Mathematics of Growth

Exponential Growth:

$N_0 =$ initial amount
$N_t =$ amount after $t$ years
$r =$ growth rate (fraction per year)

Then $N_{t+1} = N_t + r N_t = N_t (1+r)$

For example, $N_1 = N_0 (1+r)$; $N_2 = N_1 (1+r) = N_0 (1+r)^2$;

and in general, $N_t = N_0 (1+r)^t$
Continuous Compounding

Rate of change of quantity N is proportional to N
\[\frac{dN}{dt} = rN\]
\[N = N_0 e^{rt}\]

Doubling Time

The doubling time \((T_d)\) of a quantity that grows at a fixed exponential rate \(r\) is easily derived from
\[N = N_0 e^{rt}\]
Doubling Time

Doubling time can be found by setting \( N=2N_0 \) at \( t = T_d \)

\[
2N_0 = N_0 e^{rt}
\]

\( N_0 \) appears on both sides of the equation and can be canceled out and taking the natural log of both sides gives

\[
\ln 2 = r T_d
\]

\[
T_d = \ln 2/r = 0.693/r
\]

If the growth \( r \) is expressed as a percentage instead of as a fraction, we get the following important result

\[
T_d = 69.3/r(\%) = 70/r(\%)
\]
Problem

It took the world about 300 years to increase in population from 0.5 billion to 4.0 billion. If we assume exponential growth at a constant rate over the period of time, what would that growth rate be? Do it using both approach.
Logistic Growth

Figure: The logistic growth curve suggests a smooth transition from exponential growth to a steady-state population.
• Maximum Sustainable Yield

\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) \]
\[ \frac{dN}{dt} = \text{Maximum} \]

Setting the derivative equal to zero gives
\[ \frac{d}{dt} \left( \frac{dN}{dt} \right) = 0 \]
\[ \frac{d}{dt} \left( rN \left(1 - \frac{N}{K}\right) \right) = 0 \]
\[ \frac{d}{dt} \left( rN - r\frac{N^2}{K} \right) = 0 \]
\[ r \frac{dN}{dt} - \frac{r}{K} \frac{d}{dt} (N^2) = 0 \]
\[ r \frac{dN}{dt} - \frac{r}{K} 2N \frac{dN}{dt} = 0 \]
\[ r \frac{dN}{dt} \left(1 - 2\frac{N}{k}\right) = 0 \]
\[ 1 - 2\frac{N}{k} = 0 \]
\[ 2\frac{N}{k} = 1 \]
\[ N = \frac{K}{2} \]
• **Age Structure**

A graphical presentation of the data, indicating numbers of people in each age category, is called an age structure.