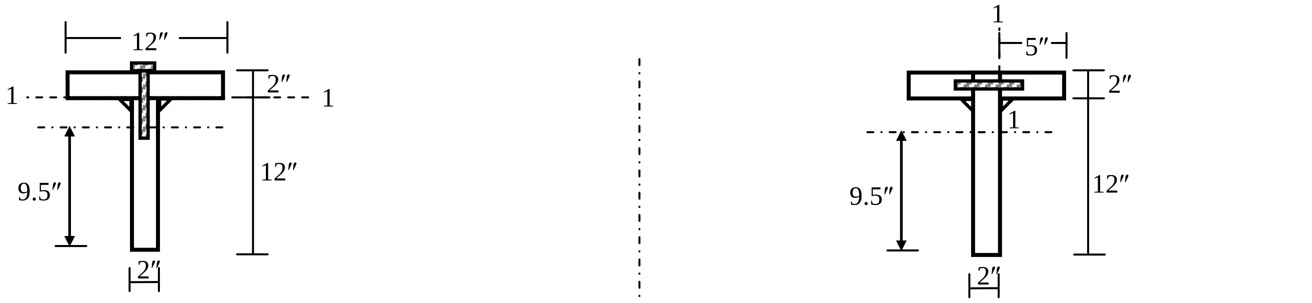


Fig. 5.5: Flexural Shear Flow at 1-1 for a T-Section joined as (a) and (b)

For the T-section joined as (a)

Area A = 48 in^2, $\bar{y}$ = 9.5″, $\bar{I}_z$ = 884 in^4

At level 1-1, Q_{11} = 24 × (2.5 + 2/2) = 84 in^3

∴ Shear flow $q_{11} = VQ_{11}/\bar{I}_z$ = 5 × 84/884 = 0.475 k/in

Allowable shear at bolts = (π/4) × (3/8)^2 × 15 = 1.66 k

∴ Required bolt spacing = 1.66/0.475 = 3.49″

∴ Bolts can be spaced @ 3.5″ c/c

Allowable weld shear = 0.707 τ_{all} t × 2 = 1.414 × 15 × t = 21.21 t k/in

∴ 21.21 t = 0.475 ⇒ t = 0.022″, which is quite nominal

∴ 1/16″ thickness will be more than sufficient [even if used in part of the beam]

For the T-section joined as (b)

Area A = 48 in^2, $\bar{y}$ = 9.5″, $\bar{I}_z$ = 884 in^4

At level 1-1, Q_{11} = 5 × 2 × (2.5 + 2/2) × 2 = 70 in^3

∴ Shear flow $q_{11} = VQ_{11}/\bar{I}_z$ = 5 × 70/884 = 0.396 k/in

Allowable shear at bolts = 1.66 k

∴ Required bolt spacing = 1.66/0.396 = 4.18″

∴ Bolts can be spaced @ 4″ c/c

Allowable weld shear = 21.21 t k/in

∴ 21.21 t = 0.396 ⇒ t = 0.019″, which is also nominal

∴ 1/16″ thickness will again be more than sufficient

Example 5.5:

Repeat the calculations of Example 5.4 for the I-section shown in Example 5.3.

Shear Center

Torsional moments are caused by forces acting off the axis of a section. Every cross-section possesses a point through which the transverse load must be applied so as to produce to torsion in the section. The point is called the *Shear Center* of the section. Torsion is produced if the transverse load is applied away from the shear center and moreover, the twist takes place around it. For this reason, the shear center is also called the center of twist.

For cross-sectional areas having one axis of symmetry (e.g., T-section, channel section), the shear center is always located on the axis of symmetry. For those having two axes of symmetry (e.g., rectangular, circular, I-section), the shear center coincides with the centroid of the cross-sectional area.

For thin-walled channel sections, of particular interest here, the location of the shear center is obtained explicitly by considering the shear flow within the section due to applied transverse load V. If b = Width of the flange, h = Height of the web, t = Thickness of the section, I = Moment of inertia of the section (shown in Fig. 5.6), the maximum shear flow in the horizontal flanges

$$q_{max} = VQ_{max}/I = V \times (b \times t \times h/2)/I \quad \text{………………..….(5.8)}$$

$\therefore$ Horizontal Force, $H = q_{max}\, b/2 = V \times (b^2 \times h \times t)/(4\, I)$ ………………..….(5.9)

$\therefore$ Torsional moment due to horizontal forces $T = H \times h = V \times (b^2 h^2 t)/(4\, I)$ ……………..….(5.10)

$\therefore$ Required distance of applied load from web, $e = T/V = (b^2 h^2 t)/(4\, I)$ ……………..….(5.11)

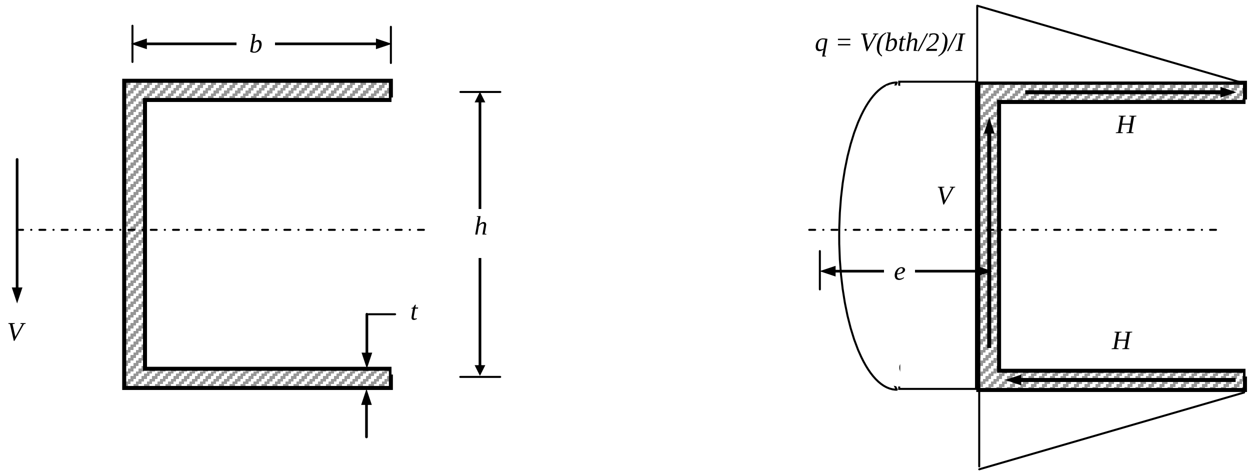


Fig. 5.6: Location of Shear Center of Thin-walled Channel section

For thin-walled cross-sections, I can be approximated by

$$I \simeq th^3/12 + bth^2/2 \quad \text{…………………….(5.12)}$$

from which $e \simeq b/(2+h/3b)$ …..……………...(5.13)

As mentioned, torsion is induced in the section if the transverse load is applied away from the shear center. Fig. 5.7 shows that in such cases, the section twists about its shear center; i.e., the torsional moment equals to the applied force times its perpendicular distance from the shear center.

Knowledge about torsional deformations is required to calculate the torsional rotation induced in such cases.

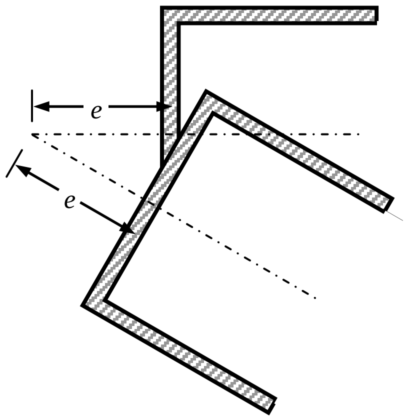
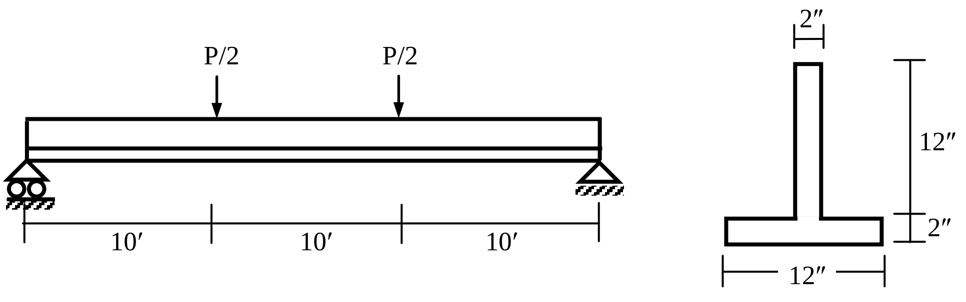


Fig. 5.7: Section twists about Shear Center

Practice Problems on Flexural Shear Stress

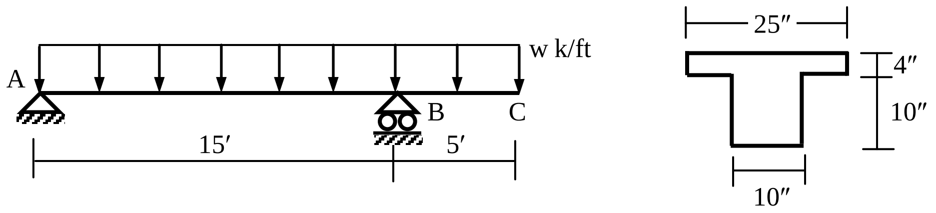
1. Compare the maximum flexural shear stresses on a square and a circular area if their cross-sectional areas the same (assume 100 in^2) and they are subjected to the same shear force (assume 10 k).

2. (i) Calculate the maximum allowable load P in the simply supported beam loaded as shown below, if the allowable shear stress in the cross-section is 12 ksi.
   (ii) For this value of P, draw the shear stress diagram over the section at support.



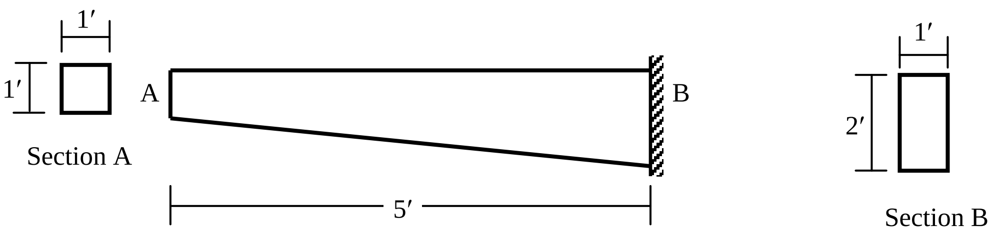
Cross Section

3. (i) Calculate the maximum allowable value of w (k/ft) if the flexural shear stress over the cross-section is not to exceed 100 psi.
   (ii) For the value of w calculated in (i), draw the shear stress diagram over the cross-section where the shear force in the beam is the maximum.

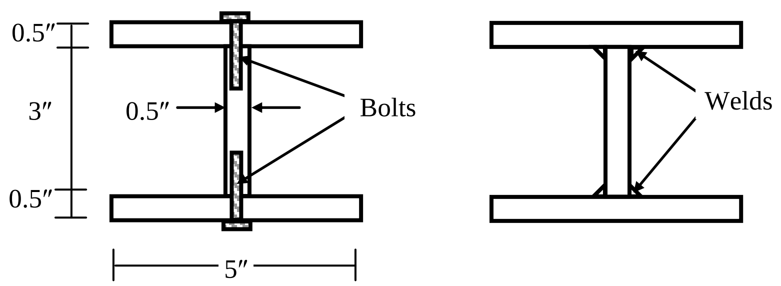


Cross Section

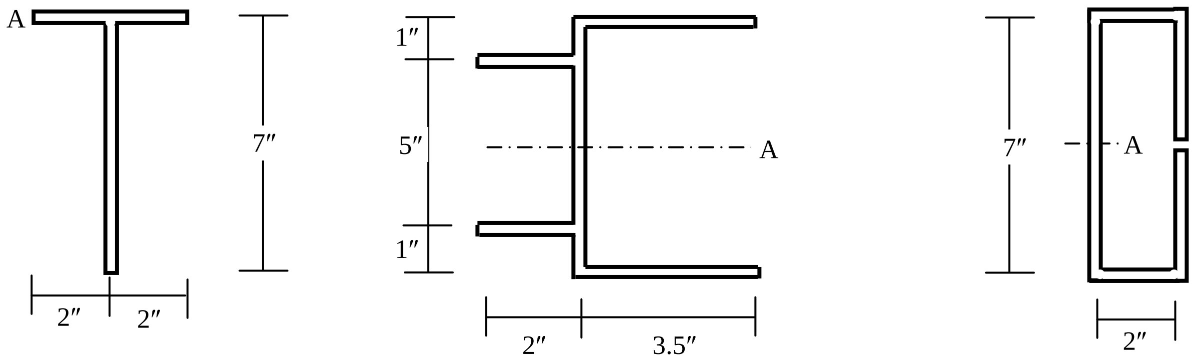
4. The figure below shows a 5′ long cantilever beam of uniformly varying cross-section. The beam is 1′ wide and its depth increases linearly from 1′ at the free end A to 2′ at the fixed end B. If the beam weighs 150 lb per ft^3, calculate the maximum shear stress at the fixed end B due to the self-weight of the beam.



5. Rectangular plates [(5″×0.5″) and (3″×0.5″)] are to be joined either by bolts of 0.25″ diameter (spaced @ 3″ c/c) or 1/8″ thick welds to form an I-section as shown below. Calculate the shear stress induced in the bolts and welds if the section is subjected to a shear force of 10 kips
[Given: Allowable shear stress = 12 ksi].



6. Locate the shear centers of the areas (thickness = 0.25″ throughout) shown below by centerline dimensions, and calculate the torsion produced by the vertical forces (10 kips) passing through points A.



7. The channel-shaped cross-sectional area of a beam is shown below by centerline dimensions. If the self-weight of the beam is 15 lb, calculate the magnitude of force P needed to avoid torsion in the section.