Water vapor in a Static Atmospheric Column

Two laws govern the properties of water vapor in a static column, the ideal gas law

$$p = \rho_a R_a T \tag{3.2.12}$$

and the *hydrostatic pressure law*

$$\frac{dp}{dz} = -\rho_a g \tag{3.2.13}$$

The variation of air temperature with altitude is described by

$$\frac{dT}{dz} = -\alpha \tag{3.2.14}$$

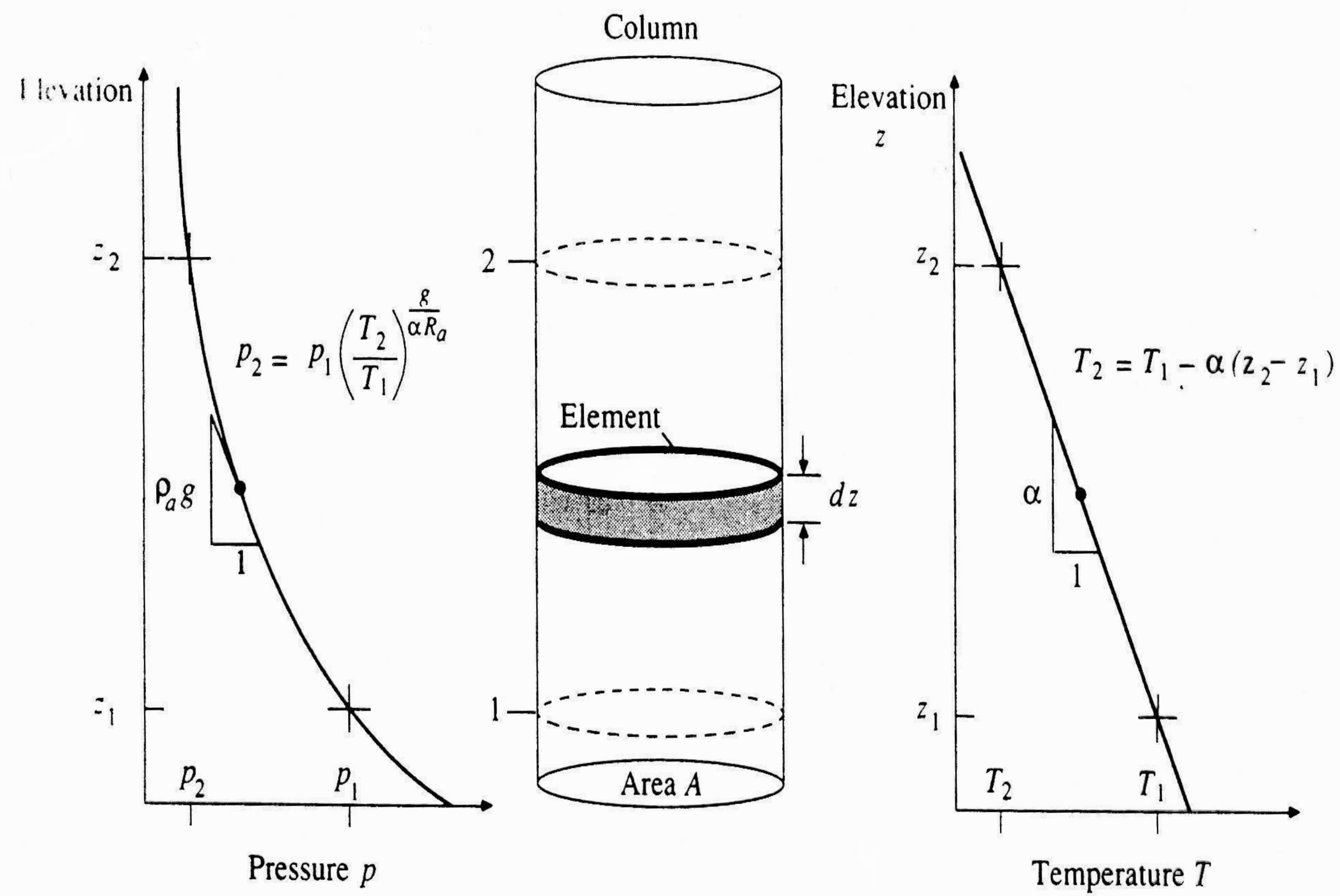
where α is the lapse rate. As shown in Fig. 3.2.2, a linear temperature variation combined with the two physical laws yields a nonlinear variation of pressure with altitude. Density and specific humidity also vary nonlinearly with altitude. From (3.2.12), $\rho_a = p/R_aT$, and substituting this into (3.2.13) yields

$$\frac{dp}{dz} = \frac{-pg}{R_aT}$$

or

$$\frac{dp}{p} = \left(\frac{-g}{R_aT}\right) dz$$

Substituting $dz = -dT/\alpha$ from (3.2.14):



Elevation
z_2
z_1
$p_2 = p_1\left(\frac{T_2}{T_1}\right)^{\frac{g}{\alpha R_a}}$
$\rho_a g$
1
p_2
p_1
Pressure p
Column
2
1
Element
dz
Area A
Elevation
z
z_2
z_1
$T_2 = T_1 - \alpha(z_2 - z_1)$
α
1
T_2
T_1
Temperature T

FIGURE 3.2.2
Pressure and temperature variation in an atmospheric column.

$$\frac{dp}{p} = \left(\frac{g}{\alpha R_a}\right)\frac{dT}{T}$$

and integrating both sides between two levels 1 and 2 in the atmosphere gives

$$\ln\left(\frac{p_2}{p_1}\right) = \left(\frac{g}{\alpha R_a}\right)\ln\left(\frac{T_2}{T_1}\right)$$

or

$$p_2 = p_1\left(\frac{T_2}{T_1}\right)^{g/\alpha R_a} \tag{3.2.15}$$

From (3.2.14) the temperature variation between altitudes z_1 and z_2 is

$$T_2 = T_1 - \alpha(z_2 - z_1) \tag{3.2.16}$$

Precipitable Water

The amount of moisture in an atmospheric column is called its precipitable water.

$$m_p = \int_{z_1}^{z_2} q_v \rho_a A \, dz$$

The integral (3.2.17) is calculated using intervals of height Δz, each witl an incremental mass of precipitable water

$$\Delta m_p = \bar{q}_v \bar{\rho}_a A \Delta z \qquad (3.2.18$$

where $\bar{q}_v$ and $\bar{\rho}_a$ are the average values of specific humidity and air density ove the interval. The mass increments are summed over the column to give the tota precipitable water.

Example 3.2.2. Calculate the precipitable water in a saturated air column 10 km high above 1 m^2 of ground surface. The surface pressure is 101.3 kPa, the surface air temperature is 30°C, and the lapse rate is 6.5°C/km.

Solution. The results of the calculation are summarized in Table 3.2.2. The increment in elevation is taken as Δz = 2 km = 2000 m. For the first increment, at $z_1 = 0$ m, T_1 = 30°C = (30 + 273) K = 303 K; at $z_2 = 2000$ m, by Eq. (3.2.16) using α = 6.5°C/km = 0.0065°C/m,

$$T_2 = T_1 - \alpha(z_2 - z_1)$$

$$= 30 - 0.0065(2000 - 0)$$

$$= 17^\circ\mathrm{C}$$

TABLE 3.2.2

Calculation of precipitable water in a saturated air column (Example 3.2.2)

Column	1 Elevation z (km)	2 Temperature (°C)	3 (°K)	4 Air pressure p (kPa)	5 Density ρ_a (kg/m^3)	6 Vapor pressur e (kPa)
	0	30	303	101.3	1.16	4.24
	2	17	290	80.4	0.97	1.94
	4	4	277	63.2	0.79	0.81
	6	–9	264	49.1	0.65	0.31
	8	–22	251	37.6	0.52	0.10
	10	–35	238	28.5	0.42	0.03

Column	7 Specific humidity q_v (kg/kg)	8 Average over increment $\bar{q}_v$ (kg/kg)	9 $\bar{\rho}_a$ (kg/m^3)	10 Incremental mass Δm (kg)	11 % of total mass
	0.0261				
	0.0150	0.0205	1.07	43.7	57
	0.0080	0.0115	0.88	20.2	26
	0.0039	0.0060	0.72	8.6	11
	0.0017	0.0028	0.59	3.3	4
	0.0007	0.0012	0.47	1.1	2
				77.0	

$$= (17 + 273)\ \text{K}$$
$$= 290\ \text{K}$$

as shown in column 3 of the table. The gas constant R_a can be taken as 287 J/kg·K in this example because its variation with specific humidity is small [see Eq. (3.2.8)]. The air pressure at 2000 m is then given by (3.2.15) with $g/\alpha R_a = 9.81/(0.0065 \times 287) = 5.26$, as

$$p_2 = p_1 \left(\frac{T_2}{T_1} \right)^{g/\alpha R_a}$$
$$= 101.3 \left(\frac{290}{303} \right)^{5.26}$$
$$= 80.4\ \text{kPa}$$

as shown in column 4.

The air density at the ground is calculated from (3.2.12):

$$\rho_a = \frac{p}{R_a T}$$
$$= \frac{101.3 \times 10^3}{(287 \times 303)}$$
$$= 1.16\ \text{kg/m}^3$$

and a similar calculation yields the air density of 0.97 kg/m^3 at 2000 m. The average density over the 2 km increment is therefore $\bar{\rho}_a = (1.16 + 0.97)/2 = 1.07$ kg/m^3 (see columns 5 and 9).

The saturated vapor pressure at the ground is determined using (3.2.9):

$$e = 611 \exp\left(\frac{17.27T}{237.3 + T}\right)$$

$$= 611 \exp\left(\frac{17.27 \times 30}{237.3 + 30}\right)$$

$$= 4244 \text{ Pa}$$

$$= 4.24 \text{ kPa}$$

The corresponding value at 2000 m where $T = 17°\text{C}$, is $e = 1.94$ kPa (column 6). The specific humidity at the ground surface is calculated by Eq. (3.2.6):

$$q_v = 0.622\frac{e}{p}$$

$$= 0.622 \times \frac{4.24}{101.3}$$

$$= 0.026 \text{ kg/kg}$$

At 2000 m $q_v = 0.015$ kg/kg. The average value of specific humidity over the 2-km increment is therefore $\bar{q}_v = (0.026 + 0.015)/2 = 0.0205$ kg/kg (column 8). Substituting into (3.2.18), the mass of precipitable water in the first 2-km increment is

$$\Delta m_p = \bar{q}_v \bar{\rho}_a A \, \Delta z$$

$$= 0.0205 \times 1.07 \times 1 \times 2000$$

$$= 43.7 \text{ kg}$$

By adding the incremental masses, the total mass of precipitable water in the column is found to be m_p = 77 kg (column 10). The equivalent depth of liquid water is $m_p / \rho_w A$ = 77/(1000 × 1) = 0.077 m = 77 mm.

The numbers in column 11 of Table 3.2.2 for percent of total mass in each increment show that more than half of the precipitable water is located in the first 2 km above the land surface in this example. There is only a very small amount of precipitable water above 10 km elevation. The depth of precipitable water in this column is sufficient to produce a small storm, but a large storm would require inflow of moisture from surrounding areas to sustain the precipitation.